2. Kinematic foundations

Mechanics of Manipulation

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Kinematic foundations.

We will focus on rigid motions in

- the Euclidean plane ($\mathbb{E}^2$)
- Euclidean three space ($\mathbb{E}^3$)
- the sphere ($\mathbb{S}^2$)

Why the sphere? Rigid motions of the sphere correspond to rotations about a given point in $\mathbb{E}^3$.

Kinematics foundations: some definitions

First, some general definitions. Let $\mathbb{X}$ be the *ambient space*, either $\mathbb{E}^2$, $\mathbb{E}^3$, or $\mathbb{S}^2$.

- A *system* is a set of points in the space $\mathbb{X}$.
- A *configuration* of a system is the location of every point in the system.
- *Configuration space* is a metric space comprising all configurations of a given system.
  (What kind of space is configuration space? Devise a metric.)
  (Note: Every metric for cspace is sort of defective.)
- The *degrees of freedom* of a system is the dimension of the configuration space. (A less precise but roughly equivalent definition: the minimum number of real numbers required to specify a configuration.)

Kinematics foundations: systems, DOFs

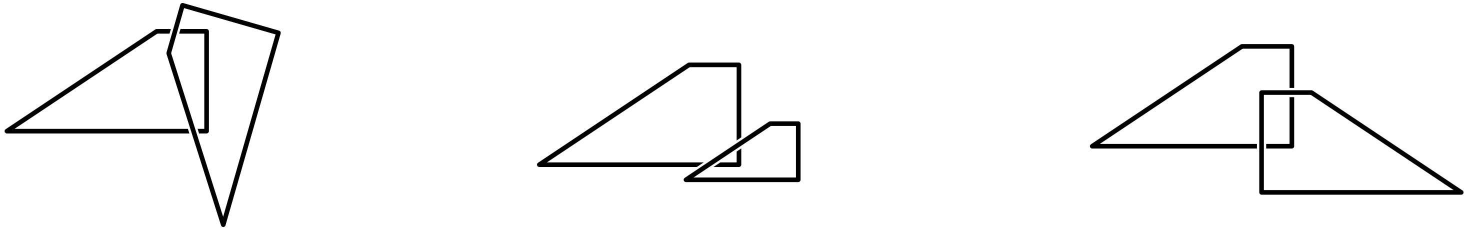
System	Configuration	DOFs
point in plane	x, y	2
point in space	x, y, z	3
rigid body in plane	x, y, θ	3
rigid body in space	x, y, z, ϕ, θ, ψ	6

Kinematics foundations: rigid bodies, displaceme

Definitions:

A *displacement* is a change of configuration that does not change the distance between any pair of points, nor does it change the handedness of the system.

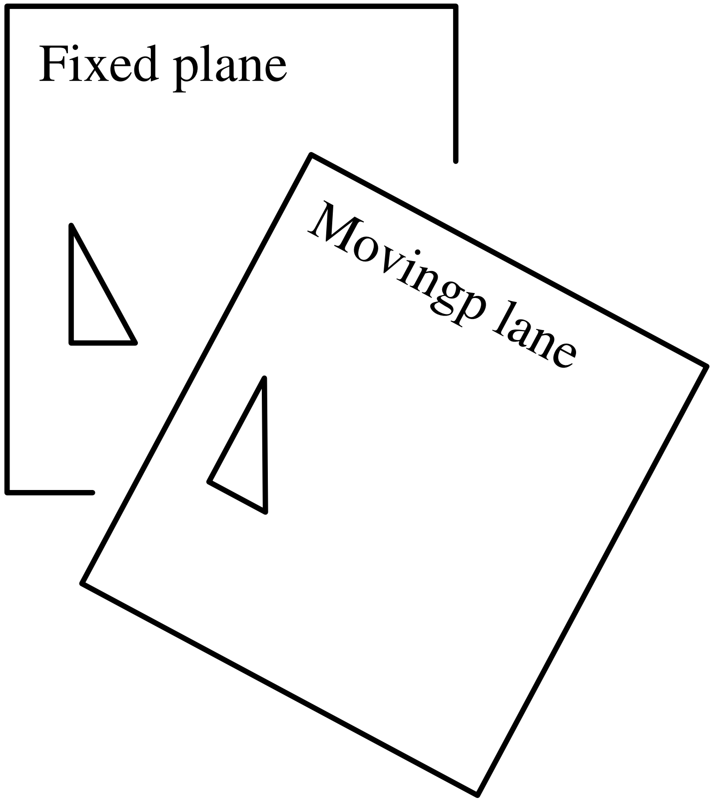
A *rigid body* is a system that is capable of displacements only.



Transformations, rigid and otherwise.

Kinematics foundations: moving and fixed planes

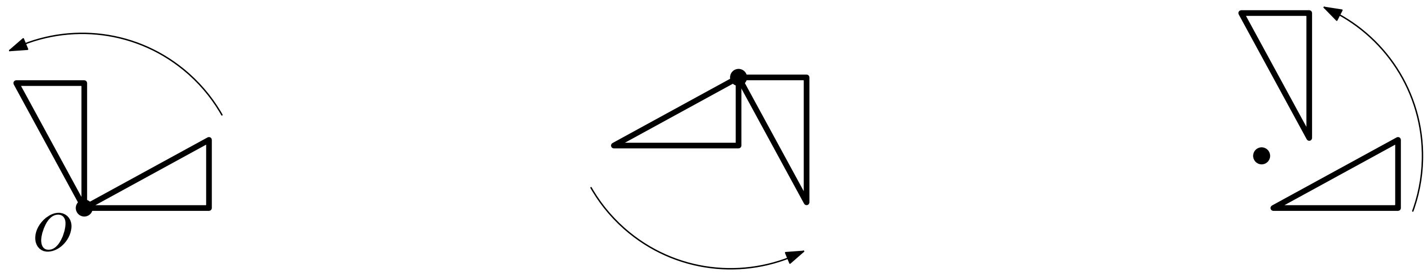
We will consider displacements to apply to *every* point in the ambient space. E.g., displacements are described as motion of *moving* plane relative to *fixed* plane.



Moving and fixed planes.

Kinematics foundations: rotations and translatio

A *rotation* is a displacement that leaves at least one point fixed.
A *translation* is a displacement for which all points move equal distances along parallel lines.



Rotation about O

Rotation about a point on the body

Rotation about a point not on the body

Kinematics foundations: digression for group the

A *group* is a set of elements X and a binary operator $\circ$ satisfying the following properties:

Closure: for all x and y in X, $x \circ y$ is in X.

Associativity: for all x, y, and z in X, $(x \circ y) \circ z$ is equal to $x \circ (y \circ z)$.

Identity: there is some element, called 1, such that for all x in X $x \circ 1 = 1 \circ x = x$.

Inverses: for all x in X, there is some element called x^{-1} such that $x \circ x^{-1} = x^{-1} \circ x = 1$.

(Did I remember them all?)
Some groups are commutative (Abelian) and some are not. The integers with addition are a commutative group. Nonsingular k by k matrices with matrix multiplication are a noncommutative group.

Kinematics foundations: Displacements as a grou

Every displacement D can be described as an operator on the ambient space $\mathbb{X}$, mapping every point x to some new point $D(x) = x'$.

The product of two displacements is the composition of the corresponding operators, i.e. $(D_2 \circ D_1)(\cdot) = D_2(D_1(\cdot))$.

The inverse of a displacement is just the operator that maps every point back to its original position.

The identity is the null displacement, which maps every point to itself.

In other words:

The displacements, with functional composition, form a group.

Kinematics foundations: $\mathbf{SE}(2)$, $\mathbf{SE}(3)$, and $\mathbf{SO}(3)$

These groups of displacements have names:

$\mathbf{SE}(2)$: The special Euclidean group on the plane.

$\mathbf{SE}(3)$: The special Euclidean group on $\mathbb{E}^3$.

$\mathbf{SO}(3)$: The special orthogonal group.

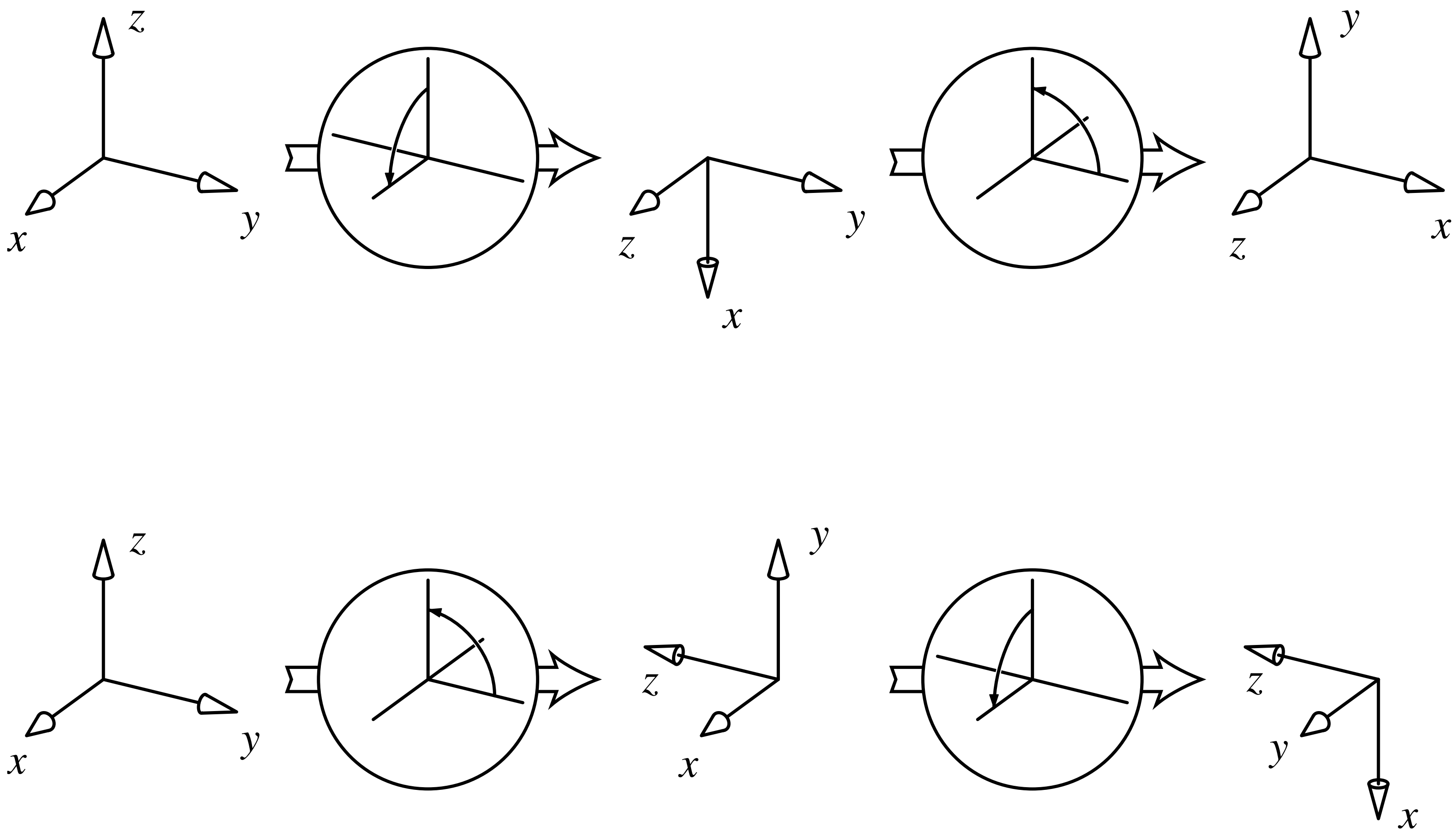
Whence the names?

Special: they preserve handedness.

Orthogonal: referring to the connection with orthogonal matrices, which will be covered later.

Kinematics foundations: do displacements comm

Does $\mathbf{SO}(3)$ commute? **NO!** No, no, no. (If you have found a commutative way of representing spatial rotations, you are confused.)



Kinematics foundations: do displacements comm

Does $\mathbf{SE}(3)$ commute?
Does $\mathbf{SE}(2)$ commute?
Does $\mathbf{SO}(2)$ commute?

Time for a digression . . .

Next we look at $\mathbf{SE}(2)$, $\mathbf{SO}(3)$, and $\mathbf{SE}(3)$.

First, it helps if we contemplate the infinite . . .

The projective plane.

The basic idea:

> Start with the Euclidean plane $\mathbb{E}^2$.
>
> Add some points, the **ideal points** or the **points at infinity**.
>
> Call the new structure the **projective plane**—$\mathbb{P}^2$.

You can do it formally by defining an ideal point for each set of parallel lines, but we will employ a more concrete method using **homogeneous coordinates**.

Homogeneous coordinates.

Let the Cartesian coordinates of some point in $\mathbb{E}^2$ be

$$(\eta, \nu)$$

Then we will say that

$$(x, y, w) \triangleq (w\eta, w\nu, w)$$

are the **homogeneous coordinates** of the point, **provided**

$$w \neq 0$$

To go from homogeneous to Cartesian:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \mapsto \begin{pmatrix} x/w \\ y/w \end{pmatrix}, w \neq 0 \quad (1)$$

Point in $\mathbb{E}^2$ versus line through origin of $\mathbb{E}^3$

Scaling the homogeneous coordinates does **not** change the point!

$$\begin{pmatrix} ax \\ ay \\ aw \end{pmatrix} \mapsto \begin{pmatrix} ax/aw \\ ay/aw \end{pmatrix} = \begin{pmatrix} x/w \\ y/w \end{pmatrix}, a, w \neq 0 \qquad (2)$$

So, homogeneous coordinates represent a point in $\mathbb{E}^2$ by a line through the origin of $\mathbb{E}^3$.

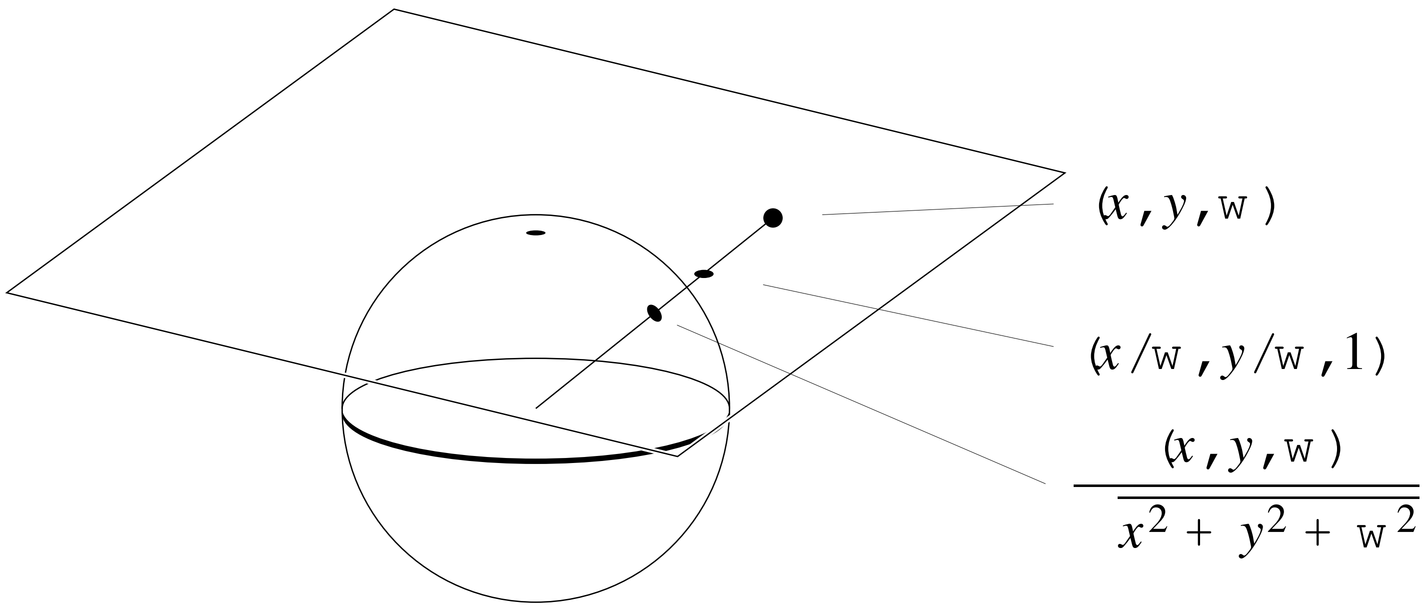
$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \left\{ \begin{pmatrix} wx \\ wy \\ w \end{pmatrix} \middle| \, w \neq 0 \right\}$$

Central projection

The Euclidean plane can be embedded as the $w = 1$ plane.
We can also embed a sphere of points satisfying $x^2 + y^2 + w^2 = 1$.
A line through the origin of $\mathbb{E}^3$

> intersects the sphere in **antipodal points**
>
> intersects the $w = 1$ plane at the appropriate point $(x/w, y/w)$.

These constructions are **central projection**, either to the sphere or to the plane.



Ideal points

The original idea: extend $\mathbb{E}^2$ by adding some ideal points.

Euclidean point: line through origin of $\mathbb{E}^3$ intersecting $w = 1$ plane.

Ideal point: line through origin of $\mathbb{E}^3$ parallel to $w = 1$ plane.

With Cartesian coords, no place to put ideal points. With homogeneous coordinates, there's a big gaping hole!

The projective plane

So . . .

define the projective plane $\mathbb{P}^2$ to be the set of lines through the origin of $\mathbb{E}^3$.

A line in $\mathbb{E}^2$ is represented by plane through origin of $\mathbb{E}^3$.

The ideal points form a line! The **line at infinity**. The equator of the embedded sphere.

“Parallel lines” intersect at infinity.

Duality. Two points determine a line. Two lines determine a point. Every axiom of the projective plane has a dual axiom by switching “line” and “point”.

Noneuclidean geometry!!!