14. The Oriented Plane

Mechanics of Manipulation

Matt Mason
matt.mason@cs.cmu.edu

http://www.cs.cmu.edu/~mason

Carnegie Mellon

Exhortations

“Hands are levers of influence on the world that made intelligence worth having. Precision hands and precision intelligence coevolved in the human lineage, and the fossil record shows that hands led the way.”—Steven Pinker, in *How the Mind Works*

Outline.

Reuleaux's space. . .what is it?

Formal definitions.

Central projection.

Convexity in the oriented plane.

Relation to polyhedral convex cones.

Rotation centers and the oriented plane.

Reuleaux's space

Reuleaux represents PCC's in planar diff'l twist space.

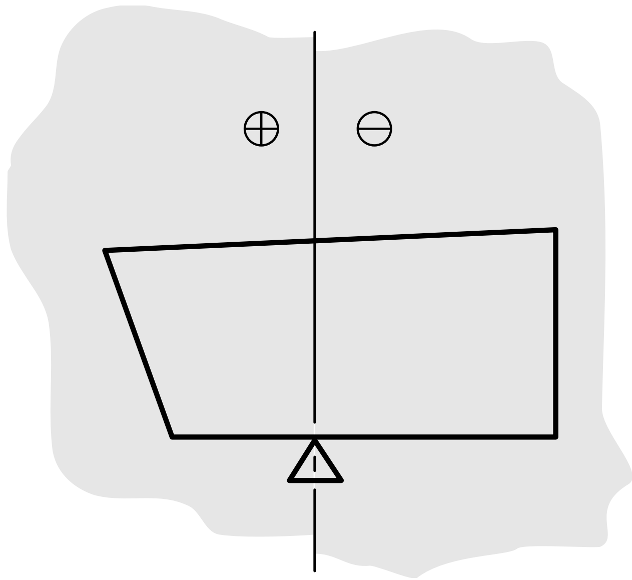
> 3D twist space is 6D.
>
> planar twist space is 3D.
>
> Reuleaux does it in 2D!

What space do signed rotation centers live in?

Plane with $+$ and/or $-$ label,

and points at infinity?

Stolfi and Guibas: the *oriented plane*.



Formal definition

Consider homogeneous coordinates $(x, y, w) \neq (0, 0, 0)$.

A **point in the oriented plane** is a ray of triples:

$$\{(kx, ky, kw) \mid k > 0\} \tag{1}$$

Three cases:

$w > 0$: Signed Euclidean point $(x/w, y/w, \oplus)$.

$w < 0$: Signed Euclidean point $(x/w, y/w, \ominus)$.

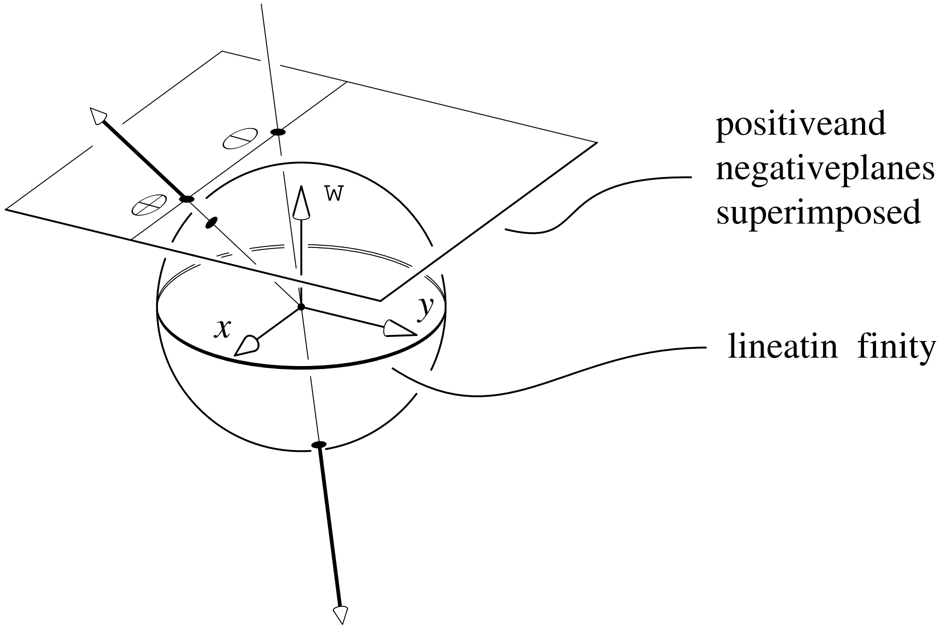
$w = 0$: Ideal point: point at infinity.

Central projection

Project each point (x, y, w) to $w = 1$ plane.

Attach appropriate sign.

Ideal points miss the plane but hit the equator.



Relation to projective plane

Maybe we should call it the *Oriented Projective Plane*.

Projective plane is set of lines through the origin of $\mathbb{E}^3$.

Oriented plane is set of *directed* lines through the origin of $\mathbb{E}^3$.

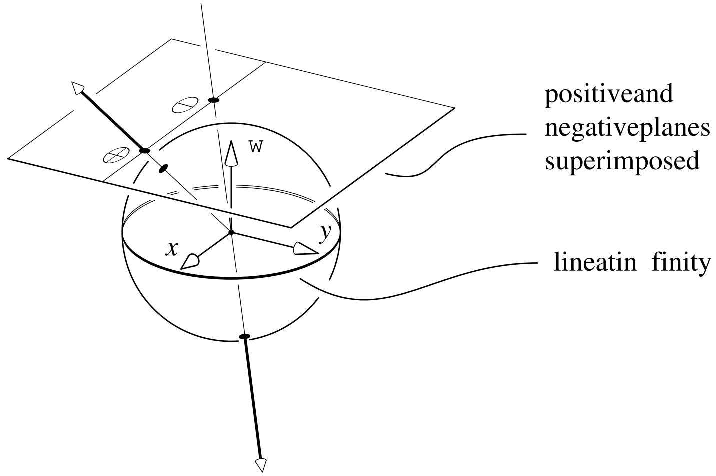
The projective plane is the sphere $\mathbb{S}(2)$ with antipodes identified.

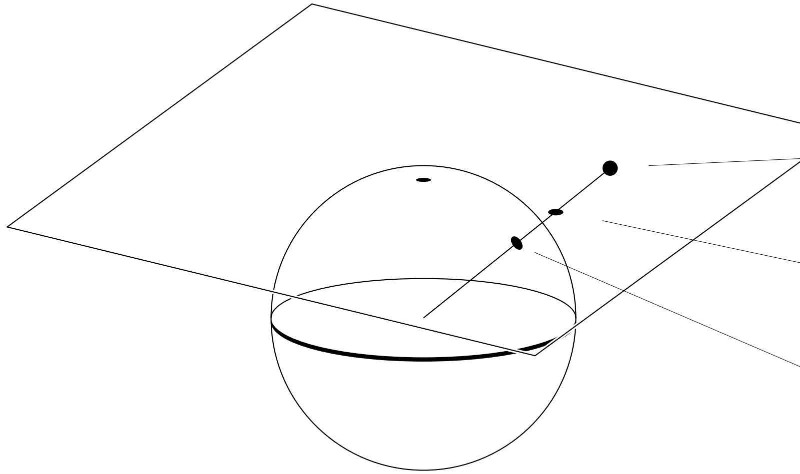
The oriented plane is the sphere $\mathbb{S}(2)$.

Northern hemisphere is the positive plane,

Southern hemisphere is the negative plane,

Equator is the ideal line.





Why do we need it?

Because, e.g. Reuleaux's method is easier than analyzing polyhedral convex cones in planar differential twist space.

> (or rather, Reuleaux's method is an easy way to analyze PCCs in planar differential twist space.)

In practice, we work directly in the oriented plane, as Reuleaux did, not worrying too much about the projection.

But to answer deep questions, refer back to the projection.

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So . . .

- Two non-antipodal points determine a line.
  (The ideal line, or a Euclidean line labeled "$\pm$".)
- Every pair of lines intersects in *two* antipodal points.

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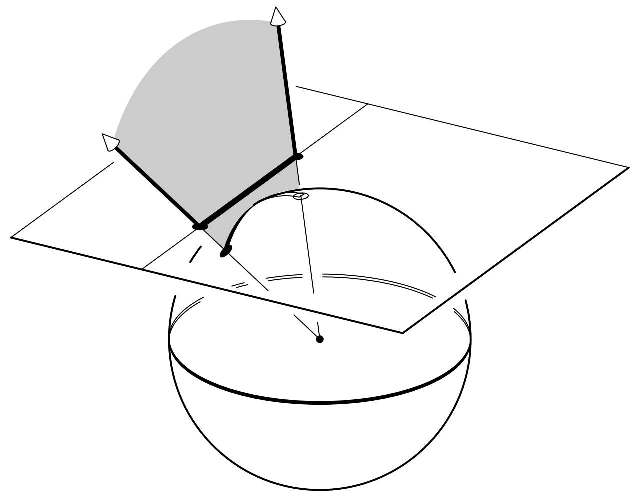
What is $\overline{ab}$ in the oriented plane? What is $\mathrm{conv}(\{a, b\})$?

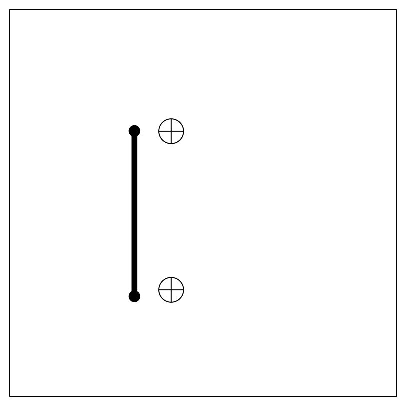
Convexity: plus and plus

Consider the polyhedral convex cone determined by the two rays.

Project it to the plane.

So, if a and b have the same sign then $\mathrm{conv}(\{a, b\})$ is the line segment between them with the same sign.

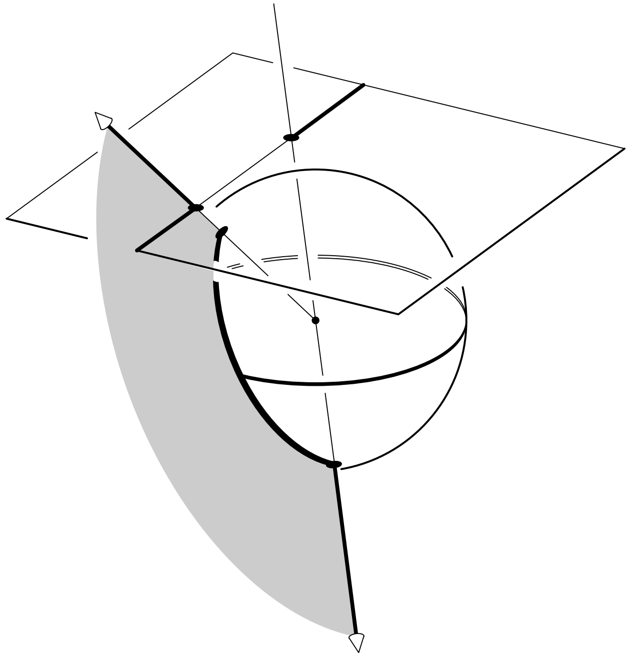


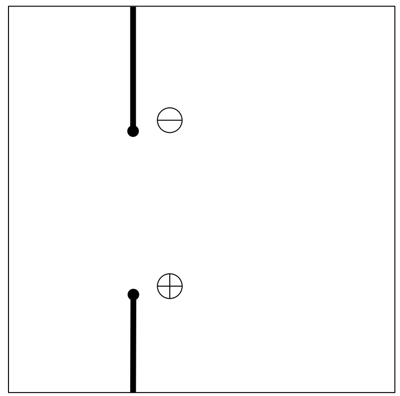


Convexity: plus and minus

Consider the PCC . . .

Define the *external* line segment.





Problem Set 4

Let's do PS4!

How do you construct the convex hull of some set of points?

1. Pick two points, and construct their convex hull.
2. Repeat step 1, ad nauseum.

Reuleaux's method

Given a differential velocity twist $(v_{0x}, v_{0y}, \omega_z)$, what is the rotation center?

Velocity at a point $\mathbf{p} = (x, y)$ is

$$\mathbf{v}_0 + \omega \times \mathbf{p}$$

$$= \begin{pmatrix} v_{0x} - \omega y \\ v_{0y} + \omega x \end{pmatrix}$$

Set equal to $(0, 0)$ and solve for $(x, y) \ldots$

Rotation center is at

$$\begin{pmatrix} -v_{0y}/\omega \\ v_{0x}/\omega \end{pmatrix}$$

Central projection, and rotation of coordinates.