CSCI 2200 - Spring 2015

Exam 1

Name: ____

Instructions:

- Write your name on the **front and back** of this cover sheet and **sign the bottom of this page**.
- You have **<u>100 minutes</u>** to complete this exam. The exam is worth a total of 100 points.
- Put away laptop computers and other electronic devices. Calculators are NOT allowed. Cheating on an exam will result in an **immediate F** for the entire course.
- A one page, two-sided crib sheet is allowed. Rulers are also allowed.
- **Write your answers clearly and completely.**
- Logical equivalences and rules of inference are given on the last page.
- Please read each question carefully several times before beginning to work and especially before asking questions. We generally will not answer questions except when there is a glaring mistake or ambiguity in the statement of a question.

I will not discuss the contents of this exam in the presence of anyone who has not yet taken the exam.

Signature: ________________________________ **Date:** ____________

Name: ______________________________

Problem 1 (12)	
Problem 2 (16)	
Problem 3 (7)	
Problem 4 (10)	
Problem 5 (20)	
Problem 6 (10)	
Problem 7 (15)	
Problem 8 (10)	
Total (100)	

1. (6+6=12 points) **Graded by Ashwin**

   (a) Using propositional logic, write a statement that contains the propositions p, q, and r that is true only when exactly one of p, q, and r is false. Your statement should only use the logical connectives $\neg$, $\vee$ and $\wedge$.

   $$(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$$

   This solution can be obtained by creating a truth table and generating the DNF.

   (b) Give a proposition that is equivalent to $(\neg p \wedge \neg q) \vee r$ that only uses the propositions p, q, and r, and the logical connectives $\neg$ and $\rightarrow$.

   $$\begin{aligned} \neg p \wedge \neg q) \vee r &\equiv \neg(p \vee q) \vee r \\ &\equiv (p \vee q) \rightarrow r \\ &\equiv (\neg p \rightarrow q) \rightarrow r \end{aligned}$$

2. (4×4=16 points) **Graded by Stacy**
Consider the following propositional functions:

$S(x,y)$: "x saw y."
$L(x,y)$: "x likes y."
$A(y)$: "y is an award winning movie."
$C(y)$: "y is a comedy."

Let the domain for the x be all people and the domain for y be all movies. Translate the following statements into predicate logic.

(a) There is an award-winning movie that is not a comedy.

$$\exists y\ (A(y) \land \neg C(y))$$

(b) Someone has seen every award-winning comedy.

$$\exists x\ \forall y\ ((C(y) \land A(y)) \to S(x,y)$$

(c) No one liked every movie he has seen.

$$\neg\ \exists x\ \forall y\ (S(x,y) \to L(x,y))$$

(d) Taz only likes movies that win awards.

$$\forall y\ (L(Taz,y) \to A(y))$$

3. (7 points) Determine whether the following argument is valid and prove your answer.

$$\begin{array}{ll} & p \to (q \vee r) \\ & q \wedge \neg r \\ \hline \therefore & p \end{array}$$

Solution: (Graded by Lingxun)

It's not valid. A counterexample is $p = False, q = True, r = False$, which satisfies both premises but with $p = False$.

4. (10 points) Prove that the square root of any (positive) irrational number is irrational.

Solution: (Graded by Lingxun)

Proof by contraposition. We need to prove that if a number x is rational, then x^2 is also rational.

Assume a number x is rational, so $x = \frac{a}{b}$, where a and b are integers and $b \neq 0$.

Thus, $x^2 = \frac{a^2}{b^2}$. Because a and b are integers, a^2 and b^2 are also integers. Because $b \neq 0$, $b^2 \neq 0$. Therefore, x^2 is rational.

5. (20 points) Let n be an integer between 0 and 99. Note that $n = 10 \cdot a + b$, for some integers a and b, with $0 \leq a \leq 9$ and $0 \leq b \leq 9$. The integers a and b are the digits of n.
Prove that n is divisible by 3 if and only if the sum of its digits is divisible by 3.

Solution: Graded by Ridwan

First we show that if $10a + b$ is divisible by 3, then $a + b$ is divisible by 3.
Assume $10a + b$ is divisible by 3. Then $10a + b = 3k$ for some integer k. Therefore,

$$a + b = 3k - 9a = 3(k - 3a).$$

So $a + b = 3j$ where $j = k - 3a$ is an integers. Therefore $a + b$ is divisible by 3.

Next we show that if $a + b$ is divisible by 3 then $10a + b$ is divisible by 3.
Assume $a + b$ is divisible by 3 Then $a + b = 3k$ for some integer k. Therefore,

$$\begin{aligned} a + b + 9a = & \quad 3k + 9a \\ 10a + b = & \quad 3(k + 3a), \end{aligned}$$

So $10a + b = 3j$ where $j = k + 3a$ is an integer. Therefore $10a + b$ is divisible by 3.

So, we have shown that n is divisible by 3 if and only if the sum of digits is divisible by 3.
QED.

6. (15 points) Prove that for all real numbers x and y, $|x-y| \geq |x| - |y|$.

Solution: (Graded by Jai)

(definition of absolute value: $|a| = a$, if $a \geq 0$ and $|a| = -a$, if $a < 0$.)

There are four cases that need be considered. They are: (i) $x \geq 0$, $y \geq 0$ (ii) $x < 0$, $y < 0$ (iii) $x \geq 0$, $y < 0$, and (iv) $x < 0$, $y \geq 0$.

Case (i) $x \geq 0$, $y \geq 0$
Since $x \geq 0$ and $y \geq 0$, $|x| = x$ and $|y| = y$.
Therefore $|x| - |y| = x - y$, By definition of absolute value, it holds that $|x-y| \geq x - y$, so $|x-y| \geq |x| - |y|$.

Case(ii) $x < 0$, $y < 0$
In this case, $|x| = -x$ and $|y| = -y$. Thus $|x| - |y| = -x - (-y) = y - x$.
Note that, by definition of absolute value, $|x-y| = |-(x-y)| = |y-x|$.
And, as in the previous case, $|y-x| \geq y - x$. Therefore, $|x-y| \geq |x| - |y|$.

Case (iii) $x \geq 0$, $y < 0$
In this case, $|x| = x$ and $|y| = -y$, so $|x| - |y| = x - (-y) = x + y$
$|x-y| = x - y$, as this number will always be positive (given that $x \geq 0$ and $y < 0$).
$x - y \geq x + y$ holds true for all $y < 0$.
Therefore, $|x-y| \geq |x| - |y|$ holds true.

Case (iv) $x < 0$, $y \geq 0$
Since $x < 0$, $|x| = -x$, and $|y| = y$, as $y \geq 0$.
Thus, $|x| - |y| = -x - y$.
$|x-y| = -(x-y)$, as $x - y$ will always be negative given that $x < 0$ and $y \geq 0$.
$y - x \geq -x - y$ holds true for all $y \geq 0$.
Therefore, $|x-y| \geq |x| - |y|$ holds true.

As $|x-y| \geq |x| - |y|$ holds true in all the cases, **it can be concluded that** $|x-y| \geq |x| - |y|$ **is true. QED**

7. (10 points) Give a proof by contradiction that for all positive real numbers x, $\frac{x}{(x+1)} < \frac{(x+1)}{(x+2)}$.

Solution (graded by Ashwin)

This is a proof by contradiction.
Assume that x is positive and $\frac{x}{(x+1)} > \frac{(x+1)}{(x+2)}$. Then, x(x+2) $>$ (x+1)(x+1), which means $x^2 + 2x > x^2 + 2x + 1$. Since $x > 0$, this inequality does not hold. Thus a contradiction has been found. So, for all positive real numbers x, $\frac{x}{(x+1)} < \frac{(x+1)}{(x+2)}$.

8. (3+3+4=10 points) **Graded by Ashwin,Lingxun**

(a) Let A, B, and C be sets. According to the inclusion-exclusion principle, what is $|A \cup B \cup C|$?

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap B|$$

(b) Let $A = \{0, 1, 2, 3, 4, 5\}$ and let $B = \{a, b, c\}$. What is $|\mathcal{P}(B) \times A|$?

The cardinality of A is $|A| = 6$.
The cardinality of B is $|B| = 3$, so $|\mathcal{P}(B)| = 2^3 = 8$.
Therefore, $|\mathcal{P}(B) \times A| = 8 \times 6 = 48$.

(c) Disprove the following claim: for all sets X and Y, $\mathcal{P}(X \cup Y) \subseteq \mathcal{P}(X) \cup \mathcal{P}(Y)$.

Counterexample:

Let $X = \{1\}$ and let $Y = \{2\}$.
Then, $\mathcal{P}(X) = \{\emptyset, \{1\}\}$, $\mathcal{P}(Y) = \{\emptyset, \{2\}\}$, and $\mathcal{P}(X) \cup \mathcal{P}(Y) = \{\emptyset, \{1\}, \{2\}\}$.
The power set of the union of X and Y is $\mathcal{P}(X \cup Y) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.
Since, $\{1, 2\} \notin \mathcal{P}(X) \cup \mathcal{P}(Y)$, the claim does not hold.

[SCRATCH PAPER]