CSCI 4260/MATH 4150 Homework 3

Due: 12pm on Monday, April 16

- The homework contains 6 questions. Total points: 15 + 10 + 15 + 15 + 20 + 25 = 100.
- Late homework policy: **Between April 16 noon and Tuesday April 17 3:30pm (end of Nikhil's office hour) 70%. Afterwards, 0%.**
- In this homework, you can discuss problems with no more than one classmate. However, you must (i) write your solutions independently, and (ii) on your solution, specify the name of your collaborator.
- You can not use any material other than the textbook and lecture notes (especially no Internet).
- Homeworks are perfect ways of preparing for the exams. Therefore, before solving homework questions, please go over the theorems and proofs we covered in class. Then, try to solve homework questions using the principles, ideas and concepts you have learned.

Question 1 (15 points):

In the standard network-flow formulation, we have capacities defined on the arcs. Suppose, in addition to arc capacities, we are given a node capacity $c(u)$ for each node u. We now add the following requirement to the definition of feasibility:

Flow f is feasible if

(i) it satisfies conservation and edge capacity constraints, and

(ii) the *net* amount of flow leaving node u is less than or equal to $c(u)$, $\forall u$.

Show how the max-flow problem with this additional constraint can be solved by using the standard max-flow algorithm as a subroutine. (Hint: transform your input to the standard flow formulation. Then establish a 1-1 correspondence between the max-flow for the transformed network and the max-flow for the input flow network with vertex capacities).

Question 2 (10 points): Prove or disprove: every tree has at most one perfect matching. (hint: symmetric difference).

Question 3 (15 points): Exercise 8.4 on Page 192 of the textbook.

Question 4 (15 points): Let $\alpha(G)$ be the size of a maximum matching on *a* graph G.

Prove that every *maximal* matching in G has at least $\frac{\alpha(G)}{2}$ edges.

Question 5 (20 points): Let M and M' be matchings in a bipartite graph $G = (X, Y, E)$. Suppose M saturates $S \subseteq X$ and that M' saturates $T \subseteq Y$. Prove that G has a matching that saturates $S \cup T$.

Question 6 (25 points): Two people play a game on a bipartite graph $G = (X, Y, E)$ with $|X| = |Y|$. The game is played in turns. At each turn, the current player chooses a vertex. Once a vertex is chosen, it can not be chosen again. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice of the other player. Thus, the chosen vertices together follow a path. The last player able to move wins.

Prove that the second player has a winning strategy if G has a perfect matching and otherwise the first player wins the game.
Hint: present a strategy for each case and show why these strategies work.