Computational Finance – Portfolio Optimization and Capital Allocation

1 Introduction

Given a set of n instruments $s_1, \ldots, s_n$ with with possibly uncertain returns, the task is to select a portfolio $\theta_1, \ldots, \theta_n$ of the instruments so as to maximize one's utility.

More formally, let $p_1, \ldots, p_n$ be the current prices of the instruments, and let the future prices be $q_1, \ldots, q_n$. The future prices q_i are random variables. There may be a risk free asset s_0 with current price p_0 and deterministic future price q_0. Let W_0 be the wealth now, available to be invested, and assume that a fraction of this wealth w_i is invested in instrument i, so that the number of units θ_i of each instrument purchased is given by

$$\theta_i = \frac{w_i W_0}{p_i}.$$

The future wealth is given by

$$\begin{aligned} W &= \sum_i \theta_i q_i, \\ &= W_0 \sum_i w_i \frac{q_i}{p_i}. \end{aligned}$$

We define the return of instrument i by $R_i = \frac{q_i}{p_i}$, which is a random variable, and the portfolio return by $R = \frac{W}{W_0}$. Then,

$$\begin{aligned} R(\mathbf{w}) &= \sum_i w_i R_i, \\ &= \mathbf{w}^{\mathrm{T}} \mathbf{R}. \end{aligned}$$

where $\mathbf{w}$ is the vector of portfolio weights, and $\mathbf{R}$ is the vector of returns. We define the expected returns $\mu_i = E[R_i]$ and the expected return vector $\boldsymbol{\mu}$ to be the vector of expected returns. Then the expected return $\mu(R) = E[R(\mathbf{w})]$ is

$$\begin{aligned} \mu &= \sum_i w_i \mu_i, \\ &= \mathbf{w}^{\mathrm{T}} \boldsymbol{\mu}. \end{aligned}$$

We define the covariance matrix Σ by $\Sigma_{ij} = Cov[R_i, R_j]$. Then the variance of the return, $\sigma^2(R) = Var[R(\mathbf{w})]$ is given by

$$\begin{aligned} \sigma^2 &= \sum_i \sum_j w_i w_j \Sigma_{ij}, \\ &= \mathbf{w}^{\mathrm{T}} \Sigma \mathbf{w}. \end{aligned}$$

2 Maximizing Expected Utility

Everyone has a utility function $U(\cdot)$ which is increasing and concave, and one wishes to maximize the expected utility[1]. Since $\sum_i w_i = 1$ Thus the solution to the portfolio optimization problem is given by

$$\begin{aligned} &\max_{\mathbf{w}} E[U(R(\mathbf{w}))] \quad , \\ \text{s.t.} \quad &\mathbf{w}^{\mathrm{T}}\mathbf{1} = 1, \end{aligned}$$

where $\mathbf{1}$ is the vector of ones. The utility function is increasing and concave, so $U' > 0$ and $U'' < 0$. Often there may be other constraints on the portfolio, for example, a long-only constraint which requires all $w_i \geq 0$. In this case the portfolio optimization problem becomes

$$\begin{aligned} &\max_{\mathbf{w}} E[U(R(\mathbf{w}))] \quad , \\ \text{s.t.} \quad &\mathbf{w}^{\mathrm{T}}\mathbf{1} = 1, \\ &\mathbf{w} \geq 0, \end{aligned}$$

Other types of common constraints are that the weights be integers ($w_i \in \mathbb{Z}$) or binary (all or nothing, $w_i \in \{0, 1\}$). We will focus on the problem above, with and without the long only constraint. discrete constraints (for example integer or binary) are generally very difficult to handle.

3 Mean-Variance Analysis

The Second Order Approximation – Quadratic Utility To second order, we can write

$$U(R) = U(\mu) + U'(\mu)(R - \mu) + \frac{1}{2}U''(\mu)(R - \mu)^2,$$

where we have not shown the dependence on $\mathbf{w}$. Taking the expectation, the middle term equates to 0, and we have (to second order)

$$E[U(R)] = U(\mu) + \frac{1}{2}U''(\mu)\sigma^2(R).$$

Since $U'' < 0$, we see that for fixed expected return μ, $E[U(R)]$ is a decreasing function of σ,

$$\frac{\partial E[U(R)]}{\partial \sigma} < 0.$$

[1]We do not get into the details but the existence of a strictly increasing concave utility function for which one solves the portfolio optimization problem by maximizing expected utility is intimitately to the non-existence of type I or II arbitrage (see for example [**?**]).

Exercise 3.1

Suppose that one wishes to maximize, over portfolios $\mathbf{w}$, a function $f(\mu, \sigma)$ which satisfies $\frac{\partial f}{\partial \sigma} < 0$.

Show that if one desires a particular expected return μ, so that $\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} = \mu$, then among all portfolios having the desired expected return, the one maximizing f is the one with the minimum variance.

Exercise 3.2

Suppose that the return $R(\mathbf{w})$ is Normally distributed, so that $R \sim N(\mu, \sigma^2)$. Then $E[U(R)]$ is a function of μ, σ^2, $E[U(R)] = U(\mu, \sigma^2)$. Show that

$$\frac{\partial V}{\partial \sigma} < 0.$$

[Hint: Show that

$$U(\mu, \sigma) = \int_{-\infty}^{\infty} d\epsilon \ \epsilon n(\epsilon) U'(\mu + \sigma\epsilon),$$

where $n(\cdot)$ is the standard normal probability density function. Now integrate by parts (note that $\int d\epsilon \ \epsilon n(\epsilon) = -n(\epsilon)$), and use the fact that $U'' < 0$.]

From the previous exercises, we conclude that for the second order approximation or for Normally distributed return,

$$\frac{\partial E[U(R)]}{\partial \sigma} < 0.$$

This means that for a fixed value of μ, the portfolio which minimizes the variance is the maximum utility portfolio having that return. We define $\sigma^2_{min}(\mu)$ as the minimum variance attainable for a given μ and $\mathbf{w}^*(\mu)$ as the portfolio which minimizes the variance for a given μ. We obtain $\sigma^2_{min}(\mu)$ and $\mathbf{w}^*(\mu)$ from the solution to the following optimization problem,

$$\begin{aligned} &\min_{\mathbf{w}} \mathbf{w}^{\mathsf{T}}\Sigma\mathbf{w} \quad , \\ \text{s.t.} \quad &\mathbf{w}^{\mathsf{T}}\mathbf{1} = 1, \\ &\mathbf{w}^{\mathsf{T}}\boldsymbol{\mu} = \mu, \\ &\mathbf{w} \geq 0, \end{aligned}$$

This is a well known optimization problem, known as a quadratic program. In fact, it is so well known that most mathematical packages come with a built in routine for solving such problems. For example in matlab 7, such a routine is provided by the function quadprog in the optimization toolbox. Typically such routines solve quadratic programs allowing one to specify the objective function and the linear equality and inequality constraints. In our case, there are $n+2$ constraints, so the solution time is $O(n^3)$.

Exercise 3.3

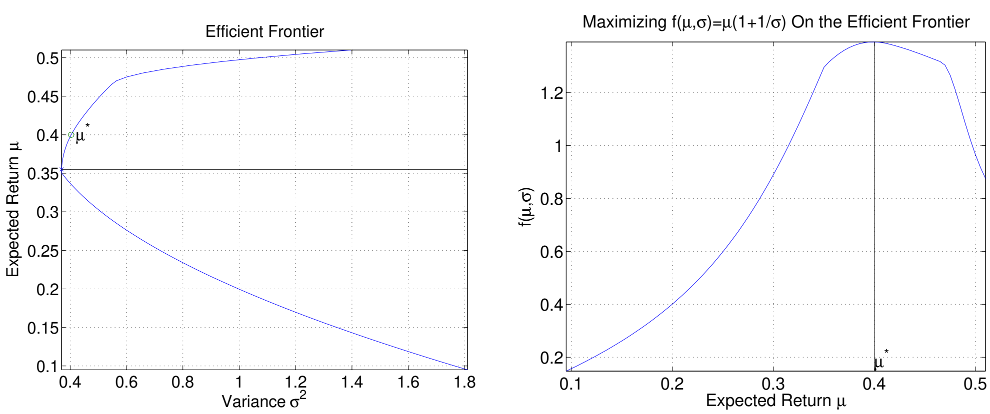
Show that for any utility function $U(\mu,\sigma)$ with $\frac{\partial V}{\partial \sigma} < 0$ the utility maximizing portfolio is given by $w^*(\mu^*)$ for some μ^*.

The previous exercise shows that the utility maximizing portfolio can be constructed by performing a one-dimensional minimization over μ. Specifically by varying μ, one may compute $\sigma^2_{min}(\mu)$, and minimize the one dimensional function

$$Q(\mu) = U(\mu, \sigma^2_{min}(\mu)).$$

This approach would work for any utility function $U(\mu,\sigma)$ with $\frac{\partial U}{\partial \sigma} < 0$. A common alternative utility function with this property (for positive μ) is the Sharpe ratio, $U(\mu,\sigma) = \frac{\mu}{\sigma}$.

As we vary the expected return μ, the minimum attainable variance $\sigma^2_{min}(\mu)$ traces a curve in σ^2–μ space which is known as the efficient frontier. Points which are above this curve are inefficient in the sence that lower variance (higher utility) portfolios exist yielding the same expected return. Points below this curve are unattainable. The figure below illustrates the efficient frontier for an example expected return vector and covariance matrix.



For this example, we have chosen $\boldsymbol{\mu}$ and Σ as

$$\boldsymbol{\mu} = \begin{bmatrix} 0.5129 \\ 0.4605 \\ 0.3504 \\ 0.0950 \\ 0.4337 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1.5036 & 0.4958 & 0.5528 & 1.4101 & 0.7228 \\ 0.4958 & 0.5479 & 0.3650 & 0.6561 & 0.7062 \\ 0.5528 & 0.3650 & 0.3699 & 0.6339 & 0.5108 \\ 1.4101 & 0.6561 & 0.6339 & 1.8100 & 1.0356 \\ 0.7228 & 0.7062 & 0.5108 & 1.0356 & 0.9733 \end{bmatrix}$$

The optimal portfolio lies on the efficiency frontier, which for our example is illustrated in the figure above where we are maximizing $f(\mu, \sigma) = \mu(1 + \frac{1}{\sigma})$. The range of attainable expected return is $[\min \mu_i, \max \mu_i]$. As can be seen,which is typically the case, the efficiency frontier has two branches, the so called lower branch and the upper branch. Notice that the upper branch is the most practical with higher expected return for a given risk σ^2.

Exercise 3.4

For the example in the previous discussion, for the choices of $\boldsymbol{\mu}$ and Σ, reconstruct the efficient frontier construct the optimal portfolios maximizing the utility functions

(a) The Sharpe ratio $U(\mu, \sigma) = \frac{\mu}{\sigma}$.

(b) The Quadratic Utility $U(\mu, \sigma) = \mu - \lambda\sigma^2$ for $\lambda \in \{0.1, 0.2, 1\}$.

3.0.1 Exact Solution When Short Selling is Allowed

When there is no sign constraint on the weights $\mathbf{w}$ we have the following optimization problem

$$\begin{aligned} & \min_{\mathbf{w}} \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} \quad , \\ \text{s.t.} \quad & \mathbf{w}^{\mathrm{T}}\mathbf{1} = 1, \\ & \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} = \mu. \end{aligned}$$

Setting up the Lagrangian, we have

$$\mathcal{L}(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w} + \lambda_1(\mathbf{w}^{\mathrm{T}}\mathbf{1} - 1) + \lambda_2(\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} - \mu).$$

Setting $\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \mathbf{0}$ at an optimum $\mathbf{w}^*$ gives

$$\mathbf{w}^* = -\frac{1}{2}\lambda_1\Sigma^{-1}\mathbf{1} - \frac{1}{2}\lambda_2\Sigma^{-1}\boldsymbol{\mu}.$$

We are assuming that $\Sigma$is invertible, which in particular means that there is no risk free asset (as then a row and column in Σ which is all 0), and no security is a linear combination of some other set of securities. The constraints require that

$$\begin{aligned}\mathbf{1}^{\mathsf{T}}\Sigma^{-1}\mathbf{1}\lambda_1+\mathbf{1}^{\mathsf{T}}\Sigma^{-1}\boldsymbol{\mu}\lambda_2 &= -2,\\ \boldsymbol{\mu}^{\mathsf{T}}\Sigma^{-1}\mathbf{1}\lambda_1+\boldsymbol{\mu}^{\mathsf{T}}\Sigma^{-1}\boldsymbol{\mu}\lambda_2 &= -2\mu.\end{aligned}$$

In matrix format we have

$$\begin{bmatrix}\mathbf{1}^{\mathsf{T}}\Sigma^{-1}\mathbf{1} & \mathbf{1}^{\mathsf{T}}\Sigma^{-1}\boldsymbol{\mu}\\ \mathbf{1}^{\mathsf{T}}\Sigma^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}^{\mathsf{T}}\Sigma^{-1}\boldsymbol{\mu}\end{bmatrix}\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}=-2\begin{bmatrix}1\\ \mu\end{bmatrix}.$$

Define $\alpha=\mathbf{1}^{\mathsf{T}}\Sigma^{-1}\mathbf{1}$, $\beta=\mathbf{1}^{\mathsf{T}}\Sigma^{-1}\boldsymbol{\mu}$ and $\gamma=\boldsymbol{\mu}^{\mathsf{T}}\Sigma^{-1}\boldsymbol{\mu}$. Taking the inverse of the matrix, one can solve for λ_1,λ_2.

Exercise 3.5

Solve for λ_1,λ_2 above.

(a) Show that the inverse of the 2×2 matrix defined by α,β,γ exists. [Hint: Show that its determinant is $\alpha\gamma-\beta^2>0$ using the Cauchy-Schwarz inequality.]

(b) Show that

$$\lambda_1=-\frac{2(\gamma-\beta\mu)}{\alpha\gamma-\beta^2},\qquad \lambda_2=-\frac{2(\alpha\mu-\beta)}{\alpha\gamma-\beta^2}.$$

(c) Show that the optimal portfolio is given by

$$\mathbf{w}^*=\frac{\gamma-\beta\mu}{\alpha\gamma-\beta^2}\Sigma^{-1}\mathbf{1}+\frac{\alpha\mu-\beta}{\alpha\gamma-\beta^2}\Sigma^{-1}\boldsymbol{\mu}.$$

(d) Let $\mathbf{w}_g$ be the portfolio with globally minimum variance satisfying the budget constraint $\mathbf{w}^{\mathsf{T}}\mathbf{1}=1$. Show that

$$\mathbf{w}_g=\frac{1}{\alpha}\Sigma^{-1}\mathbf{1}.$$

(e) Define a second portfolio $\mathbf{w}_{\boldsymbol{\mu}}=\frac{1}{\beta}\Sigma^{-1}\boldsymbol{\mu}$. Verify that

$$\mathbf{w}_g^{\mathsf{T}}\mathbf{1}=1,\qquad \mathbf{w}_{\boldsymbol{\mu}}^{\mathsf{T}}\mathbf{1}=1.$$

(f) Show that

$$\mathbf{w}^*=\rho\mathbf{w}_g+(1-\rho)\mathbf{w}_{\boldsymbol{\mu}},$$

where $\rho=\frac{\alpha\gamma-\alpha\beta\mu}{\alpha\gamma-\beta^2}$.

The result of the previous exercise is an example of a separation result which says that the efficient portfolios on the efficient frontier are a linear combination of two portfolios, $\mathbf{w}_g, \mathbf{w}_{\boldsymbol{\mu}}$. The portfolios $\mathbf{w}_g, \mathbf{w}_{\boldsymbol{\mu}}$ can be viewed as two "mutual funds" and this result says that in the mean-variance optimal world, two mutual funds suffice $(\mathbf{w}_g, \mathbf{w}_{\boldsymbol{\mu}})$ – the global minimum variance portfolio, and a second portfolio $\mathbf{w}_{\boldsymbol{\mu}}$. Every optimal portfolio can be expressed as a linear combination of these. In fact, the efficient frontier has a very simple parabolic form.

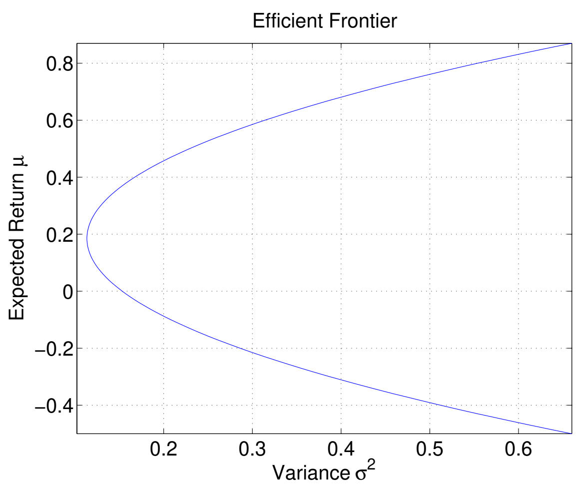
Exercise 3.6

Show that the efficient frontier in mean-variance space has a parabolic form. In particular, show that

$$\sigma_{min}(\mu) = \sqrt{\frac{1}{\alpha}\left(\frac{(\alpha\mu - \beta)^2}{\alpha\gamma - \beta^2} + 1\right)},$$

and plot the mean variance frontier for the example shown in the previous subsection, this time without using the long only constraint.

[Answer:



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Exercise 3.7

This problem investigates what happens when there is a risk free instrument s_0 with expected return μ_0. If we treat all instruments equally, then the Σ matrix becomes singular and the previous prescription does not work. Let w_0 be the weighting of s_0 in the portfolio, and as before, let $\mathbf{w}$ be the weighting in the other n instruments.

(a) Show that the mean variance optimization problem without the long only constraint becomes

$$\begin{aligned} &\min_{\mathbf{w}} \mathbf{w}^{\intercal}\Sigma\mathbf{w} \qquad , \\ \text{s.t.} \quad &w_0 + \mathbf{w}^{\intercal}\mathbf{1} = 1, \\ &w_0\mu_0 + \mathbf{w}^{\intercal}\boldsymbol{\mu} = \mu. \end{aligned}$$

Note that w_0 does not appear in the objective, but it appears in the constraints.

(b) Solve this problem and plot the mean-variance efficient frontier for the same parameters as the previous exercise and $\mu_0 = 0.05$.

3.1 Estimating $\boldsymbol{\mu}$ and Σ

All the previous discussion assumed knowledge of $\boldsymbol{\mu}$ and Σ. In practice, one has to estimate $\boldsymbol{\mu}$ and Σ using estimates $\hat{\boldsymbol{\mu}}$ and $\hat{\Sigma}$ which have been generated over historical data. In particular, suppose that we have observed $s_i(t)$ for $t = 1, \ldots, T$. We can compute

$$\hat{R}_i(t) = \frac{s_i(t)}{s_i(t-1)}.$$

One can then estimate the expected return $\hat{\mu}_i$ as the sample mean of $\hat{R}_i(t)$, and similarly, one can estimate the covariance matrix $\hat{\Sigma}_{ij}$ as the sample covariance between $\hat{R}_i(t)$ and $\hat{R}_j(t)$.

4 Portfolio Optimization with Maximum Drawdown Constraints

In order to construct the mean-variance efficient frontier, one has to use historical data to estimate the covariances and expected returns, which then become inputs to a quadratic program. Since we need historical data we may as well discuss alternate approaches to allocating the portfolio which use a risk measure other than the variance. One natural risk measure is the maximum drawdown (MDD). Suppose that an instrument s_i has values $s_i(t)$. Let $\mu_i(t)$ be the return up to time t for instrument i,

$$\mu_i(t) = \frac{s_i(t)}{s_i(0)}.$$

Suppose we have a portfolio which invests a fraction w_i of the wealth at time 0 in instrument s_i. Then for the portfolio, we have that the return up to time t is given by

$$\begin{aligned} R_{\mathbf{w}}(t) &= \sum_{i=1}^{n} w_i \mu_i(t), \\ &= \mathbf{w}^{\mathrm{T}} \boldsymbol{\mu}(t), \end{aligned}$$

where $\boldsymbol{\mu}(t)$ is the vector containing the returns of the instruments up to time t. We define the MDD of the portfolio over the time period $[0, T]$ as the MDD of the return curve $\boldsymbol{R}_{\mathbf{w}}$, where the return curve for the portfolio is given by,

$$\begin{aligned} \boldsymbol{R}_{\mathbf{w}} &= \begin{bmatrix} R_{\mathbf{w}}(1) \\ R_{\mathbf{w}}(2) \\ \vdots \\ R_{\mathbf{w}}(T) \end{bmatrix}, \\ MDD &= MDD(\boldsymbol{R}_{\mathbf{w}}). \end{aligned}$$

The final return of the portfolio is

$$\begin{aligned} R(\mathbf{w}) &= R_{\mathbf{w}}(T), \\ &= \mathbf{w}^{\mathrm{T}} \boldsymbol{\mu}(T). \end{aligned}$$

As with the mean-variance analysis, we may consider maximizing a utility function

$$U(R(\mathbf{w}), MDD(\boldsymbol{R}_{\mathbf{w}})),$$

which for fixed $R(\mathbf{w})$ is decreasing in $MDD(\boldsymbol{R}_{\mathbf{w}})$. In this case, we need to construct the mean-MDD efficient frontier, which corresponds to constructing the portfolio $\mathbf{w}^*(\mu)$ which minimizes the MDD while attaining a specified return μ. Equivalently, we can consider the problem of constructing the optimal portfolio $\mathbf{w}^*(MDD)$ which maximizes the return while constrained to having a specified MDD.

Exercise 4.8

Show that the efficient frontier obtained by minimizing the MDD for a specified return and the efficient frontier obtained by maximizing the return for a specified MDD are equivalent (under some mild conditions).

We will obtain the mean-MDD efficient frontier by constructing the optimal portfolio which maximizes the return while attaining a specified MDD. For a particular instrument, we define its return series as

$$\boldsymbol{R}_i = \begin{bmatrix} \mu_i(1) \\ \mu_i(2) \\ \vdots \\ \mu_i(T) \end{bmatrix},$$

and we define the *return matrix* as the matrix whose columns are the individual instrument return series,

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{R}_1 & \boldsymbol{R}_2 & \cdots & \boldsymbol{R}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^{\mathrm{T}}(1) \\ \boldsymbol{\mu}^{\mathrm{T}}(2) \\ \vdots \\ \boldsymbol{\mu}^{\mathrm{T}}(T) \end{bmatrix}.$$

Notice that

$$\boldsymbol{R}_{\mathbf{w}} = \boldsymbol{R}\mathbf{w}.$$

To obtain the efficient frontier, we need to maximize the return subject to the MDD constraint. Specifically, we need to solve the optimization problem

$$\begin{aligned} \max_{\mathbf{w}} \ & \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu}(T), \\ \text{s.t.} \ & MDD(\boldsymbol{R}_{\mathbf{w}}) \leq mdd, \\ & \mathbf{w}^{\mathrm{T}}\mathbf{1} = 1, \\ & \mathbf{w} \geq 0. \end{aligned}$$

Except for the ugly looking MDD constraint, this looks like a linear program. Fortunately, it turns out that the MDD constraint can be rephrased in terms of linear inequalities, and so we will end up with a linear program. Specifically lets introduce the drawdown to time t, $DD(t)$,

$$DD(t) = \max_{i \leq t} R_{\mathbf{w}}(i) - R_{\mathbf{w}}(t).$$

$DD(t)$ is the drawdown from the previous high in getting to time t. The reason for introducing the function $DD(t)$ is that the MDD is equal to $DD(t^*)$ for some time t^*. In particular, the time for which the $DD(t)$ is maximized.

Exercise 4.9

Show that $MDD = \max_t DD(t)$.

Exercise 4.10

Show that the constraint $MDD \le mdd$ is equivalent to the T constraints

$$\begin{aligned} DD(1) &\le mdd \quad , \\ DD(2) &\le mdd \quad , \\ &\vdots \quad\quad\quad , \\ DD(T) &\le mdd \quad . \end{aligned}$$

Thus, we may replace the MDD constraint with T constraints on the DD. Let z_t be the maximum of $R_{\mathbf{w}}(i)$ up to t. Then

$$DD(t) = z_t - R_{\mathbf{w}}(t).$$

The z_t can be defined recursively by $z_0 = 0$, and

$$z_t = \max(z_{t-1}, R_{\mathbf{w}}(t)).$$

Combining the MDD constraint with this recursion, we obtain a set of inequalities which need to be satisfied for $t = 1, \ldots, T$,

$$\begin{aligned} z_t - R_{\mathbf{w}}(t) &\le mdd, \\ z_{t-1} &\le z_t, \\ R_{\mathbf{w}}(t) &\le z_t. \end{aligned}$$

We introduce the auxiliary variable $\mathbf{z}$, and then in vector form, these inequalities become $z_0 = 0$ and

$$\begin{aligned} \mathbf{z} - \boldsymbol{R}\mathbf{w} &\le mdd \cdot \mathbf{1}, \\ z_{t-1} &\le z_t \qquad \text{for } t = 1, \ldots, T, \\ \boldsymbol{R}\mathbf{w} &\le \mathbf{z}. \end{aligned}$$

These inequalities are implied by the MDD constraint. In particular any set of $z_0, \mathbf{z}$ which satisfy these constraints implies that $MDD(\boldsymbol{R}_{\mathbf{w}}) \le mdd$.

Exercise 4.11

Show that for any set of $z_0, \mathbf{z}$ satisfying the above constraints, $DD(t) \le z_t - R_{\mathbf{w}}(t) \le mdd$ and hence $MDD(\boldsymbol{R}_{\mathbf{w}}) \le mdd$.

For the z_t to actually equal the maxima and for this set of constraints to be equivalent to the MDD constraint, the z_t should be as small as possible. It turns out that by trying to maximize the return, this is implicitly accomplished. We thus have the following equivalent optimization problem to compute the efficient frontier,

$$\begin{aligned} &\max_{\mathbf{w},\mathbf{z},z_0} \ \mathbf{w}^{\mathrm{T}}\boldsymbol{\mu}(T), \\ \text{s.t.} \quad & z_0 = 0, \\ & \mathbf{z} - \boldsymbol{R}\mathbf{w} \le mdd \cdot \mathbf{1}, \\ & z_{t-1} \le z_t \qquad \text{for } t = 1, \ldots, T, \\ & \boldsymbol{R}\mathbf{w} \le \mathbf{z}, \\ & \mathbf{w}^{\mathrm{T}}\mathbf{1} = 1, \\ & \mathbf{w} \ge 0. \end{aligned}$$

The variables $z_0, \mathbf{z}$ are called auxilliary variables, and it is common to introduce such variables in transforming a more complex optimization problem into a linear program. We now have a linear program with $3T + 1$ additional constraints for a total of $n + 3T + 2$ constraints. Solving this LP is thus in $O((n + 3T)^3)$ using standard packages like linprog in matlab.

Exercise 4.12

While we have argued that the solution to the LP is equivalent to the solution to the original constrained MDD problem, we have not actually shown it.

(a) Show that any solution $\mathbf{w}, \mathbf{z}, z_0$ to this LP results in an allocation $\mathbf{w}$ which is feasible for the original MDD-constrained optimization.

(b) Show that for any solution $\mathbf{w}$ to the original MDD-constrained optimization, there is a feasible point having the same $\mathbf{w}$ and appropriately chosen $\mathbf{z}, z_0$ which is feasible for this LP.

(c) Show that the allocation $\mathbf{w}$ which results from a solution to this LP is a solution to the MDD-constrained return optimization problem.

4.1 Integer Portfolio Constraints and Mixed Integer Linear Programming

One natural way in which an integer constraint appears is when securities can only be bought in integral amounts. Another which we present through an example is when a particular investment s_i may be invested in at some minimum level m_i up to some maximum level M_i and there is a fixed cost f_i to taking on this investment. We assume that all these costs are expressed as fractions of the initial wealth W_0. In such cases, it is typically useful to introduce the binary variables $y_i \in \{0, 1\}$ which indicate whether or not the investment is undertaken. In this case, we have that $w_i \in \{0\} \cup [m_i, M_i]$, which can be summarized by the linear constraints

$$M_i y_i \geq w_i \geq m_i y_i.$$

These constraints automatically imply that if $y_i = 0$, then $w_i = 0$ and if $y_i = 1$ then $w_i \in [m_i, M_i]$. The budget constraint is $\sum_i w_i + f_i y_i \leq 1$. Thus we see that all the constraints and the objective can be expressed in terms of linear inequalities.The only catch is that some of the variables are binary. The resulting optimization problem is called a *mixed program.* If the objective and all constraints are linear equality or inequality constraints, it is a *mixed integer linear program (MILP).* For example, when one is trying to simply maximize the return under such constraints, we have

$$\begin{aligned} & \max_{\mathbf{w},\mathbf{y}} \ \mathbf{w}^{\mathrm{T}} \boldsymbol{\mu}, \\ \text{s.t.} \qquad & y_i \in \{0, 1\}, \\ & w_i \leq M_i y_i, \\ & m_i y_i \leq w_i, \\ & \textstyle\sum_{i=1}^{n} w_i + f_i y_i \leq 1, \\ & \mathbf{w} \geq 0. \end{aligned}$$

Such problems are typically very hard to solve exactly (even in the computational sense) and one has to resort to approximation algorithms. We will not elaborate further, except to say that there is a vast literature on approximate solution of integer linear programs.