CSCI 2200 - Spring 2015

Exam 2

Name: ______________________________

Instructions:

- Write your name on the **front and back** of this cover sheet and **sign the bottom of this page**.
- You have **100 minutes** to complete this exam. The exam is worth a total of 100 points.
- Put away laptop computers and other electronic devices. Calculators are NOT allowed. Cheating on an exam will result in an **immediate F** for the entire course.
- A one page, two-sided crib sheet is allowed. Rulers are also allowed.
- **Write your answers clearly and completely.**
- Useful summation formulae are given on the last page.
- Please read each question carefully several times before beginning to work and especially before asking questions. We generally will not answer questions except when there is a glaring mistake or ambiguity in the statement of a question.

I will not discuss the contents of this exam in the presence of anyone who has not yet taken the exam.

Signature: ______________________________ **Date:** ____________

Name: ______________________________

Problem 1 (8)	
Problem 2 (4)	
Problem 3 (15)	
Problem 4 (15)	
Problem 5 (20)	
Problem 6 (20)	
Problem 7 (10)	
Problem 8 (8)	
Total (100)	

1. ($8 = 4 \times 2$ points) Let A be the set of all integers. Let B be the set of all positive multiples of 4. Let $C = \{1, 2, 3, 4, 5\}$. Let $D = \{0, 1\}$.
Circle TRUE or FALSE

(a) $\|A\| > \|B\|$	TRUE	FALSE
(b) There is at least one surjective function from B to A.	TRUE	FALSE
(c) There are exactly 5! functions from C to A.	TRUE	FALSE
(d) The function $f(x) = x^2$ is an injective function from D to A.	TRUE	FALSE

Solution:

(a) FALSE: A and B both have cardinality aleph-naught.
(b) TRUE: Since A and B have the same cardinality, there exists a bijection between them.
(c) FALSE: There are infinitely many functions that map C to A.
(d) TRUE: $f(0) = 0$ and $f(1) = 1$. Therefore, f is an injective function from D to A.

2. (4 points) A set S of numbers is defined recursively by
Basis step: $2 \in S$
Recursive step: If $x \in S$ then $x + 3 \in S$ and $2x \in S$.

Which of the following numbers are elements of S? Circle **all** that apply.

(a) 6
(b) 7
(c) 19
(d) 12

Solution: (b) and (c) are elements of S.
7 can be generated from the recursive definition by $2 \cdot 2 + 3$.
19 can be generated from the recursive definition by $2 \cdot 2 \cdot 2 \cdot 2 + 3$.

3. (15 points) (**Graded by Jai**)
Prove by mathematical induction that, for every positive integer n, $3^{2n} - 1$ is divisible by 8.

Solution:
This is proof by induction.

Basis step: ($n = 1$) $3^{2 \cdot 1} - 1 = 9 - 1 = 8$, and 8 is divisible by 8.

Inductive step: Assume $3^{2k} - 1$ is divisible by 8, or equivalently, $3^{2k} - 1 = 8r$ for some integer r.
We will show this implies, $3^{2(k+1)} - 1$ is divisible by 8, or equivalently, that $3^{2(k+1)} - 1 = 8s$ for some integer s.

$$3^{2(k+1)} - 1 = 3^2 \cdot 3^{2k} - 1 = 9(3^{2k} - 1) + 9 - 1 = 9(3^{2k} - 1) + 8.$$

Applying the inductive hypothesis, we obtain

$$9(3^{2k} - 1) + 8 = 9(8r) + 8 = 8(9r + 1).$$

Therefore, $3^{2(k+1)} - 1 = 8s$, where $s = 9r + 1$ is an integer. So $3^{2(k+1)} - 1$ is divisible by 8.
QED

4. (15 points) (**Graded by Ashwin**)
McDonald's sells Chicken McNuggets in 4, 6, 9 and 20 piece boxes. Give a proof by strong induction that for any integer $n \geq 24$, we can always obtain exactly n McNuggets by buying multiple boxes.

Solution:
This is a proof by strong induction.
Basis step:
For $n = 24$, we can buy one 20 piece box and one 4 piece box.
For $n = 25$, we can buy one 4 piece box, two 6 piece boxes, and one 9 piece box.
For $n = 26$, we can buy two 4 piece boxes and three six piece boxes.
For $n = 27$, we can buy three 9 piece boxes.

Inductive step: Assume we can get j McNuggets by buying multiple 4, 6, 9 and 20 piece boxes, where $24 \leq j \leq k$, and $k \geq 27$.
We will show this implies we can get $k+1$ McNuggets by buying multiple 4, 6, 9 and 20 piece boxes.
By the inductive hypothesis, we can get $k-3$ McNuggets by buying multiple 4, 6, 9 and 20 piece boxes.
We buy one more 4 piece box to obtain $k+1$ McNuggets.
QED

5. (8 + 12 = 20 points) (**Graded by Lingxun**)

(a) Consider a full binary tree T. Let $L(T)$ denote the number of leaf nodes in T, i.e., the number of nodes in T without children. Give a recursive definition of $L(T)$.

Solution:
Basis step: For a full binary tree T consisting of a single node, $L(T) = 1$.
Recursive step: If T_1 and T_2 are disjoint full binary trees, then for $T = T = T_1 \circ T_2$,

$$L(T) = L(T_1) + L(T_2).$$

(b) Recall the recursive definition of the number of nodes in a full binary tree.
Basis step: The number of nodes in a full binary tree T consisting of a single node is $N(T) = 1$.
Recursive step: If T_1 and T_2 are full binary trees, then the number of nodes in the full binary tree $T = T_1 \circ T_2$ is $N(T) = 1 + N(T_1) + N(T_2)$.
Prove by structural induction that for any full binary tree T, $L(T) = \frac{N(T)+1}{2}$.

Solution:
This is a proof by structural induction.
Basis step: For a full binary tree T consisting of a single node, $L(T) = 1$, $N(T) = 1$, and $1 = \frac{1+1}{2}$.

Recursive step: Assume that for disjoint full binary trees T_1 and T_2, $L(T_1) = \frac{N(T_1)+1}{2}$ and $L(T_2) = \frac{N(T_2)+1}{2}$.
We will show this implies that for $T = T_1 \circ T_2$, $L(T) = \frac{N(T)+1}{2}$.
By the definition of $L(T)$, $L(T) = L(T_1) + L(T_2)$.
Applying the inductive hypothesis, we obtain

$$\begin{aligned} L(T) &= \frac{N(T_1)+1}{2} + \frac{N(T_2)+1}{2} \\ &= \frac{N(T_1)+N(T_2)+1+1}{2} \\ &= \frac{N(T)+1}{2}, \end{aligned}$$

where the last equality follows from the recursive definition of $N(T)$.
QED

6. (8 + 12 = 20 points) (**Graded by Ridwan**)
Consider the following recursive algorithm:

```
procedure mystery(n: positive integer)
    if n = 1 then return 1
    else return n^2 + mystery(n − 1)
```

(a) Conjecture a closed formula (without a summation) for the number returned by `mystery(n)`.

Solution: *mystery*(n) returns $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

(b) Give a proof by induction that your conjecture is correct.

Solution:
This is a proof by induction.
Basis step: $(n = 1)$ *mystery*(1) returns 1, and $\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = 1$.

Inductive step:
Assume *mystery(k)* returns $\frac{k(k+1)(2k+1)}{6}$. We will show this implies *mystery(k+1)* returns $\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$.
According to the pseudocode, *mystery(k + 1)* returns $(k+1)^2 +$ *mystery(k)*.
It follows from the inductive hypothesis that *mystery(k+1)* returns $(k+1)^2 + \frac{k(k+1)(2k+1)}{6}$.
Simplifying this expression, we obtain,

$$
\begin{aligned}
(k+1)^2 + \frac{k(k+1)(2k+1)}{6} &= \frac{6(k+1)(k+1) + k(k+1)(2k+1)}{6} \\
&= \frac{(k+1)(6(k+1) + k(2k+1)}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}
$$

QED

7. (10 points) (**Graded by Stacy**)
A factory makes automobile parts. Each part has a serial number consisting of a digit (from 0 to 9), an uppercase letter (from the English alphabet), and another digit (from 0 to 9), where the digits must be distinct. Example serial numbers are 5C7, 1P6, and 3Z0.

   (a) Suppose the factory manufactured 5,000 parts last week. What is the largest integer m for which we can guarantee that at least m of those parts had identical serial numbers?

   Solution: 3

   (b) Prove your answer to part (a).

   Solution:
   The number of distinct serial numbers is $10 \cdot 26 \cdot 9 = 2340$.
   By the generalized pigeonhole principle, there is at at least one serial number for which at least $\lceil \frac{5000}{2340} \rceil = 3$ parts use that serial number.

8. (8 points) Circle **<u>all</u>** that are equivalent to $\binom{26}{5}$.

   (a) The number of distinct strings of length 5 that can be generated from the uppercase English alphabet.

   (b) The number of ways I can select 22 scoops of ice cream from 5 ice cream flavors.

   (c) The number of distinct bit strings of length 21 that contain exactly 5 ones.

   (d) The coefficient of x^5y^{21} in the binomial expansion of $(x+y)^{26}$.

Solution: (d) is the only item that is equivalent to $\binom{26}{5}$

   (a) The number of distinct strings of length 5 that can be generated from the uppercase English alphabet is 26^5, which is not equal to $\binom{26}{5}$.
   (b) The number of ways I can select 22 scoops of ice cream from 5 ice cream flavors is $\binom{26}{4}$.
   (c) The number of distinct bit strings of length 21 that contain exactly 5 ones is $\binom{21}{5}$.
   (d) According to the binomial theorem, the coefficient of x^5y^{21} is $\binom{26}{21} = \binom{26}{5}$.