

6. Representing Rotation

Mechanics of Manipulation

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Outline.

- Generalities
- Axis-angle
- Rodrigues's formula
- Rotation matrices
- Euler angles

Why representing rotations is hard.

- Rotations do not commute.
- The topology of spatial rotations does not permit a smooth embedding in Euclidean three space.

Choices

- More than three numbers
 - Rotation matrices
 - Unit quaternions. (aka Euler parameters)
- Many-to-one
 - Axis times angle (matrix exponential)
- Unsmooth and many-to-one
 - Euler angles
- Unsmooth and many-to-one and more than three numbers
 - Axis-angle

Axis-angle

Recall Euler's theorem: every spatial rotation leaves a line of fixed points: the rotation axis.

Let O , $\hat{\mathbf{n}}$, θ , be ...

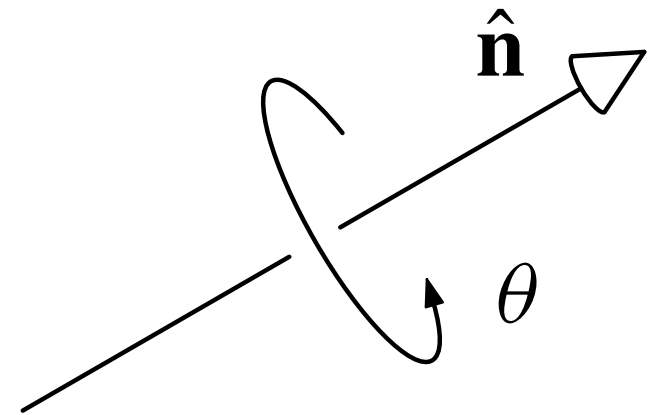
Let $\text{rot}(\hat{\mathbf{n}}, \theta)$ be the corresponding rotation.

Many to one:

$$\text{rot}(-\hat{\mathbf{n}}, -\theta) = \text{rot}(\hat{\mathbf{n}}, \theta)$$

$$\text{rot}(\hat{\mathbf{n}}, \theta + 2k\pi) = \text{rot}(\hat{\mathbf{n}}, \theta), \text{ for any integer } k.$$

When $\theta = 0$, the rotation axis is indeterminate, giving an infinity-to-one mapping.



Representation

What do we want from a representation? For a start:

- Rotate points;
Rodrigues's formula
- Compose rotations;
Using axis-angle? Ugh.
- (Convert to other representations.)

Rodrigues's formula

Others derive Rodrigues's formula using rotation matrices, missing the geometrical aspects.

Given point \mathbf{x} , decompose into components parallel and perpendicular to the rotation axis

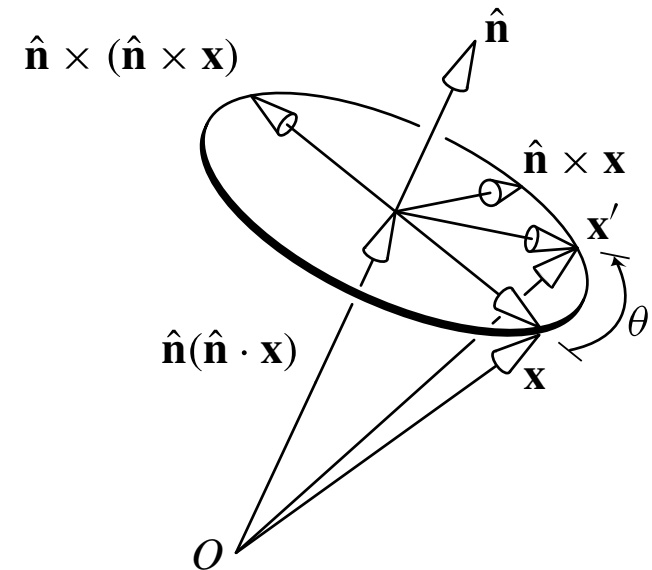
$$\mathbf{x} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) + \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

Only \mathbf{x}_\perp is affected by the rotation, yielding *Rodrigues's formula*:

$$\mathbf{x}' = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) + \sin \theta (\hat{\mathbf{n}} \times \mathbf{x}) + \cos \theta \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

A common variation:

$$\mathbf{x}' = \mathbf{x} + (\sin \theta) \hat{\mathbf{n}} \times \mathbf{x} + (1 - \cos \theta) \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$



Rotation matrices

Choose O on rotation axis. Choose frame $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$.

Let $(\hat{\mathbf{u}}'_1, \hat{\mathbf{u}}'_2, \hat{\mathbf{u}}'_3)$ be the image of that frame.

Write the $\hat{\mathbf{u}}'_i$ vectors in $\hat{\mathbf{u}}_i$ coordinates, and collect them in a matrix:

$$\hat{\mathbf{u}}'_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}'_1 \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}'_1 \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}'_1 \end{pmatrix}$$

$$\hat{\mathbf{u}}'_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}'_2 \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}'_2 \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}'_2 \end{pmatrix}$$

$$\hat{\mathbf{u}}'_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}'_3 \\ \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}'_3 \\ \hat{\mathbf{u}}_3 \cdot \hat{\mathbf{u}}'_3 \end{pmatrix}$$

So many numbers

A rotation matrix has nine numbers,
but spatial rotations have only three degrees of freedom,
leaving six excess numbers ...

There are six constraints that hold among the nine numbers.

$$|\hat{\mathbf{u}}'_1| = |\hat{\mathbf{u}}'_2| = |\hat{\mathbf{u}}'_3| = 1$$
$$\hat{\mathbf{u}}'_3 = \hat{\mathbf{u}}'_1 \times \hat{\mathbf{u}}'_2$$

i.e. the $\hat{\mathbf{u}}'_i$ are unit vectors forming a right-handed coordinate system.

Such matrices are called *orthonormal* or *rotation* matrices.

Rotating a point

Let (x_1, x_2, x_3) be coordinates of \mathbf{x} in frame $(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2, \hat{\mathbf{u}}_3)$.

Then \mathbf{x}' is given by the same coordinates taken in the $(\hat{\mathbf{u}}'_1, \hat{\mathbf{u}}'_2, \hat{\mathbf{u}}'_3)$ frame:

$$\begin{aligned}\mathbf{x}' &= x_1 \hat{\mathbf{u}}'_1 + x_2 \hat{\mathbf{u}}'_2 + x_3 \hat{\mathbf{u}}'_3 \\ &= x_1 A \hat{\mathbf{u}}_1 + x_2 A \hat{\mathbf{u}}_2 + x_3 A \hat{\mathbf{u}}_3 \\ &= A(x_1 \hat{\mathbf{u}}_1 + x_2 \hat{\mathbf{u}}_2 + x_3 \hat{\mathbf{u}}_3) \\ &= A\mathbf{x}\end{aligned}$$

So rotating a point is implemented by ordinary matrix multiplication.

Rotating a point

Let A and B be coordinate frames. Notation:

x a point

\mathbf{x} a geometrical vector, directed from an origin O to the point x
or, a vector of three numbers, representing x in an unspecified frame

${}^A\mathbf{x}$ a vector of three numbers, representing x in the A frame

Let ${}^B_A R$ be the rotation matrix that rotates frame B to frame A .

Then (see previous slide) ${}^B_A R$ represents the rotation of the point x :

$${}^B\mathbf{x}' = {}^B_A R {}^B\mathbf{x}$$

Note prescripts all match. Both points, and xform, must be written in same coordinate frame.

Coordinate transform

There is another use for ${}^B_A R$:

${}^A \mathbf{x}$ and ${}^B \mathbf{x}$ represent the same point, in frames A and B resp.

To transform from A to B :

$${}^B \mathbf{x} = {}^B_A R {}^A \mathbf{x}$$

For coord xform, matrix subscript and vector superscript “cancel”.

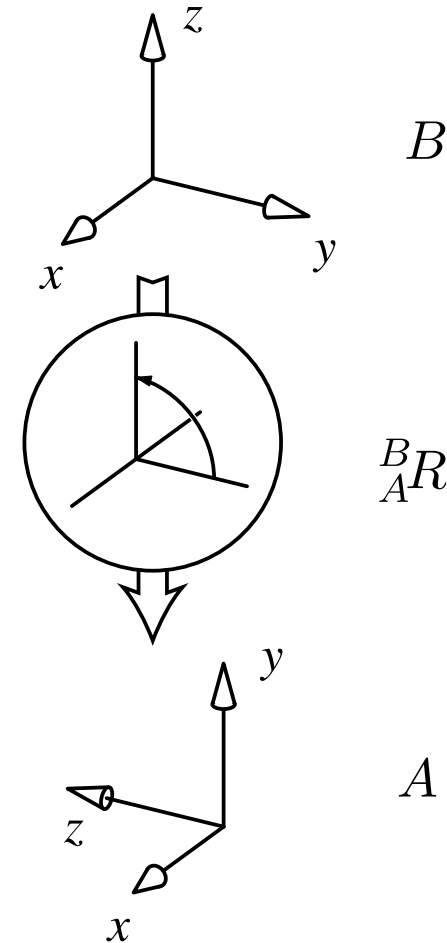
Rotation from B to A is the same as coordinate transform from A to B .

Example rotation matrix

$$\begin{aligned} {}^B_A R &= \left(\begin{array}{c|c|c} B\mathbf{x}_A & B\mathbf{y}_A & B\mathbf{z}_A \end{array} \right) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

How to remember what ${}^B_A R$ does?
Pick a coordinate axis and see. The x axis isn't very interesting, so try y :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Nice things about rotation matrices

- Composition of rotations: $\{R_1; R_2\} = R_2R_1$.
($\{x; y\}$ means do x then do y .)
- Inverse of rotation matrix is its transpose ${}^B_A R^{-1} = {}^A_B R = {}^B_A R^T$.
- Coordinate xform of a rotation matrix:

$${}^B R = {}^B_A R {}^A R {}^A_B R$$

Converting $\text{rot}(\hat{\mathbf{n}}, \theta)$ to R

Ugly way: define frame with $\hat{\mathbf{z}}$ aligned with $\hat{\mathbf{n}}$, use coordinate xform of previous slide.

Keen way: Rodrigues's formula!

$$\mathbf{x}' = \mathbf{x} + (\sin \theta) \hat{\mathbf{n}} \times \mathbf{x} + (1 - \cos \theta) \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{x})$$

Define "cross product matrix" N :

$$N = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

so that

$$N\mathbf{x} = \hat{\mathbf{n}} \times \mathbf{x}$$

... using Rodrigues's formula ...

Substituting the cross product matrix N into Rodrigues's formula:

$$\mathbf{x}' = \mathbf{x} + (\sin \theta)N\mathbf{x} + (1 - \cos \theta)N^2\mathbf{x}$$

Factoring out \mathbf{x}

$$R = I + (\sin \theta)N + (1 - \cos \theta)N^2$$

That's it! Rodrigues's formula in matrix form. If you want to you could expand it:

$$\begin{pmatrix} n_1^2 + (1 - n_1^2)c\theta & n_1n_2(1 - c\theta) - n_3s\theta & n_1n_3(1 - c\theta) + n_2s\theta \\ n_1n_2(1 - c\theta) + n_3s\theta & n_2^2 + (1 - n_2^2)c\theta & n_2n_3(1 - c\theta) - n_1s\theta \\ n_1n_3(1 - c\theta) - n_2s\theta & n_2n_3(1 - c\theta) + n_1s\theta & n_3^2 + (1 - n_3^2)c\theta \end{pmatrix}$$

where $c\theta = \cos \theta$ and $s\theta = \sin \theta$. Ugly.

Rodrigues's formula for differential rotations

Consider Rodrigues's formula for a differential rotation $\text{rot}(\hat{\mathbf{n}}, d\theta)$.

$$\begin{aligned}\mathbf{x}' &= (I + \sin d\theta N + (1 - \cos d\theta)N^2)\mathbf{x} \\ &= (I + d\theta N)\mathbf{x}\end{aligned}$$

so

$$\begin{aligned}d\mathbf{x} &= N\mathbf{x} d\theta \\ &= \hat{\mathbf{n}} \times \mathbf{x} d\theta\end{aligned}$$

It follows easily that differential rotations are vectors: you can scale them and add them up. We adopt the convention of representing angular velocity by the unit vector $\hat{\mathbf{n}}$ times the angular velocity.

Converting from R to $\text{rot}(\hat{\mathbf{n}}, \theta) \dots$

Problem: $\hat{\mathbf{n}}$ isn't defined for $\theta = 0$.

We will do it indirectly. Convert R to a unit quaternion (next lecture), then to axis-angle.

Euler angles

Three numbers to describe spatial rotations. ZYZ convention:

$$(\alpha, \beta, \gamma) \mapsto \text{rot}(\gamma, \hat{\mathbf{z}}'') \text{rot}(\beta, \hat{\mathbf{y}}') \text{rot}(\alpha, \hat{\mathbf{z}})$$

Can we represent an arbitrary rotation?

Rotate α about $\hat{\mathbf{z}}$ until

$$\hat{\mathbf{y}}' \perp \hat{\mathbf{z}}'';$$

Rotate β about $\hat{\mathbf{y}}'$ until

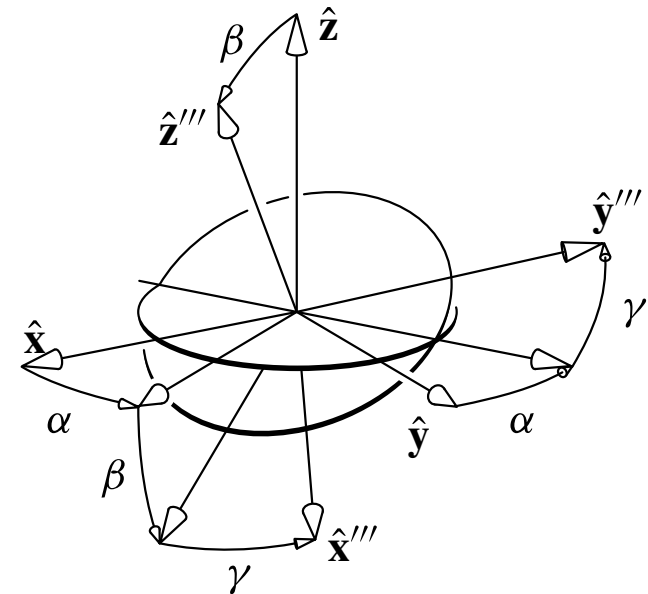
$$\hat{\mathbf{z}}'' \parallel \hat{\mathbf{z}}''';$$

Rotate γ about $\hat{\mathbf{z}}''$ until

$$\hat{\mathbf{y}}'' = \hat{\mathbf{y}}'''.$$

Note two choices for $\hat{\mathbf{y}}'$...

... except sometimes infinite choices.



From (α, β, γ) to R

Expand $\text{rot}(\alpha, \hat{\mathbf{z}}) \text{rot}(\beta, \hat{\mathbf{y}}) \text{rot}(\gamma, \hat{\mathbf{z}})$

(Why is that the right order?)

$$\begin{aligned} & \begin{pmatrix} \mathbf{c}\alpha & -\mathbf{s}\alpha & 0 \\ \mathbf{s}\alpha & \mathbf{c}\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{c}\beta & 0 & \mathbf{s}\beta \\ 0 & 1 & 0 \\ -\mathbf{s}\beta & 0 & \mathbf{c}\beta \end{pmatrix} \begin{pmatrix} \mathbf{c}\gamma & -\mathbf{s}\gamma & 0 \\ \mathbf{s}\gamma & \mathbf{c}\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{c}\alpha \mathbf{c}\beta \mathbf{c}\gamma - \mathbf{s}\alpha \mathbf{s}\gamma & -\mathbf{c}\alpha \mathbf{c}\beta \mathbf{s}\gamma - \mathbf{s}\alpha \mathbf{c}\gamma & \mathbf{c}\alpha \mathbf{s}\beta \\ \mathbf{s}\alpha \mathbf{c}\beta \mathbf{c}\gamma + \mathbf{c}\alpha \mathbf{s}\gamma & -\mathbf{s}\alpha \mathbf{c}\beta \mathbf{s}\gamma + \mathbf{c}\alpha \mathbf{c}\gamma & \mathbf{s}\alpha \mathbf{s}\beta \\ -\mathbf{s}\beta \mathbf{c}\gamma & \mathbf{s}\beta \mathbf{s}\gamma & \mathbf{c}\beta \end{pmatrix} \quad (1) \end{aligned}$$

From R to (α, β, γ) the ugly way

Case 1: $r_{33} = 1, \beta = \pi$. $\alpha - \gamma$ is indeterminate.

$$R = \begin{pmatrix} \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) & 0 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Case 2: $r_{33} = -1, \beta = -\pi$. $\alpha + \gamma$ is indeterminate.

$$R = \begin{pmatrix} -\cos(\alpha - \gamma) & -\sin(\alpha - \gamma) & 0 \\ -\sin(\alpha - \gamma) & \cos(\alpha - \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For generic case: solve 3rd column for β . (Sign is free choice.)
Solve third column for α and third row for γ .

... but there are numerical issues ...

From R to (α, β, γ) the clean way

Let

$$\sigma = \alpha + \gamma$$

$$\delta = \alpha - \gamma$$

Then

$$r_{22} + r_{11} = \cos \sigma (1 + \cos \beta)$$

$$r_{22} - r_{11} = \cos \delta (1 - \cos \beta)$$

$$r_{21} + r_{12} = \sin \delta (1 - \cos \beta)$$

$$r_{21} - r_{12} = \sin \sigma (1 + \cos \beta)$$

(No special cases for $\cos \beta = \pm 1$?)

Solve for σ and δ , then for α and γ , then finally

$$\beta = \tan^{-1}(r_{13} \cos \alpha + r_{23} \sin \alpha, r_{33})$$

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