CSCI 4260/MATH 4150 Homework 3

Due: 12pm on Monday, April 30

- The homework contains 5 questions, each worth 20 points. There is also a bonus question worth 15 points.
- Late homework policy: **Between April 30 noon and Tuesday May 1st 3:30pm (end of Nikhil's office hour) 70%. Afterwards, 0%.**
- In this homework, you can discuss problems with no more than one classmate. However, you must (i) write your solutions independently, and (ii) on your solution, specify the name of your collaborator.
- You can not use any material other than the textbook and lecture notes (especially no Internet).
- Homeworks are perfect ways of preparing for the exams. Therefore, before solving homework questions, please go over the theorems and proofs we covered in class. Then, try to solve homework questions using the principles, ideas and concepts you have learned.

Question 1 (20 points): Show that every triangulation of a polygon (without holes) with n vertices has exactly $(n-3)$ diagonals and $(n-2)$ triangles.

Question 2 (20 points): A private club has 90 rooms and 100 members. Keys are given to the members so that any 90 members have access to the rooms in the sense that each of these 90 members will have a key to a different room. (They do not share their keys.) Prove that at least 990 keys are needed and that 990 suffice.

Question 3 (20 points): Let G be a graph such that $\chi(G-u-v) = \chi(G) - 2$ for all distinct pairs of vertices u and v. Prove that G is a complete graph. Note: χ denotes the chromatic number.

Question 4 (20 points): A dominating set of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that every vertex in V is adjacent to at least one vertex from from S. Show that every maximal independent set is a dominating set.

Question 5 (20 points): Consider the following greedy algorithm to compute a maximum independent set of a graph G.

$S \leftarrow \emptyset$; Let v be a vertex of minimum degree. Add v to S. Delete v and all of its neighbors from G. Repeat until all vertices are deleted. Output S as an independent set.

Clearly this algorithm outputs an independent set. In this problem, you are asked to show that its performance is quite bad: Show that for any given k, we can construct a graph with $O(k)$ vertices where there is an independent set of size k but the greedy algorithm outputs only a constant number of vertices.

Bonus Question (3+12 = 15 points): Let k be the genus of K_6. What is k? Prove your answer by:

(a) showing that K_6 can not be embedded on a surface with $k-1$ handles, and

(b) presenting a drawing of K_6 on a surface with k handles.