Announcements

- Quiz 5

- HW4 due today
- HW5 is out
  - More advanced Scheme programming
  - Team assignment
    - Maximal team size is 2

Lambda Calculus

Reading: Scott, Ch. 11 on CD

Lecture Outline

- Lambda calculus
  - Introduction
  - Syntax and semantics
  - Free and bound variables
  - Substitution, formally

Lambda Calculus

- A theory of functions
  - Theory behind functional programming
  - Turing complete: any computable function can be expressed and evaluated using the calculus
  - "Lingua franca" of PL research
- Lambda (λ) calculus expresses function definition and function application
  - **f(x)=x*x** becomes **λx. x*x**
  - **g(x)=x+1** becomes **λx. x+1**
  - **f(5)** becomes **(λx. x*x) 5 → 5*5 → 25**

Syntax of Pure Lambda Calculus

- **$E ::= x \mid (\lambda x.\ E_1) \mid (E_1\ E_2)$**
  - A λ-expression is one of
    - Variable: **x**
    - Abstraction (i.e., function definition): **$\lambda x.\ E_1$**
    - Application: **$E_1\ E_2$**

Convention:
notation f, x, y, z for variables;
E, M, N, P, Q for expressions

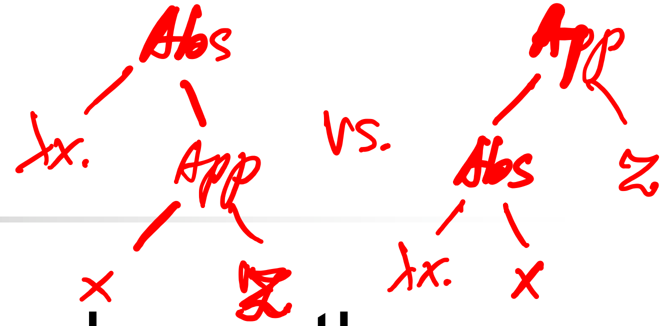
- λ-calculus formulae (e.g., **$(\lambda x.\ (x\ y))$**) are called expressions or terms
- **$(\lambda x.\ (x\ y))$** corresponds to (lambda (x) (x y)) in Scheme!

Syntactic Conventions

- Parentheses may be dropped from **(E_1 E_2)** or **(λx.E)**
  - E.g., **(f x)** may be written as **f x**
- Function application groups from left-to-right (i.e., it is left-associative)
  - E.g., **x y z** abbreviates **((x y) z)**
  - E.g., **E_1 E_2 E_3 E_4** abbreviates **(((E_1 E_2) E_3) E_4)**
  - Parentheses in **x (y z)** are necessary! Why?

x y z abbreviates (x y) z ≠ x (y z)

Syntactic Conventions



- Application has higher precedence than abstraction
  - Another way to say this is that the scope of the dot extends as far to the right as possible
  - E.g., **λx. x z** = **λx. (x z)** = **(λx. (x z))** = ≠ **((λx. x) z)**
- WARNING: This is the most common syntactic convention (e.g., Pierce 2002). Some books give abstraction higher precedence.

Terminology

- Parameter (also, formal parameter)
  - E.g., **x** is the parameter in **λx. x z**
- Argument (also, actual argument)
  - E.g., expression **λz. z** is the argument in **(λx. x) (λz. z)**

    Can you guess what this evaluates to?

Currying

Haskell Curry

- In lambda calculus, all functions have one parameter
  - How do we express n-ary functions?
  - Currying expresses an n-ary function in terms of n unary functions

f(x,y) = x+y, becomes **(λx.λy. x + y)**

(λx.λy. x + y) 2 3 → (λy. 2 + y) 3 → 2 + 3 = 5

Currying in Scheme

(define (curried-plus a)
(lambda (b) (+ a b))
)

(define curried-plus

(lambda (a) (lambda (b) (+ a b))))

(curried-plus 3 2) ERROR.

- (curried-plus 3) returns what?
  - Returns the plus-3 function (or more precisely, it returns a closure)
- ((curried-plus 3) 2) returns what?
  - 5

Currying

plus :: Int -> Int -> Int = Int -> (Int -> Int)

plus x y = x+y

$$f(x_1, x_2, \ldots, x_n) = g\ x_1\ x_2\ \ldots\ x_n$$

$$g_1\ x_2$$

$$g_2\ x_3$$

...

Function g is said to be the curried form of f.

Semantics of Pure Lambda Calculus

- An expression has as its meaning the value that results after evaluation is carried out
  - Somewhat informally, evaluation is the process of reducing expressions

  E.g., **(λx.λy. x + y) 3 2 → (λy. 3 + y) 2 → 3 + 2 = 5**

  (Note: this example is just an informal illustration. There is no **+** in the pure lambda calculus!)

- **λx.λy. x** is assigned the meaning of TRUE
- **λx.λy. y** is assigned the meaning of FALSE

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Free and Bound Variables

- Reducing expressions

$(\lambda x. \lambda y.\ x\ y)\ y = (\lambda z.\ x\ z)[y/x] = \lambda z. y\ z$

- Consider expression **(λx.λy. x y) (y w)**
- Try 1:
  - Reducing this expression results in the following

  (λy. x y) [(y w)/x] = (λy. (y w) y)

  The above notation means: we substitute argument **(y w)** for every occurrence of parameter **x** in body **(λy. x y)**.

  But what is wrong here?
- **(λx.λy. x y) (y w)**: different y's! If we substitute (y w) for x, the "free" y will become "bound"!

Free and Bound Variables

- Try 2:
  - Rename "bound" **y** in **λy. x y** to **z**: **λz. x z**

(λx.λy. x y) (y w) => (λx.λz. x z) (y w)

  - E.g., in C, **int id(int p) { return p; }** is exactly the same as **int id(int q) { return q; }**

  - Applying the reduction rule results in

(λz. x z) [(y w)/x] => (λz. (y w) z)

Free and Bound Variables

- Abstraction **(λx. E)** is also referred as binding

- Variable **x** is said to be bound in **λx. E**
- The set of free variables of **E** is the set of variables that are unbound in **E**
- Defined by cases on **E**
  - Var **x**: free(**x**) = {**x**}
  - App $\mathbf{E_1}$ $\mathbf{E_2}$: free($\mathbf{E_1}$ $\mathbf{E_2}$) = free($\mathbf{E_1}$) U free($\mathbf{E_2}$)
  - Abs λ**x.E**: free(λ**x.E**) = free(**E**) - {**x**}

Free and Bound Variables

- A variable **x** is bound if it is in the scope of a lambda abstraction: as in **λx. E**
- Variable is free otherwise

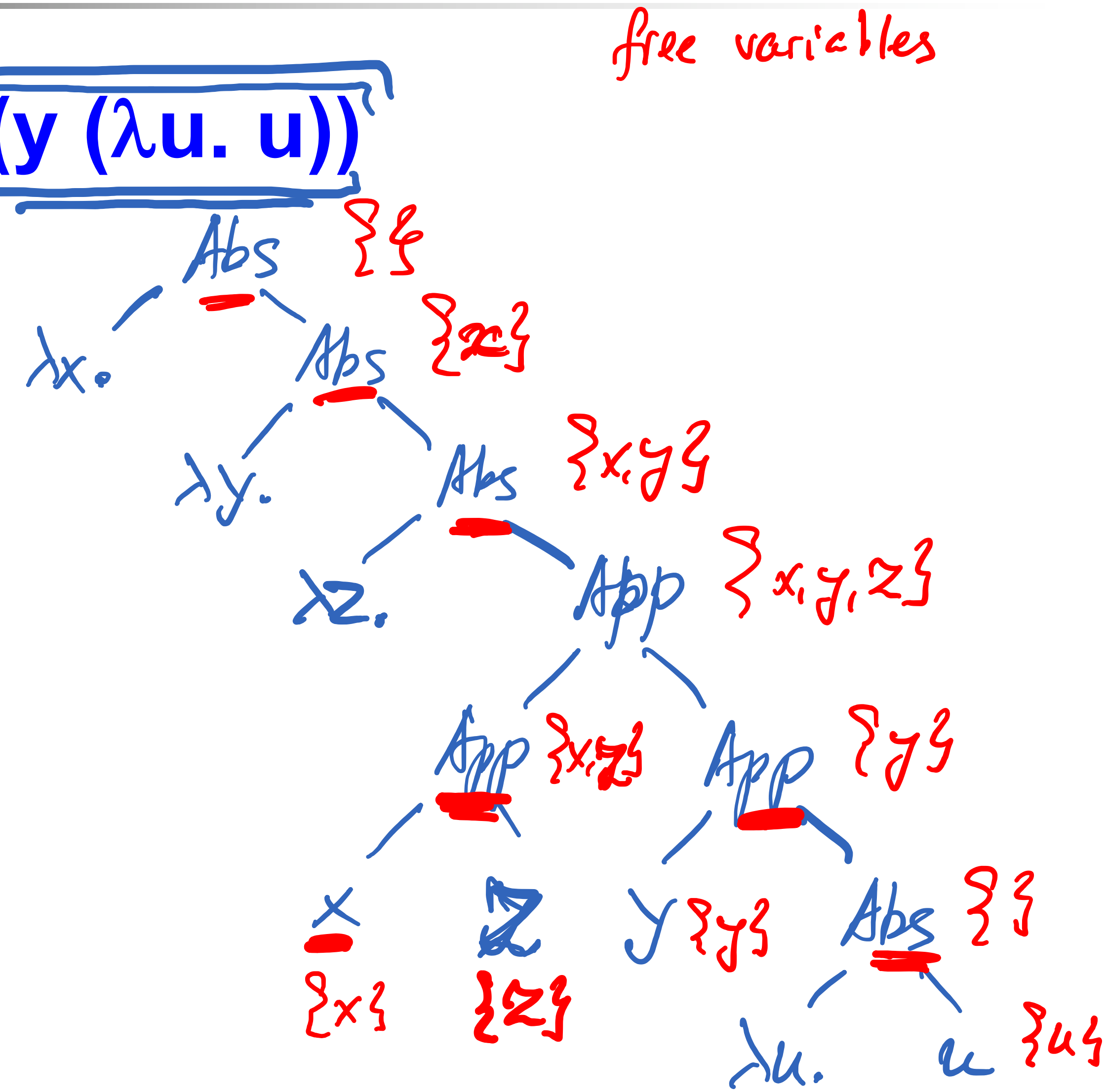
1. (λx. x) y {y}

2. (λz. z z) (λx. x) { }

3. λx.λy.λz. x z (y (λu. u)) { }

Free and Bound Variables

- $\lambda x.\lambda y.\lambda z.\ x\ z\ (y\ (\lambda u.\ u))$



Free and Bound Variables

- We must take free and bound variables into account when reducing expressions

  E.g., **(λx.λy. x y) (y w)**

  - Reduction rule defined in terms of substitution:

  (λy. x y) [(y w)/x]

  - First, rename bound **y** in **λy. x y** to **z**: **λz. x z**
    (more precisely, we have to rename to a variable that is NOT free in either **(y w)** or **(x y)**)
  - Second, replace **x** with argument **(y w)** safely:
    (λz. (y w) z) = λz. y w z

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Substitution, formally

- $(\lambda \mathbf{x.E})\ \mathbf{M} \rightarrow \mathbf{E[M/x]}$ replaces all free occurrences of **x** in **E** by **M**
- **E[M/x]** is defined by cases on **E**:
  - Var: $\mathbf{y[M/x]} = \mathbf{M}$ if $\mathbf{x = y}$

    $\mathbf{y[M/x]} = \mathbf{y}$ otherwise
  - App: $\mathbf{(E_1\ E_2)[M/x]} = \mathbf{(E_1[M/x]\ E_2[M/x])}$
  - Abs: $\mathbf{(\lambda y.E_1)[M/x]} = \mathbf{\lambda y.E_1}$ if $\mathbf{x = y}$

    $\mathbf{(\lambda y.E_1)[M/x]} = \mathbf{\lambda z.((E_1[z/y])[M/x])}$ otherwise,

    where **z** NOT in free($\mathbf{E_1}$) U free(**M**) U {**x**}

Substitution, formally

(λx.λy. x y) (y w)

(λy. E1)[M/x] ⇒ λz. ((E1[z/y])[M/x])

→ (λy. x y)[(y w)/x]

→ λ1_. (((x y)[1_/y])[(y w)/x])

→ λ1_. ((x 1_)[(y w)/x])

→ λ1_. ((y w) 1_)

→ λ1_. y w 1_ = λz. y w z

You will have to implement this substitution algorithm in Haskell

Substitution, formally

FUN!!!

$$(\lambda x.\lambda y.\ \lambda z.\ x\ z\ (y\ z))\ (\lambda x.x)$$

The End