Chapter 8

Nonholonomic System Theory

This chapter deals with the analysis of problems that involve differential constraints. One fundamental result is the Frobenius theorem, which allows one to determine whether the state transition equation represents a system is actually nonholonomic. In some cases, it may be possible to integrate the state transition equation, resulting in a problem that can be described without differential models. Another result is Chow's theorem, which indicates whether a system is controllable. Intuitively, this means that the differential constraints can be completely overcome by generating arbitrarily short maneuvers. The car-like robot enjoys the controllability property, which enables it to move itself sideways by performing parallel parking maneuvers.

8.1 Vector Fields and Distributions

A special form of the state transition equation Most of the concepts in this chapter are developed under the assumption that the state transition equation, $\dot{x} = f(x,u)$ has the following form:

$$\dot{x} = \alpha^1(x)u_1 + \alpha^2(x)u_2 + \cdots + \alpha^m(x)u_m, \tag{8.1}$$

in which each $\alpha^i(x)$ is a vector-valued function of x, and m is the dimension of U (or the number of inputs). The α^i functions can also be arranged in an $n \times m$ matrix,

$$A(x) = [\alpha^1(x) \;\; \alpha^2(x) \;\; \cdots \;\; \alpha^m(x)].$$

In this case, the state transition equation can be expressed as

$$\dot{x} = A(x)u.$$

For the rest of the chapter, it will be assumed that the matrix $A(x)$ is nonsingular. In other words, the rows of $A(x)$ are linearly independent for all x. To determine if $A(x)$ is nonsingular, one must find at least one $n \times n$ cofactor (or submatrix) of $A(x)$ which has a nonzero determinant.

Vector fields A vector field, $\vec{V}$, on a manifold X, is a function that associates with each $x \in X$, a vector, $\vec{V}(x)$. The *velocity field* is a special vector field that will be used extensively. Each vector $\vec{V}(x)$ in a velocity field represents the infinitesimal change in state with respect to time,

$$\dot{x} = \left[\frac{dx_1}{dt} \;\; \frac{dx_2}{dt} \;\; \cdots \;\; \frac{dx_n}{dt} \right], \tag{8.2}$$

evaluated at the point $x \in X$.

Note that for a fixed u, any state transition equation, $\dot{x} = f(x,u)$ defines a vector field because $\dot{x}$ is expressed as a function of x.

Distributions Each input $u \in U$ can be used to define a vector field. It will be convenient to define the set of all vector fields that can be generated using inputs. Assume that a state transition equation of the form in (8.1) is given for a state space X, and an input space $U = \mathbb{R}^m$. The set of all vector fields that can be generated using inputs $u \in U$ is called the *distribution*, and is denoted by $\triangle(X)$ or $\triangle$.

The distribution can be considered as a vector space. Note that each α^i can be interpreted as a vector field. Any vector field in $\triangle$ can be expressed as a linear combination of the α^i functions, which serve as a basis of the vector space. Consider the effect of inputs of the form $[0 \cdots 0\ 1\ 0 \cdots 0]$. If $u_i = 1$, and $u_j = 0$ for $j \neq i$, then the state transition equation yields $\dot{x} = \alpha^i(x)$. Thus, each input in this form can be used to generate a basis vector field. The dimension of the distribution the number of vector fields in its basis (in other words, the maximum number of linearly-independent vector fields that can be generated).

In terms of basis vector fields, a distribution is expressed as

$$\triangle = span\{\alpha^1(x), \alpha^2(x), \ldots, \alpha^n(x)\}$$

Example: Differential Drive The state transition equation (7.3) for the differential drive can be expressed in the form of (8.1) as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} (\frac{r}{2}\cos\theta)u_l & + & (\frac{r}{2}\cos\theta)u_r \\ (\frac{r}{2}\sin\theta)u_l & + & (\frac{r}{2}\sin\theta)u_r \\ (-\frac{r}{\ell})u_l & + & (\frac{r}{\ell})u_r \end{pmatrix} = \begin{pmatrix} \frac{r}{2}\cos\theta & \frac{r}{2}\cos\theta \\ \frac{r}{2}\sin\theta & \frac{r}{2}\sin\theta \\ -\frac{r}{\ell} & \frac{r}{\ell} \end{pmatrix} \begin{pmatrix} u_l \\ u_r \end{pmatrix} = A(x,y,\theta) \begin{pmatrix} u_l \\ u_r \end{pmatrix}.$$

The matrix $A(x, y, \theta)$ is nonsingular because all three 2×2 cofactors of $A(x, y, \theta)$ have nonzero determinants for all states.

To simplify the characterization of the distribution, a linear transformation will be performed on the inputs. Let $u_1 = u_l + u_r$ and $u_2 = u_r - u_l$. Intuitively, u_1 means "go straight" and u_2 means "rotate". Note that the original u_l and u_r can be easily recovered from u_1 and u_2. For additional simplicity, assume that $\ell = 2$ and $r = 2$. The state transition equation becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Using input $u = [1\ \ 0]$, the vector field $\vec{V} = [\cos\theta\ \ \sin\theta\ \ 0]$ is obtained. Using $u = [0\ \ 1]$, the vector field $\vec{W} = [0\ \ 0\ \ 1]$ is obtained. Any other vector field that can be generated using inputs can be constructed as a linear combination of $\vec{V}$ and $\vec{W}$. The distribution $\triangle$ has dimension two, and is expressed as $span\{\vec{V}, \vec{W}\}$.

8.2 The Lie Bracket

The Lie bracket attempts to generate velocities that are not directly permitted by the state transition equation. For the car-like robot, it will produce a vector field that can move the car sideways (it is achieved through combinations of vector fields, and therefore does not violate the nonholonomic constraint). This operation is called the *Lie bracket* (pronounced as "Lee"), and for given vector fields $\vec{V}$ and $\vec{W}$, it is denote by $[\vec{V}, \vec{W}]$. The Lie bracket is computed by

$$[\vec{V}, \vec{W}] = D\vec{W} \cdot \vec{V} - D\vec{V} \cdot \vec{W}$$

in which $\cdot$ denotes a matrix-vector multiplication,

$$D\vec{V} = \begin{pmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \cdots & \frac{\partial V_1}{\partial x_n} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \cdots & \frac{\partial V_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial V_n}{\partial x_1} & \frac{\partial V_n}{\partial x_2} & \cdots & \frac{\partial V_n}{\partial x_n} \end{pmatrix},$$

and

$$D\vec{W} = \begin{pmatrix} \frac{\partial W_1}{\partial x_1} & \frac{\partial W_1}{\partial x_2} & \cdots & \frac{\partial W_1}{\partial x_n} \\ \frac{\partial W_2}{\partial x_1} & \frac{\partial W_2}{\partial x_2} & \cdots & \frac{\partial W_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial W_n}{\partial x_1} & \frac{\partial W_n}{\partial x_2} & \cdots & \frac{\partial W_n}{\partial x_n} \end{pmatrix}.$$

In the expressions above, V_i and W_i denote the i^{th} components of $\vec{V}$ and $\vec{W}$, respectively.

It is sometimes convenient for computation of the Lie bracket to directly use the expression for each component of the new vector field (obtained by performing the multiplication indicated above). The i^{th} component of the Lie bracket is given by

$$\sum_{j=1}^{n} \left(V_j \frac{\partial W_i}{\partial x_j} - W_j \frac{\partial V_i}{\partial x_j} \right).$$

Two well-known properties of the Lie bracket are:

1. (skew-symmetry) $[\vec{V}, \vec{W}] = -[\vec{W}, \vec{V}]$ for any two vector fields, $\vec{V}$ and $\vec{W}$
2. (Jacobi identity) $[[\vec{V}, \vec{W}], \vec{U}] + [[\vec{W}, \vec{U}], \vec{V}] + [[\vec{U}, \vec{V}], \vec{W}] = 0$

8.3 Integrability and Controllability

The Lie bracket can be used to generate vector fields that potentially lie outside of $\triangle$. There are two theorems that express useful system properties that can be inferred using the vector fields generated by Lie brackets.

The Control Lie Algebra (CLA) For a given state transition equation of the form (8.1), consider the set of all vector fields that can be generated by taking Lie brackets, $[\alpha^i(x), \alpha^j(x)$, of vector fields $\alpha^i(x)$ and $\alpha^j(x)$ for $i \neq j$. Next, consider taking Lie brackets of the new vector fields with each other, and with the original vector fields. This process can be repeated indefinitely by iteratively applying the Lie bracket operations to new vector fields. The resulting set of vector fields can be considered as a kind of algebraic closure with respect to the Lie bracket operation. Let the *control Lie algebra*, $CLA(\triangle)$, denote the set of all vector fields that are obtained by this process.

In general, $CLA(\triangle)$ can be considered as a vector space, in which the basis elements are the vector fields $\alpha^1(x)$, ..., $\alpha^m(x)$, and all new, linearly-independent vector fields that were generated from the Lie bracket operations.

The process of finding the basis of $CLA(\triangle)$ is generally a tedious process. There are several systematic approaches for generating the basis, one of which is called the Phillip-Hall basis. The vector fields that should be generated for the first two steps of the Phillip-Hall approach are given below. Each Lie bracket has the opportunity to generate a vector field that is linearly-independent; however, it is not guaranteed to generate one. In fact, all Lie bracket operations may fail to generate a vector field that is independent of the original vector fields. Consider for example, the case in which the original vector fields, α^i, are all constant. All Lie brackets will be zero.

Integrability In some cases, it is possible that the state transition equation is integrable. This implies that is can be expressed purely as a function of x and u, and not of $\dot{x}$. In the case of an integrable state transition equation, the motions is actually restricted to a lower-dimensional subset of X, which is a global constraints as opposed to a local constraint.

CIRCLE EXAMPLE

The Frobenius theorem gives an interesting condition that may be applied to determine whether the state transition equation is integrable.

Theorem 1 *(Frobenius) The state transition equation is integrable if and only if all vectors fields that can be obtained by Lie bracket operations are contained in* $\triangle$.

Intuitively, if the Lie bracket operation is unable to produce any new (linearly-independent) vector fields that lie outside of $\triangle$, then the state transition equation can be integrated. Thus, the equation is not needed, and the problem can be reformulated without using $\dot{x}$. This is, however, a theoretical result; it may be a difficult or impossible task in general to actually integrate the state transition equation.

The Frobenius theorem can also be expressed in terms of dimensions. If $dim(CLA(\triangle)) = dim(\triangle)$, then the state transition equation is integrable. Note that the dimension of $CLA(\triangle)$ can never be greater than n.

If the state transition equation is not integrable, then it is called *nonholonomic*. These equations are of greatest interest.

Controllability In addition to integrability, another important property of a state transition equation is controllability. Intuitively, controllability implies that the robot is able to overcome its differential constraints by using Lie brackets to compose new motions. The controllability concepts assume that there are no obstacles.

Two kinds of controllability will be considered. A point, x', is *reachable* from x, if there exists an input that can be applied to bring the state from x to x'. Let $R(x)$ denote the set of all points reachable from x. A system is *locally controllable* if for all $x \in X$, $R(x)$ contains an open set that contains x. This implies that any state can be reached from any other state.

Let $R(x, \Delta t)$ denote the set of all points reachable in time Δt. A system is *small-time controllable* if for all $x \in X$ and any Δt, then $R(x, \Delta t)$ contains an open set that contains x.

The Dubins car is an example of a system that is locally controllable, but not small-time controllable. If there are no obstacles, it is possible to bring the car to any desired configuration from any initial configuration. This implies that the car is locally controllable. Suppose one would like to move the car to a position that would be obtained by the Reeds-Shepp car by moving a small amount in reverse. Because the Dubins car must drive forward to reach this configuration, it could require time larger than some small Δt. Hence, the Dubins care is not small-time controllable.

However, a substantial amount of time might be required to drive the care

Chow's theorem is used to determine small-time controllability.

Theorem 2 *(Chow) A system is small-time controllable if and only if the dimension of* $CLA(\triangle)$ *is* n, *the dimension of* X.

Example of integrability and controllability As an example of controllability and integrability, recall the differential drive model. From the example in Section 8.1, the original vector fields are $\alpha^1(x) = [\cos\theta \;\; \sin\theta \;\; 0]$ and $\alpha^2(x) = [0 \;\; 0 \;\; 1]$.

Let $\vec{V}$ denote $\alpha^1(x)$, and let $\vec{W}$ denote $\alpha^2(x)$. To determine integrability and controllability, the first step is to compute the Lie bracket, $\vec{Z} = [\vec{V}, \vec{W}]$. The components are

$$Z_1 = V_1 \frac{\partial W_1}{\partial x} - W_1 \frac{\partial V_1}{\partial x} + V_2 \frac{\partial W_1}{\partial y} - W_2 \frac{\partial V_1}{\partial y} + V_3 \frac{\partial W_1}{\partial \theta} - W_3 \frac{\partial V_1}{\partial \theta} = \sin\theta,$$

$$Z_2 = V_1 \frac{\partial W_2}{\partial x} - W_1 \frac{\partial V_2}{\partial x} + V_2 \frac{\partial W_2}{\partial y} - W_2 \frac{\partial V_2}{\partial y} + V_3 \frac{\partial W_2}{\partial \theta} - W_3 \frac{\partial V_2}{\partial \theta} = -\cos\theta,$$

and

$$Z_3 = V_1 \frac{\partial Y_3}{\partial x} - W_1 \frac{\partial V_2}{\partial x} + V_2 \frac{\partial Y_3}{\partial y} - W_2 \frac{\partial V_2}{\partial y} + V_3 \frac{\partial Y_3}{\partial \theta} - W_3 \frac{\partial V_2}{\partial \theta} = 0.$$

The resulting vector field is $\vec{Z} = [\sin\theta \;\; -\cos\theta \;\; 0]$.

We immediately observe that $\vec{Z}$ is linear independent from $\vec{V}$ and $\vec{W}$. This can be seen by noting that the determinant of the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ \sin\theta & -\cos\theta & 0 \end{pmatrix}$$

in nonzero for all (x, y, θ). This implies that the dimension of $CLA(\triangle) = 3$. Using the Frobenius theorem, it can be inferred that the state transition equation is not integrable, and the system is nonholonomic. From Chow's theorem, it is known that the system is small-time controllable.