

## 2. Kinematic foundations

# *Mechanics of Manipulation*

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**Chapter 1 Manipulation 1**

- 1.1 Case 1: Manipulation by a human 1
- 1.2 Case 2: An automated assembly system 3
- 1.3 Issues in manipulation 5
- 1.4 A taxonomy of manipulation techniques 7
- 1.5 Bibliographic notes 8
- Exercises 8

**Chapter 2 Kinematics 11**

- 2.1 Preliminaries 11
- 2.2 Planar kinematics 15
- 2.3 Spherical kinematics 20
- 2.4 Spatial kinematics 22
- 2.5 Kinematic constraint 25
- 2.6 Kinematic mechanisms 34
- 2.7 Bibliographic notes 36
- Exercises 37

**Chapter 3 Kinematic Representation 41**

- 3.1 Representation of spatial rotations 41
- 3.2 Representation of spatial displacements 58
- 3.3 Kinematic constraints 68
- 3.4 Bibliographic notes 72
- Exercises 72

**Chapter 4 Kinematic Manipulation 77**

- 4.1 Path planning 77
- 4.2 Path planning for nonholonomic systems 84
- 4.3 Kinematic models of contact 86
- 4.4 Bibliographic notes 88
- Exercises 88

**Chapter 5 Rigid Body Statics 93**

- 5.1 Forces acting on rigid bodies 93
- 5.2 Polyhedral convex cones 99
- 5.3 Contact wrenches and wrench cones 102
- 5.4 Cones in velocity twist space 104
- 5.5 The oriented plane 105
- 5.6 Instantaneous centers and Reuleaux's method 109
- 5.7 Line of force; moment labeling 110
- 5.8 Force dual 112
- 5.9 Summary 117
- 5.10 Bibliographic notes 117
- Exercises 118

**Chapter 6 Friction 121**

- 6.1 Coulomb's Law 121
- 6.2 Single degree-of-freedom problems 123
- 6.3 Planar single contact problems 126
- 6.4 Graphical representation of friction cones 127
- 6.5 Static equilibrium problems 128
- 6.6 Planar sliding 130
- 6.7 Bibliographic notes 139
- Exercises 139

**Chapter 7 Quasistatic Manipulation 143**

- 7.1 Grasping and fixturing 143
- 7.2 Pushing 147
- 7.3 Stable pushing 153
- 7.4 Parts orienting 162
- 7.5 Assembly 168
- 7.6 Bibliographic notes 173
- Exercises 175

**Chapter 8 Dynamics 181**

- 8.1 Newton's laws 181
- 8.2 A particle in three dimensions 181
- 8.3 Moment of force; moment of momentum 183
- 8.4 Dynamics of a system of particles 184
- 8.5 Rigid body dynamics 186
- 8.6 The angular inertia matrix 189
- 8.7 Motion of a freely rotating body 195
- 8.8 Planar single contact problems 197
- 8.9 Graphical methods for the plane 203
- 8.10 Planar multiple-contact problems 205
- 8.11 Bibliographic notes 207
- Exercises 208

**Chapter 9 Impact 211**

- 9.1 A particle 211
- 9.2 Rigid body impact 217
- 9.3 Bibliographic notes 223
- Exercises 223

**Chapter 10 Dynamic Manipulation 225**

- 10.1 Quasidynamic manipulation 225
- 10.2 Briefly dynamic manipulation 229
- 10.3 Continuously dynamic manipulation 230
- 10.4 Bibliographic notes 232
- Exercises 235

**Appendix A Infinity 237**

# Kinematic foundations.

We will focus on rigid motions in

the Euclidean plane ( $\mathbb{E}^2$ )

Euclidean three space ( $\mathbb{E}^3$ )

the sphere ( $\mathbb{S}^2$ )

Why the sphere? Rigid motions of the sphere correspond to rotations about a given point in  $\mathbb{E}^3$ .

# Kinematics foundations: some definitions

First, some general definitions. Let  $\mathbb{X}$  be the *ambient space*, either  $\mathbb{E}^2$ ,  $\mathbb{E}^3$ , or  $\mathbb{S}^2$ .

- A *system* is a set of points in the space  $\mathbb{X}$ .
- A *configuration* of a system is the location of every point in the system.
- *Configuration space* is a metric space comprising all configurations of a given system.  
(What kind of space is configuration space? Devise a metric.)  
(Note: Every metric for cspace is sort of defective.)
- The *degrees of freedom* of a system is the dimension of the configuration space. (A less precise but roughly equivalent definition: the minimum number of real numbers required to specify a configuration.)

# Kinematics foundations: systems, DOFs

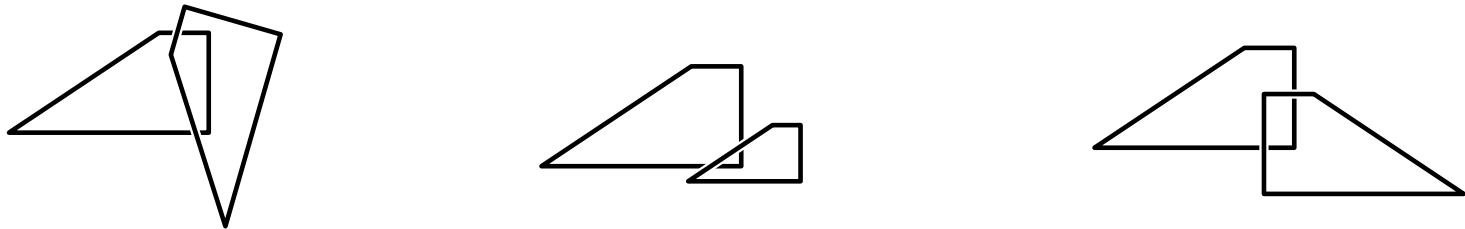
<b>System</b>	<b>Configuration</b>	<b>DOFs</b>
point in plane	$x, y$	2
point in space	$x, y, z$	3
rigid body in plane	$x, y, \theta$	3
rigid body in space	$x, y, z, \phi, \theta, \psi$	6

# Kinematics foundations: rigid bodies, displacements

Definitions:

A *displacement* is a change of configuration that does not change the distance between any pair of points, nor does it change the handedness of the system.

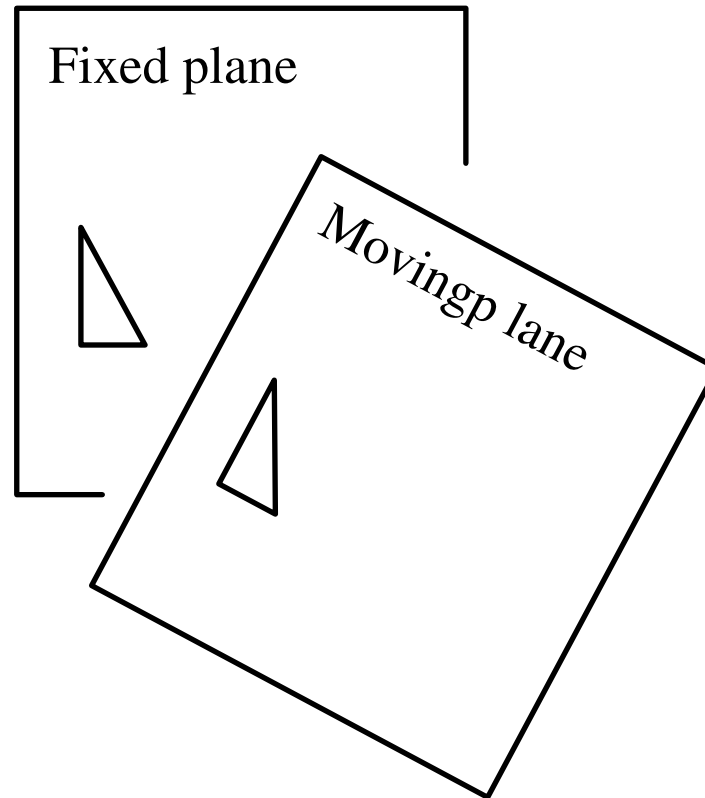
A *rigid body* is a system that is capable of displacements only.



Transformations, rigid and otherwise.

# Kinematics foundations: moving and fixed planes

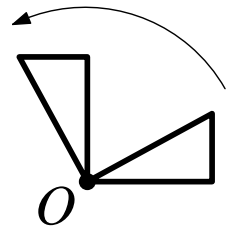
We will consider displacements to apply to *every* point in the ambient space. E.g., displacements are described as motion of *moving* plane relative to *fixed* plane.



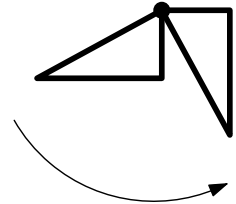
Moving and fixed planes.

# Kinematics foundations: rotations and translation

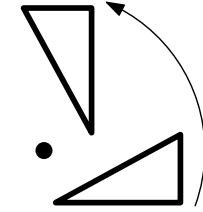
A *rotation* is a displacement that leaves at least one point fixed.  
A *translation* is a displacement for which all points move equal distances along parallel lines.



Rotation about  $O$



Rotation about a point on the body



Rotation about a point not on the body



# Kinematics foundations: digression for group theory

A *group* is a set of elements  $X$  and a binary operator  $\circ$  satisfying the following properties:

**Closure:** for all  $x$  and  $y$  in  $X$ ,  $x \circ y$  is in  $X$ .

**Associativity:** for all  $x$ ,  $y$ , and  $z$  in  $X$ ,  $(x \circ y) \circ z$  is equal to  $x \circ (y \circ z)$ .

**Identity:** there is some element, called 1, such that for all  $x$  in  $X$   $x \circ 1 = 1 \circ x = x$ .

**Inverses:** for all  $x$  in  $X$ , there is some element called  $x^{-1}$  such that  $x \circ x^{-1} = x^{-1} \circ x = 1$ .

(Did I remember them all?)

Some groups are commutative (Abelian) and some are not. The integers with addition are a commutative group. Nonsingular  $k$  by  $k$  matrices with matrix multiplication are a noncommutative group.

# Kinematics foundations: Displacements as a group

Every displacement  $D$  can be described as an operator on the ambient space  $\mathbb{X}$ , mapping every point  $x$  to some new point  $D(x) = x'$ .

The product of two displacements is the composition of the corresponding operators, i.e.  $(D_2 \circ D_1)(\cdot) = D_2(D_1(\cdot))$ .

The inverse of a displacement is just the operator that maps every point back to its original position.

The identity is the null displacement, which maps every point to itself.

In other words:

The displacements, with functional composition, form a group.

# Kinematics foundations: $SE(2)$ , $SE(3)$ , and $SO(3)$

These groups of displacements have names:

**$SE(2)$** : The special Euclidean group on the plane.

**$SE(3)$** : The special Euclidean group on  $\mathbb{E}^3$ .

**$SO(3)$** : The special orthogonal group.

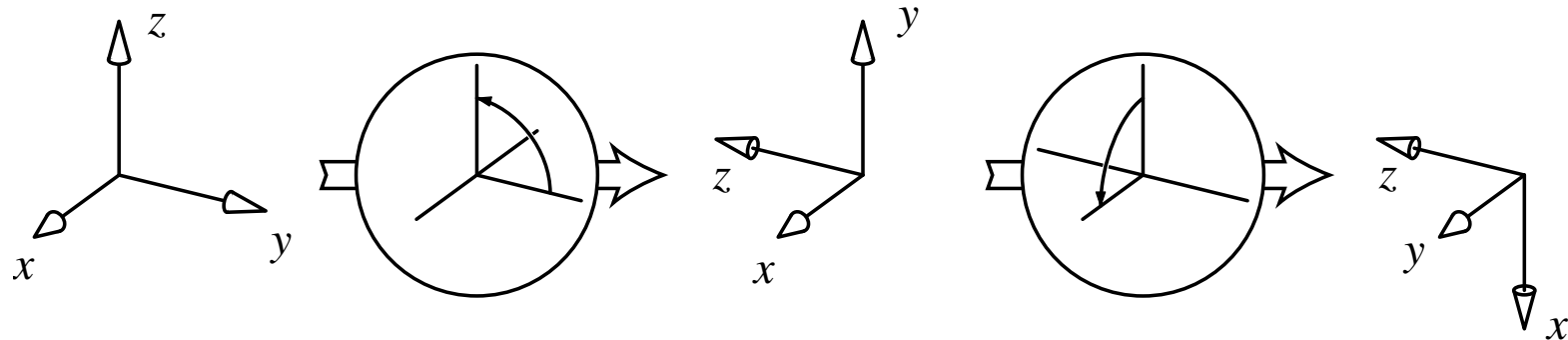
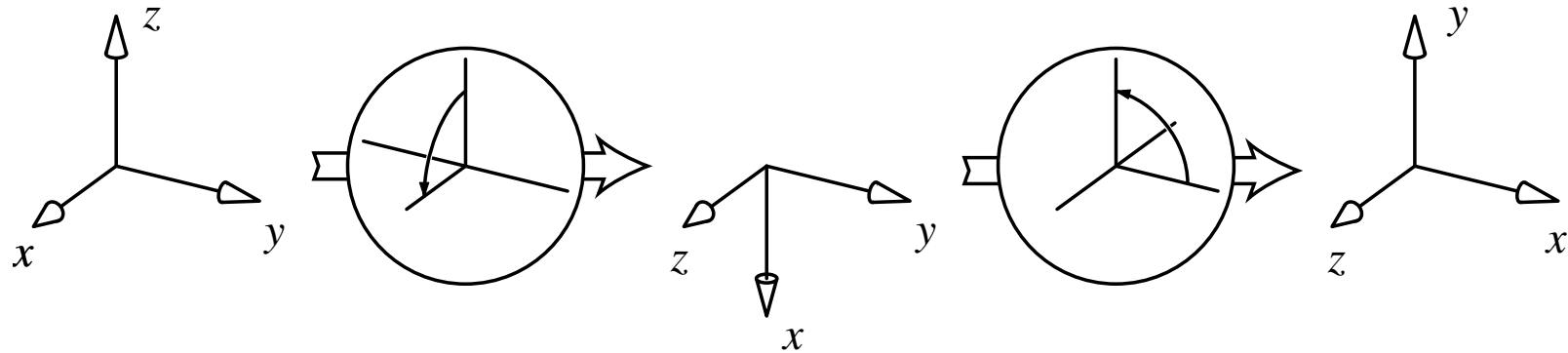
Whence the names?

**Special**: they preserve handedness.

**Orthogonal**: referring to the connection with orthogonal matrices, which will be covered later.

# Kinematics foundations: do displacements comm

Does  $SO(3)$  commute? **NO!** No, no, no. (If you have found a commutative way of representing spatial rotations, you are confused.)



# Kinematics foundations: do displacements commute

Does  $\mathbf{SE}(3)$  commute?

Does  $\mathbf{SE}(2)$  commute?

Does  $\mathbf{SO}(2)$  commute?

# Time for a digression . . .

Next we look at  $\mathbf{SE}(2)$ ,  $\mathbf{SO}(3)$ , and  $\mathbf{SE}(3)$ .

First, it helps if we contemplate the infinite . . .

# The projective plane.

The basic idea:

Start with the Euclidean plane  $\mathbb{E}^2$ .

Add some points, the **ideal points** or the **points at infinity**.

Call the new structure the **projective plane**— $\mathbb{P}^2$ .

You can do it formally by defining an ideal point for each set of parallel lines, but we will employ a more concrete method using **homogeneous coordinates**.

# Homogeneous coordinates.

Let the Cartesian coordinates of some point in  $\mathbb{E}^2$  be

$$(\eta, \nu)$$

Then we will say that

$$(x, y, w) \triangleq (w\eta, w\nu, w)$$

are the **homogeneous coordinates** of the point, **provided**

$$w \neq 0$$

To go from homogeneous to Cartesian:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} x/w \\ y/w \end{pmatrix}, w \neq 0 \quad (1)$$



# Point in $\mathbb{E}^2$ versus line through origin of $\mathbb{E}^3$

Scaling the homogeneous coordinates does **not** change the point!

$$\begin{pmatrix} ax \\ ay \\ aw \end{pmatrix} \mapsto \begin{pmatrix} ax/aw \\ ay/aw \end{pmatrix} = \begin{pmatrix} x/w \\ y/w \end{pmatrix}, a, w \neq 0 \quad (2)$$

So, homogeneous coordinates represent a point in  $\mathbb{E}^2$  by a line through the origin of  $\mathbb{E}^3$ .

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \left\{ \begin{pmatrix} wx \\ wy \\ w \end{pmatrix} \mid w \neq 0 \right\}$$

# Central projection

The Euclidean plane can be embedded as the  $w = 1$  plane.

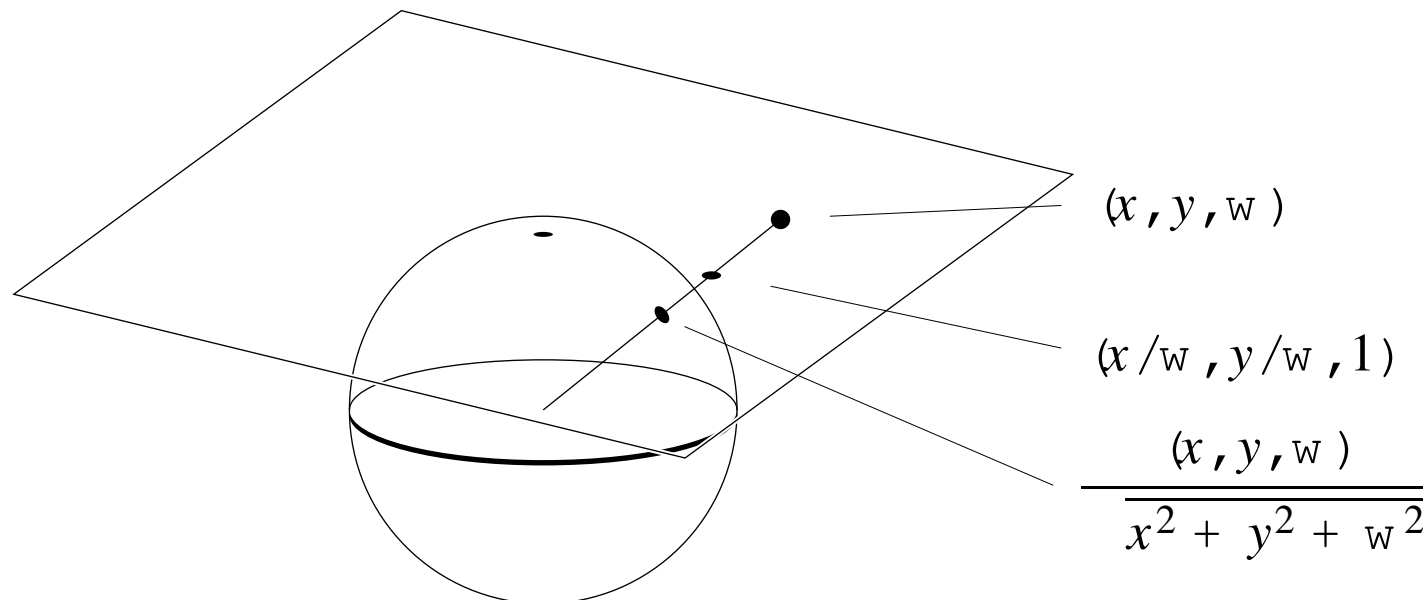
We can also embed a sphere of points satisfying  $x^2 + y^2 + w^2 = 1$ .

A line through the origin of  $\mathbb{E}^3$

intersects the sphere in **antipodal points**

intersects the  $w = 1$  plane at the appropriate point  $(x/w, y/w)$ .

These constructions are **central projection**, either to the sphere or to the plane.



# Ideal points

The original idea: extend  $\mathbb{E}^2$  by adding some ideal points.

Euclidean point: line through origin of  $\mathbb{E}^3$  intersecting  $w = 1$  plane.

Ideal point: line through origin of  $\mathbb{E}^3$  parallel to  $w = 1$  plane.

With Cartesian coords, no place to put ideal points. With homogeneous coordinates, there's a big gaping hole!

# The projective plane

So ...

define the projective plane  $\mathbb{P}^2$  to be the set of lines through the origin of  $\mathbb{E}^3$ .

A line in  $\mathbb{E}^2$  is represented by plane through origin of  $\mathbb{E}^3$ .

The ideal points form a line! The **line at infinity**. The equator of the embedded sphere.

“Parallel lines” intersect at infinity.

Duality. Two points determine a line. Two lines determine a point. Every axiom of the projective plane has a dual axiom by switching “line” and “point”.

Noneuclidean geometry!!!

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- 1.5 Bibliographic notes 8
- Exercises 8

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- 2.1 Preliminaries 11
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- 2.3 Spherical kinematics 20
- 2.4 Spatial kinematics 22
- 2.5 Kinematic constraint 25
- 2.6 Kinematic mechanisms 34
- 2.7 Bibliographic notes 36
- Exercises 37

**Chapter 3 Kinematic Representation 41**

- 3.1 Representation of spatial rotations 41
- 3.2 Representation of spatial displacements 58
- 3.3 Kinematic constraints 68
- 3.4 Bibliographic notes 72
- Exercises 72

**Chapter 4 Kinematic Manipulation 77**

- 4.1 Path planning 77
- 4.2 Path planning for nonholonomic systems 84
- 4.3 Kinematic models of contact 86
- 4.4 Bibliographic notes 88
- Exercises 88

**Chapter 5 Rigid Body Statics 93**

- 5.1 Forces acting on rigid bodies 93
- 5.2 Polyhedral convex cones 99
- 5.3 Contact wrenches and wrench cones 102
- 5.4 Cones in velocity twist space 104
- 5.5 The oriented plane 105
- 5.6 Instantaneous centers and Reuleaux's method 109
- 5.7 Line of force; moment labeling 110
- 5.8 Force dual 112
- 5.9 Summary 117
- 5.10 Bibliographic notes 117
- Exercises 118

**Chapter 6 Friction 121**

- 6.1 Coulomb's Law 121
- 6.2 Single degree-of-freedom problems 123
- 6.3 Planar single contact problems 126
- 6.4 Graphical representation of friction cones 127
- 6.5 Static equilibrium problems 128
- 6.6 Planar sliding 130
- 6.7 Bibliographic notes 139
- Exercises 139

**Chapter 7 Quasistatic Manipulation 143**

- 7.1 Grasping and fixturing 143
- 7.2 Pushing 147
- 7.3 Stable pushing 153
- 7.4 Parts orienting 162
- 7.5 Assembly 168
- 7.6 Bibliographic notes 173
- Exercises 175

**Chapter 8 Dynamics 181**

- 8.1 Newton's laws 181
- 8.2 A particle in three dimensions 181
- 8.3 Moment of force; moment of momentum 183
- 8.4 Dynamics of a system of particles 184
- 8.5 Rigid body dynamics 186
- 8.6 The angular inertia matrix 189
- 8.7 Motion of a freely rotating body 195
- 8.8 Planar single contact problems 197
- 8.9 Graphical methods for the plane 203
- 8.10 Planar multiple-contact problems 205
- 8.11 Bibliographic notes 207
- Exercises 208

**Chapter 9 Impact 211**

- 9.1 A particle 211
- 9.2 Rigid body impact 217
- 9.3 Bibliographic notes 223
- Exercises 223

**Chapter 10 Dynamic Manipulation 225**

- 10.1 Quasidynamic manipulation 225
- 10.2 Brie y dynamic manipulation 229
- 10.3 Continuously dynamic manipulation 230
- 10.4 Bibliographic notes 232
- Exercises 235

**Appendix A Infinity 237**