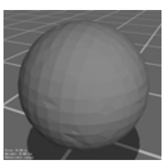
Curves & Surfaces

Today

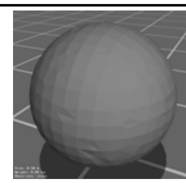
- Motivation
  - Limitations of Polygonal Models
  - Some Modeling Tools & Definitions
- Curves
- Surfaces / Patches
- Subdivision Surfaces

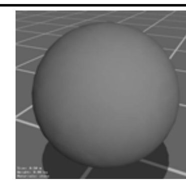
Limitations of Polygonal Meshes

- Planar facets (& silhouettes)
- Fixed resolution
- Deformation is difficult
- No natural parameterization (for texture mapping)



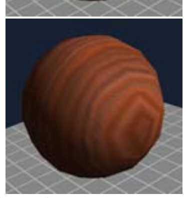
Can We Disguise the Facets?





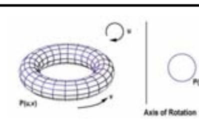
Better, but not always good enough

- Still low, fixed resolution (missing fine details)
- Still have polygonal silhouettes
- Intersection depth is planar (e.g. ray visualization)
- Collisions problems for simulation
- Solid Texturing problems
- ...

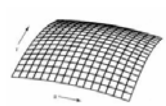


Some Non-Polygonal Modeling Tools

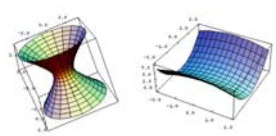
Extrusion



Surface of Revolution



Spline Surfaces/Patches



Quadrics and other implicit polynomials

Continuity definitions:

- C^0 continuous
  - curve/surface has no breaks/gaps/holes
- G^1 continuous
  - tangent at joint has same direction
- C^1 continuous
  - curve/surface derivative is continuous
  - tangent at join has same direction *and* magnitude
- C^n continuous
  - curve/surface through n^{th} derivative is continuous
  - important for shading

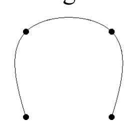
Questions?

Today

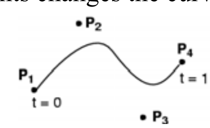
- Motivation
- Curves
  - What's a Spline?
  - Linear Interpolation
  - Interpolation Curves vs. Approximation Curves
  - Bézier
  - BSpline (NURBS)
- Surfaces / Patches
- Subdivision Surfaces

Definition: What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve



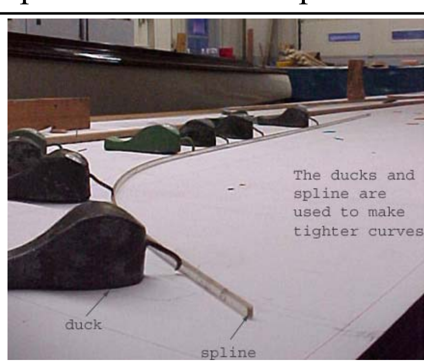
Interpolation



Bézier (approximation)

BSpline (approximation)

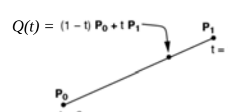
Interpolation Curves / Splines

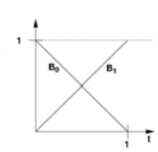


www.abm.org

Linear Interpolation

- Simplest "curve" between two points





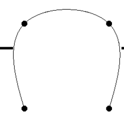
Spline Basis Functions

a.k.a. Blending Functions

$$Q(t) = \begin{pmatrix} Q_x(t) \\ Q_y(t) \\ Q_z(t) \end{pmatrix} = \begin{pmatrix} (P_0) & (P_1) \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

Interpolation Curves

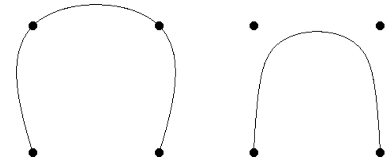


- Curve is constrained to pass through all control points
- Given points P_0, P_1, ... P_n, find lowest degree polynomial which passes through the points

$$x(t) = a_{n-1}t^{n-1} + ... + a_2t^2 + a_1t + a_0$$
$$y(t) = b_{n-1}t^{n-1} + ... + b_2t^2 + b_1t + b_0$$

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

Interpolation vs. Approximation Curves



Interpolation

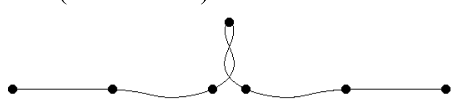
curve must pass through control points

Approximation

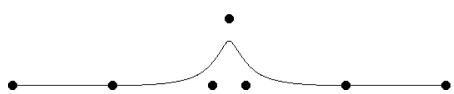
curve is influenced by control points

Interpolation vs. Approximation Curves

- Interpolation Curve – over constrained → lots of (undesirable?) oscillations

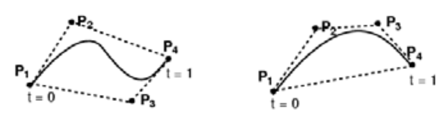


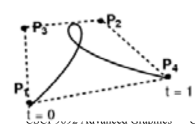
- Approximation Curve – more reasonable?



Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $\mathbf{P_0}$ to ($\mathbf{P_0}$-$\mathbf{P_1}$) and at $\mathbf{P_4}$ to ($\mathbf{P_4}$-$\mathbf{P_3}$)

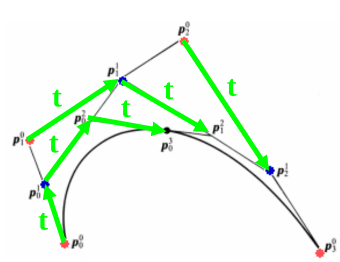




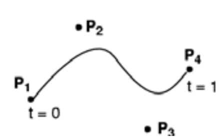
A Bézier curve is bounded by the convex hull of its control points.

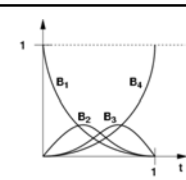
Cubic Bézier Curve

- de Casteljau's algorithm for constructing Bézier curves



Cubic Bézier Curve





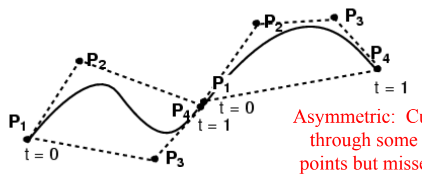
$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

$$Q(t) = \mathbf{GBT(t)} \qquad B_{Bezier} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Bernstein Polynomials

$$B_1(t) = (1-t)^3;\ B_2(t) = 3t(1-t)^2;\ B_3(t) = 3t^2(1-t);\ B_4(t) = t^3$$

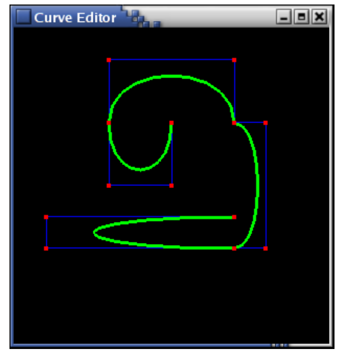
Connecting Cubic Bézier Curves



Asymmetric: Curve goes through some control points but misses others

- How can we guarantee C^0 continuity?
- How can we guarantee G^1 continuity?
- How can we guarantee C^1 continuity?
- Can't guarantee higher C^2 or higher continuity

Connecting Cubic Bézier Curves



- Where is this curve
  - C^0 continuous?
  - G^1 continuous?
  - C^1 continuous?
- What's the relationship between:
  - the # of control points, and
  - the # of cubic Bézier subcurves?

Higher-Order Bézier Curves

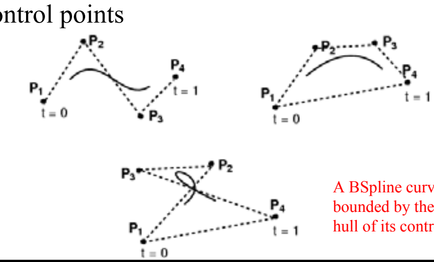
- > 4 control points
- Bernstein Polynomials as the basis functions

$$B_i^n(t) = \frac{n!}{i!(n-i)!}t^i(1-t)^{n-i}, \qquad 0 \le i \le n$$

- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling

Cubic BSplines

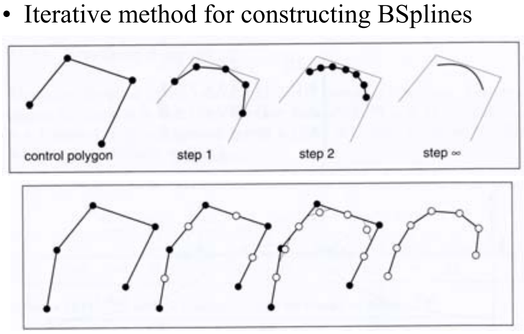
- $\ge$ 4 control points
- Locally cubic
- Curve is not constrained to pass through any control points



A BSpline curve is also bounded by the convex hull of its control points.

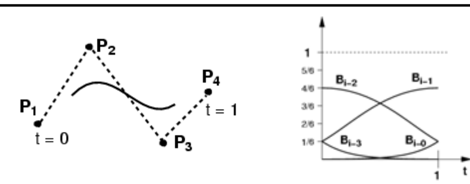
Cubic BSplines

- Iterative method for constructing BSplines



Shirley, Fundamentals of Computer Graphics

Cubic BSplines

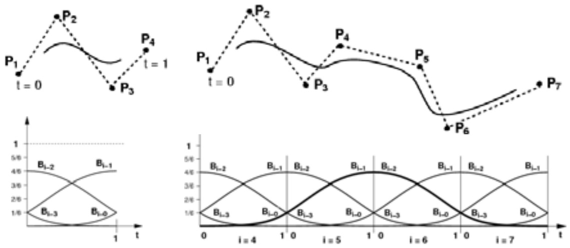


$$Q(t) = \frac{(1-t)^3}{6}P_{i-3} + \frac{3t^3-6t^2+4}{6}P_{i-2} + \frac{-3t^3+3t^2+3t+1}{6}P_{i-1} + \frac{t^3}{6}P_i$$

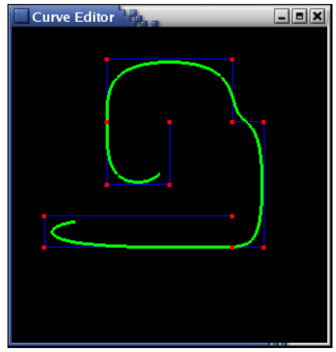
$$Q(t) = \mathbf{GBT(t)} \qquad B_{B-Spline} = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Cubic BSplines

- Can be chained together
- Better control locally (windowing)

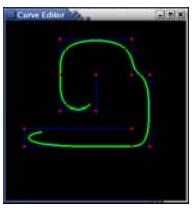


Connecting Cubic BSpline Curves

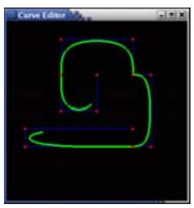


- What's the relationship between
  - the # of control points, and
  - the # of cubic BSpline subcurves?

BSpline Curve Control Points

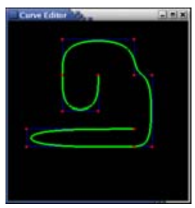


Default BSpline



BSpline with Discontinuity

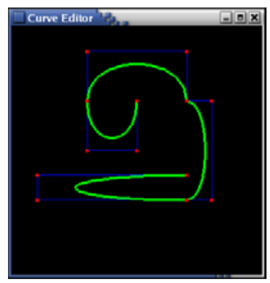
Repeat interior control point



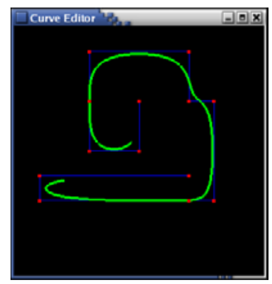
BSpline which passes through end points

Repeat end points

Bézier is not the same as BSpline



Bézier

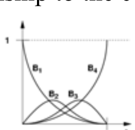


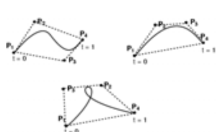
BSpline

Bézier is not the same as BSpline

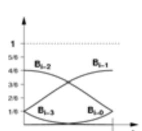
- Relationship to the control points is different

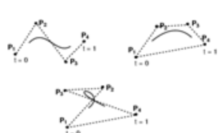
Bézier





BSpline

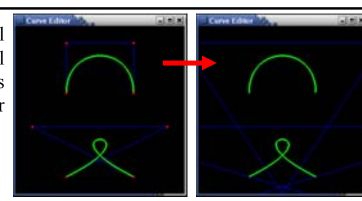




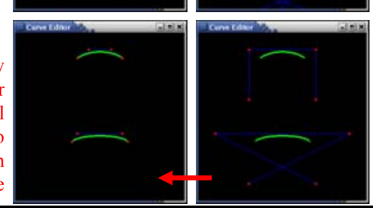
Converting between Bézier & BSpline

original control points as Bézier

new BSpline control points to match Bézier



new Bézier control points to match BSpline



original control points as BSpline

Converting between Bézier & BSpline

- Using the basis functions:

$$B_{Bezier} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B_{B-Spline} = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

NURBS (generalized BSplines)

- BSpline: uniform cubic BSpline
- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)

Questions?

Today

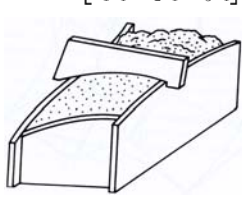
- Motivation
- Spline Curves
- Spline Surfaces / Patches
  - Tensor Product
  - Bilinear Patches
  - Bezier Patches
- Subdivision Surfaces

Tensor Product

- Of two vectors:

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \otimes \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_2b_1 & a_3b_1 \\ a_1b_2 & a_2b_2 & a_3b_2 \\ a_1b_3 & a_2b_3 & a_3b_3 \\ a_1b_4 & a_2b_4 & a_3b_4 \end{bmatrix}$$

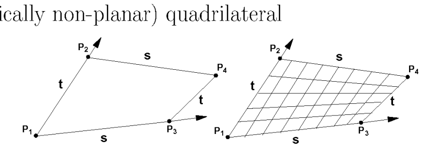
- Similarly, we can define a surface as the tensor product of two curves....



Farin, Curves and Surfaces for Computer Aided Geometric Design

Bilinear Patch

Bi-lerp a (typically non-planar) quadrilateral

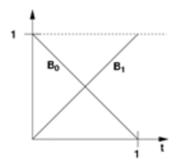


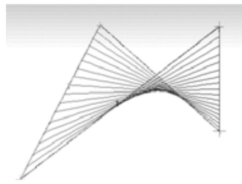
Notation: $\mathbf{L}(P_1, P_2, \alpha) \equiv (1-\alpha)P_1 + \alpha P_2$

$$Q(s,t) = \mathbf{L}(\mathbf{L}(P_1, P_2, t), L(P_3, P_4, t), s)$$

Bilinear Patch

- Smooth version of quadrilateral with non-planar vertices...





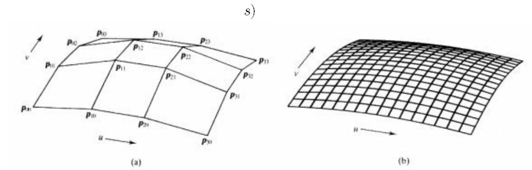
  - But will this help us model smooth surfaces?
  - Do we have control of the derivative at the edges?

Bicubic Bezier Patch

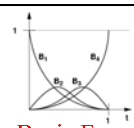
Notation: $\mathbf{CB}(P_1, P_2, P_3, P_4, \alpha)$ is Bézier curve with control points P_i evaluated at α

Define "Tensor-product" Bézier surface

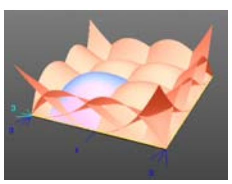
$$Q(s,t) = \mathbf{CB}(\; \mathbf{CB}(P_{00}, P_{01}, P_{02}, P_{03}, t), \mathbf{CB}(P_{10}, P_{11}, P_{12}, P_{13}, t), \mathbf{CB}(P_{20}, P_{21}, P_{22}, P_{23}, t), \mathbf{CB}(P_{30}, P_{31}, P_{32}, P_{33}, t), s)$$



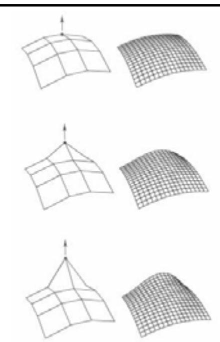
Editing Bicubic Bezier Patches



Curve Basis Functions

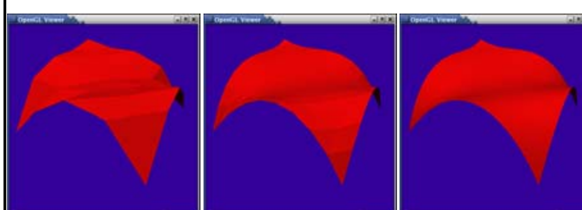


Surface Basis Functions



Bicubic Bezier Patch Tessellation

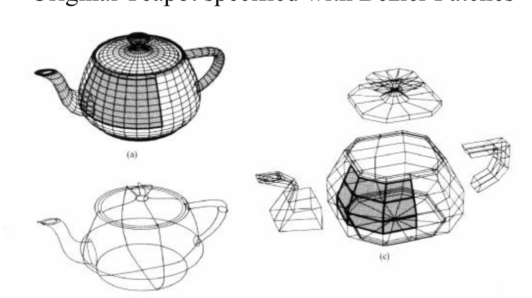
- Given 16 control points and a tessellation resolution, we can create a triangle mesh



resolution: 5x5 vertices	resolution: 11x11 vertices	resolution: 41x41 vertices

Modeling with Bicubic Bezier Patches

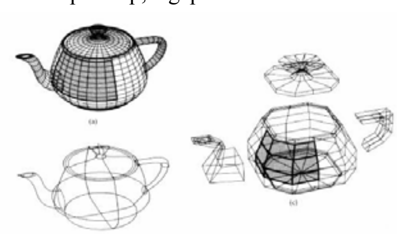
- Original Teapot specified with Bezier Patches



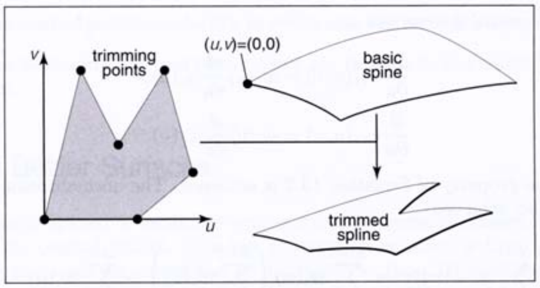
Modeling Headaches

- Original Teapot model is not "watertight":

  intersecting surfaces at spout & handle, no bottom, a hole at the spout tip, a gap between lid & base



Trimming Curves for Patches



Shirley, Fundamentals of Computer Graphics

Questions?

- Bezier Patches?

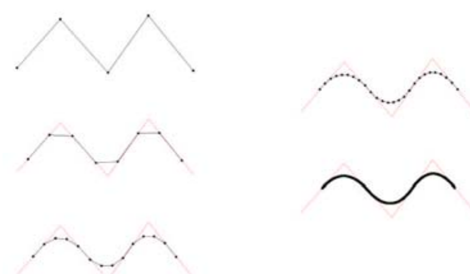
  or

- Triangle Mesh?

Today

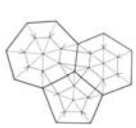
- Review
- Motivation
- Spline Curves
- Spline Surfaces / Patches
- Subdivision Surfaces

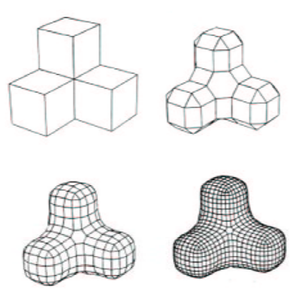
Chaikin's Algorithm



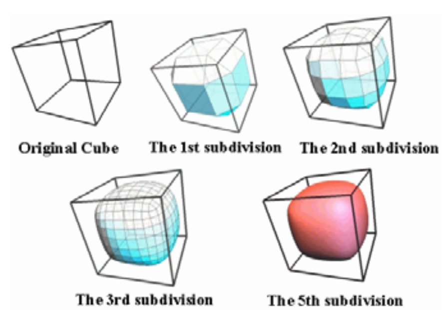
Doo-Sabin Subdivision

Idea: introduce a new vertex for each face
At the midpoint of old vertex, face centroid



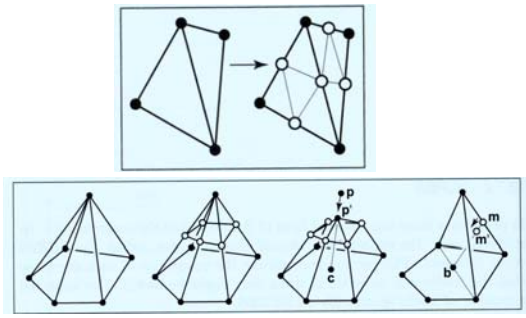


Doo-Sabin Subdivision



http://www.ke.ics.saitama-u.ac.jp/xuz/pic/doo-sabin.gif

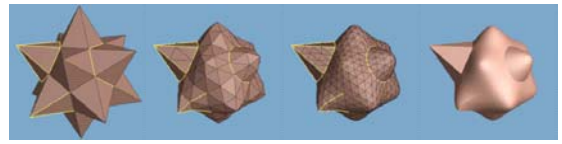
Loop Subdivision



Shirley, Fundamentals of Computer Graphics

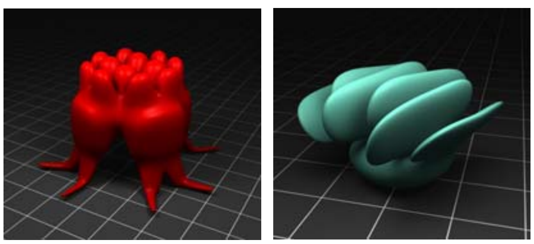
Loop Subdivision

- Some edges can be specified as crease edges



http://grail.cs.washington.edu/projects/subdivision/

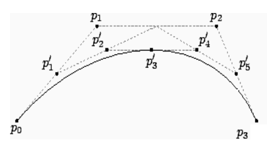
Questions?



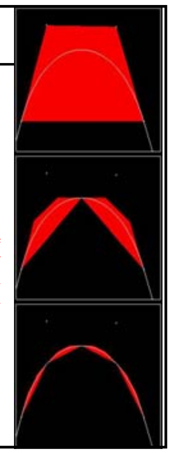
Justin Legakis

Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  - P_0 P_1 P_2 P_3
- Can be split into 2 new Bezier curves:
  - P_0 P'_1 P'_2 P'_3
  - P'_3 P'_4 P'_5 P_3



A Bézier curve is bounded by the convex hull of its control points.



Reading for Friday:

- "Subdivision Surfaces in Character Animation", DeRose, Kass & Truong SIGGRAPH 1998
- Additional Reference:
  SIGGRAPH 99 course notes
  Subdivision for Modeling and Animation