13. Polyhedral Convex Cones

Mechanics of Manipulation

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Outline.

1. Positive linear span
2. Types of cones
3. Edge and face representation
4. Supplementary cones; polar
5. Representing frictionless contact
6. Cones in wrench space
   - force closure
7. Cones in velocity twist space

Positive linear span

For now, use n-dimensional vector space $\mathbf{R}^n$. Later, wrench space and velocity twist space.

Let $\mathbf{v}$ be any non-zero vector in $\mathbf{R}^n$. Then the set of vectors

$$\{k\mathbf{v} \mid k \geq 0\} \tag{1}$$

describes a *ray*.

Let $\mathbf{v}_1$, $\mathbf{v}_2$ be non-zero and non-parallel. Then the set of positively scaled sums

$$\{k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \mid k_1, k_2 \geq 0\} \tag{2}$$

is a planar cone—sector of a plane.

Generalize by defining the *positive linear span* of a set of vectors $\{\mathbf{v}_i\}$:

$$\mathrm{pos}(\{\mathbf{v}_i\}) = \{\sum k_i\mathbf{v}_i \mid k_i \geq 0\} \tag{3}$$

(The positive linear span of the empty set is the origin.)

Relatives of positive linear span

The *linear span*

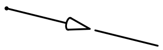
$$\mathrm{lin}(\{\mathbf{v}_i\}) = \{\sum k_i \mathbf{v}_i \mid k_i \in \mathbf{R}\} \tag{4}$$

The *convex hull*

$$\mathrm{conv}(\{\mathbf{v}_i\}) = \{\sum k_i \mathbf{v}_i \mid k_i \geq 0, \sum k_i = 1\} \tag{5}$$

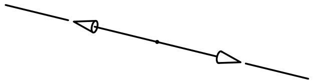
Varieties of cones in three space

1 edge

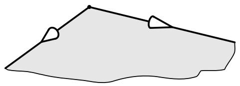


a. ray

2 edges

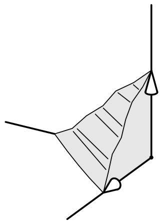


b. line

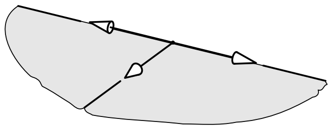


c. planar cone

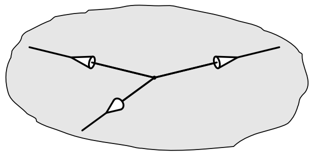
3 edges



d. solid cone

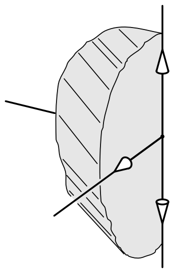


e. half plane

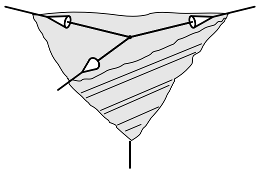


f. plane

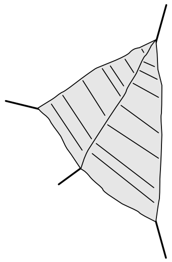
4 edges



g. wedge



h. half space



i. whole space

Spanning all of $\mathbf{R}^n$

Theorem: A set of vectors $\{\mathbf{v}_i\}$ positively spans the entire space $\mathbf{R}^n$ if and only if the origin is in the interior of the convex hull:

$$\mathrm{pos}(\{\mathbf{v}_i\}) = \mathbf{R}^n \leftrightarrow \mathbf{0} \in \mathrm{int}(\mathrm{conv}(\{\mathbf{v}_i\})) \tag{6}$$

Theorem: It takes at least $n+1$ vectors to positively span $\mathbf{R}^n$.

Representing cones

Two ways to represent cones: *edge representation* and *face representation*.

Edge representation uses positive linear span. Given a set of edges $\{\mathbf{e}_i\}$, the cone is given by $\mathrm{pos}(\{\mathbf{e}_i\})$.

Face representation of cones

First represent *planar half-space* by inward pointing normal vector $\mathbf{n}$.

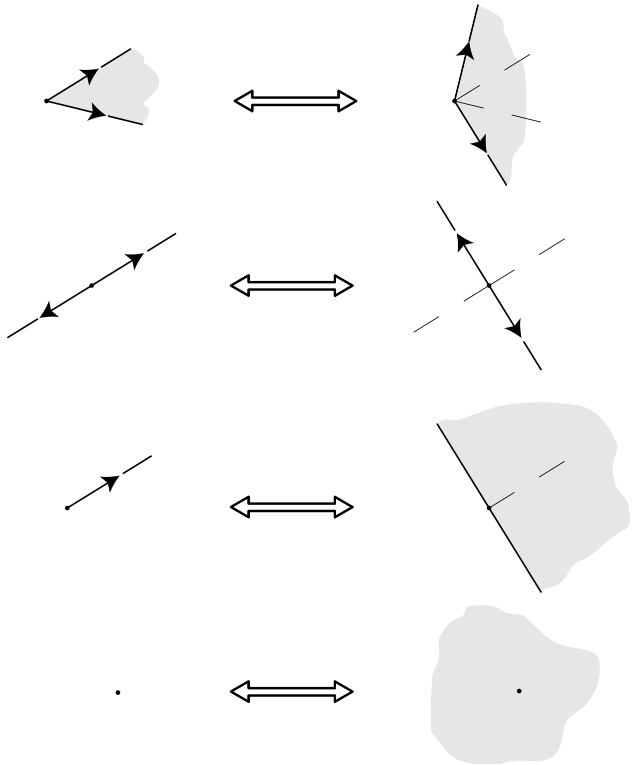
$$\mathrm{half}(\mathbf{n}) = \{\mathbf{v} \mid \mathbf{n} \cdot \mathbf{v} \geq 0\} \tag{7}$$

(Here we use dot product, but when working with twists and wrenches we will use reciprocal product.)

Consider a cone with face normals $\{\mathbf{n}_i\}$. Then the cone is the intersection of the half-spaces:

$$\cap\{\mathrm{half}(\mathbf{n}_i)\} \tag{8}$$

Supplementary cone; polar



Supplementary cone $\mathrm{supp}(V)$ (also known as *polar*) comprises the vectors that make non-negative dot products with vectors in V:

$$\{u \in \mathbf{R}^n \mid u \cdot v \geq 0 \; \forall v \in V\} \quad (9)$$

The supplementary cone's *edges* are the original cone's *face normals*, and vice versa. So if

$$V = \mathrm{pos}(\{\mathbf{e}_i\}) = \cap\{\mathrm{half}(\mathbf{n}_i)\} \quad (10)$$

then

$$\mathrm{supp}(V) = \mathrm{pos}(\{\mathbf{n}_i\}) = \cap\{\mathrm{half}(\mathbf{e}_i)\} \quad (11)$$

Frictionless contact

Characterize contact by *set of possible wrenches*.

Assume uniquely determined contact normal.

Assume frictionless contact can give arbitrary magnitude force along inward-pointing normal.

Then a frictionless contact gives a ray in wrench space, $\mathrm{pos}(\mathbf{w})$, where $\mathbf{w} = (\mathbf{c}, \mathbf{c}_0)$ is the contact screw.

Two contacts

Given two frictionless contacts $\mathbf{w}_1$ and $\mathbf{w}_2$, total wrench is the sum of possible positive scalings of $\mathbf{w}_1$ and $\mathbf{w}_2$:

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2; k_1, k_2 \geq 0 \tag{12}$$

i.e. the positive linear span $\mathrm{pos}(\{\mathbf{w}_1, \mathbf{w}_2\})$.

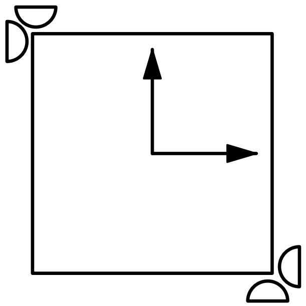
Generalizing:
Theorem: If a set of frictionless contacts on a rigid body is described by the contact normals $\mathbf{w}_i = (\mathbf{c}_i, \mathbf{c}_{0i})$ then the set of all possible wrenches is given by the positive linear span $\mathrm{pos}(\{\mathbf{w}_i\})$.

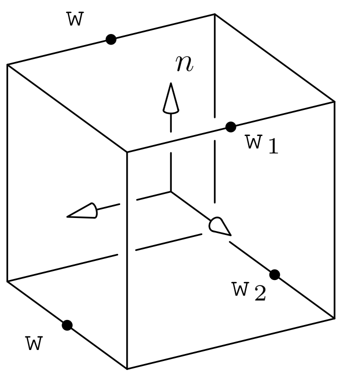
Force closure

Definition: **Force closure** means that the set of possible wrenches exhausts all of wrench space.

It follows from theorem ? that a frictionless force closure requires at least 7 contacts. Or, since planar wrench space is only three-dimensional, frictionless force closure in the plane requires at least 4 contacts.

Example wrench cone





Construct unit magnitude force at each contact.

Write screw coords of wrenches.

Take positive linear span.

Exhausts wrench space?

Cones in velocity twist space

Cannot use finite displacement twists. They are not vectors.

Velocity twists are vectors!

Let $\{\mathbf{w}_i\}$ be a set of contact normals.

Let $W = \mathrm{pos}(\{\mathbf{w}_i\})$ be set of possible wrenches.

First order analysis: velocity twists T must be *reciprocal or repelling* to contact wrenches: $T = \mathrm{supp}(W)$.