8. Representing displacements

Mechanics of Manipulation

Matt Mason
matt.mason@cs.cmu.edu

http://www.cs.cmu.edu/~mason

Carnegie Mellon

Outline.

- Review of spatial displacements
- Homogeneous coordinates
- Plücker coordinates of a line
- Screw coordinates

Review of spatial displacements

Definition: rigid motion

Theorem 2.2: any displacement of $\mathbf{E}^n$ can be represented as a rotation composed with a translation.

Definition: a **screw** is a line plus a pitch.

Definition: a **twist** is a motion along a screw.

Theorem 2.7 (Chasles's theorem): every displacement of $\mathbf{E}^3$ is a twist.

These can guide design of representations.

Homogeneous coordinates

Recall theorem 2.2: a displacement can be decomposed into a rotation followed by a translation.

$$\mathbf{x}' = R\mathbf{x} + \mathbf{d}$$

We can write it more compactly. Add a fourth component to points:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}$$

(Remember, we did this before. If the fourth element is 0 we get points at infinity. Now we're focusing on ordinary points.)

Transforms using homogeneous coords

Define the *homogeneous coordinate transform matrix* T:

$$T = \left(\begin{array}{ccc|c} & R & & \mathbf{d} \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

And write

$$\mathbf{x}' = T\mathbf{x}$$

It's just a more compact way of writing:

$$\mathbf{x}' = R\mathbf{x} + \mathbf{d}$$

Especially useful for expressions such as $\mathbf{x}' = T_3 T_2 T_1 \mathbf{x}$.

Plücker coordinates

Screw coordinates are built on top of Plücker coordinates, which are a way of representing lines.

Let $\mathbf{p}$ be a point on the line;

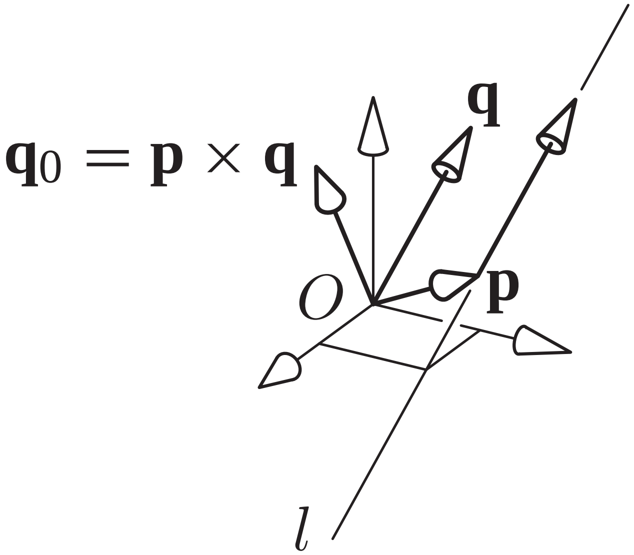
Let $\mathbf{q}$ be the **direction vector**;

Let $\mathbf{q}_0 = \mathbf{p} \times \mathbf{q}$, the **moment vector**;

Then $(\mathbf{q}, \mathbf{q}_0)$ gives the six **Plücker coordinates**;

(Note choice of $\mathbf{p}$ doesn't matter:

$$\begin{aligned}\mathbf{p}' \times \mathbf{q} &= \mathbf{p} \times \mathbf{q} + (\mathbf{p}' - \mathbf{p}) \times \mathbf{q} \\ &= \mathbf{p} \times \mathbf{q}\end{aligned}$$



Plücker's excess numbers

Plücker coordinates give six numbers. A line requires only four.

First, since $\mathbf{q}_0 = \mathbf{p} \times \mathbf{q}$, there is a constraint:

$$\mathbf{q} \cdot \mathbf{q}_0 = 0$$

Second, scaling gives same the line

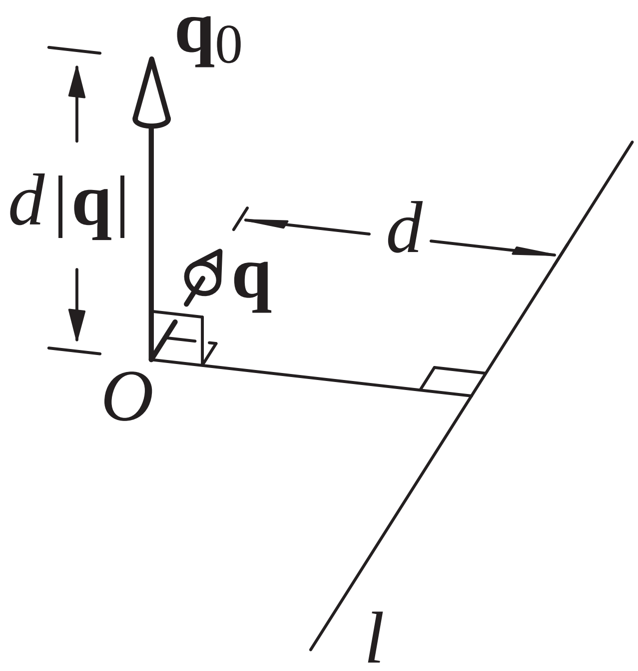
$$(\mathbf{q}, \mathbf{q}_0) \equiv k(\mathbf{q}, \mathbf{q}_0)$$

(So, why not normalize, scaling by $1/|\mathbf{q}|$? Sometimes, as we shall see, $|\mathbf{q}| = 0$!)

Reading Plücker coordinates: generic case

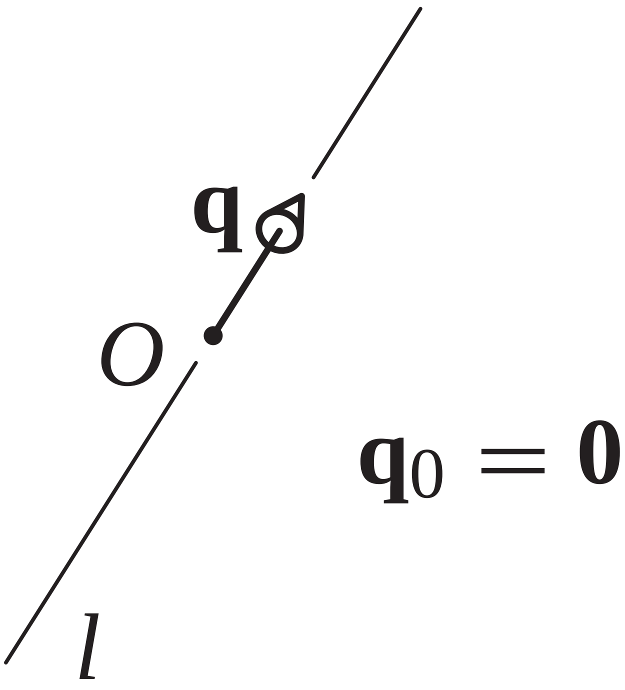
Nonzero $\mathbf{q}_0$ is orthogonal to a plane containing the line.

Magnitude $|\mathbf{q}_0|/|\mathbf{q}|$ gives distance to line.

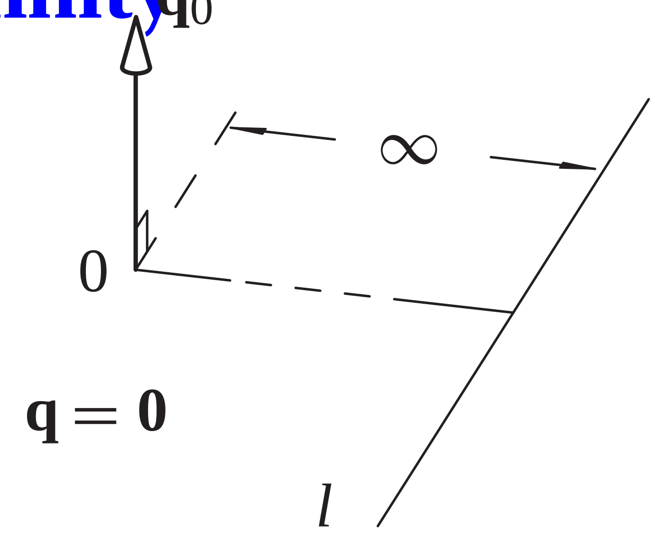


Plücker coords of line through origin

Zero $\mathbf{q}_0$: Line passes through origin.



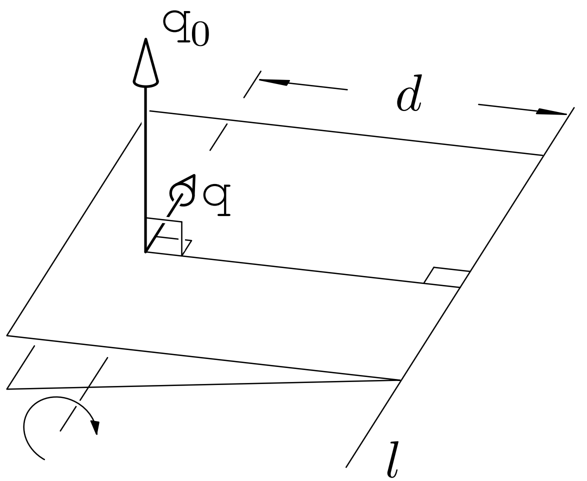
Plücker coords of line at infinity



Nonzero $\mathbf{q}_0$ is orthogonal to a plane containing the line.

Magnitude of $|\mathbf{q}_0|/|\mathbf{q}|$ gives distance to line.

Work it out as a limiting process. Hold $\mathbf{q}_0$ constant as line goes to infinity.



Using Plücker coordinates

Direction of line : $\mathbf{q}$.

Distance of line from O: $|\mathbf{q}_0|/|\mathbf{q}|$.

Point-on-line test for point x:

$$(\mathbf{x} - \mathbf{p}) \times \mathbf{q} = 0$$

$$\mathbf{x} \times \mathbf{q} - \mathbf{p} \times \mathbf{q} = 0$$

$$\mathbf{x} \times \mathbf{q} = \mathbf{q}_0$$

Find point on line closest to O:

$$\mathbf{q} \times \mathbf{q}_0 / \mathbf{q} \cdot \mathbf{q}, \quad \text{for } \mathbf{q} \neq 0$$

A topical example.

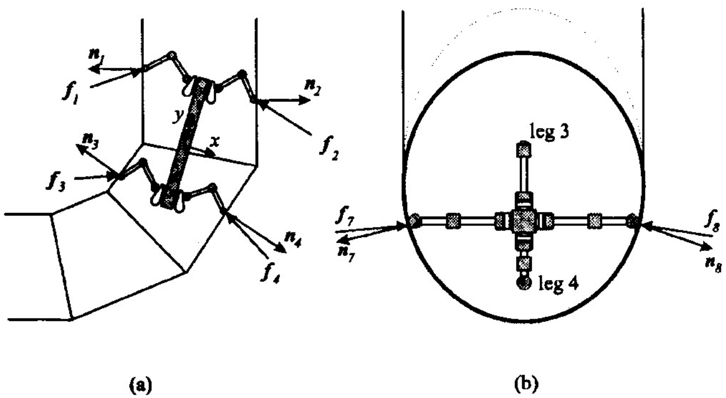


Fig. 2. Lateral and top drawings of a pipe crawling robot getting into a curve.

the authors report the design of a quadruped walking robot with passive articulated feet that adapt to the ground, thus being able to detect the orientation of the local ground surface by means of two angle sensors integrated at each robot's ankle.

Most tactile sensors, whether based on conductive silicone rubber, pressure-sensitive semiconductors, or piezoelectric elements, detect contact position using surface-mounted arrays of force-sensitive elements [24]. In this paper, a simpler system for the estimation of the normal vectors using a five-axis force/torque sensor is presented. This use of force sensors was first pointed out by Salisbury [30] in the context of manipulation systems. It is usually called *intrinsic* contact sensing for the use of internal force and torque measurements [1], [2], [6]. Force-based contact sensors have been actually implemented in robotic hands [4], [21], [35] and object shape detection systems [33]. To the authors' best knowledge, no existing legged robots implement this technique.

II. INTRINSIC TACTILE SENSING METHOD

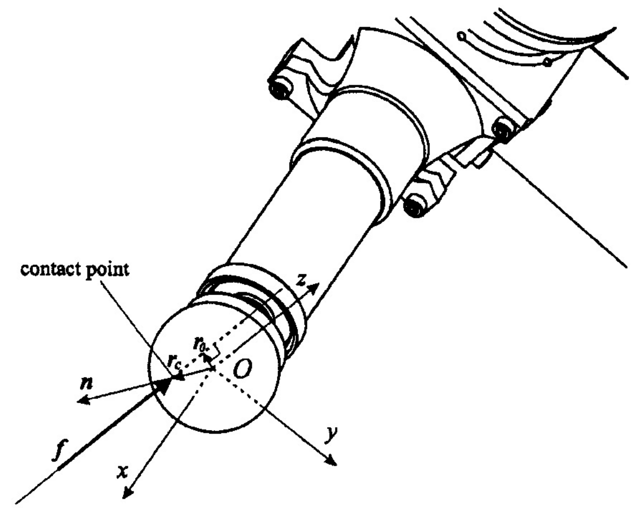


Fig. 3. Problem statement and reference system.

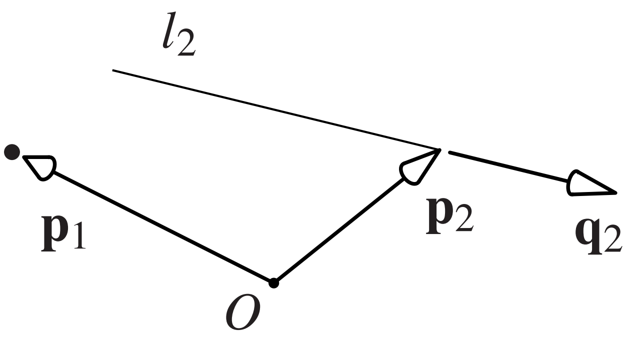
where

$$\mathbf{r}_0 = \frac{\mathbf{F} \times \mathbf{M}}{\|\mathbf{F}\|^2}. \qquad (5)$$

The line of action of the force or wrench axis is a line through $\mathbf{r}_0$ and parallel to $\mathbf{F}$ parameterized by λ. This line intersects the foot surface in two locations: one corresponding to a force pulling out of the surface and one corresponding to a force pushing into the surface. Because adhesive forces are not allowed, the contact point is determined as the intersection point for which the contact force points inwardly at the foot surface, that is

Moment about a point

Moment of a line l_2 about a point p_1. In analogy with unit force in direction $\mathbf{q}_2$. What would the torque be?



$$(\mathbf{p}_2 - \mathbf{p}_1) \times \left(\frac{\mathbf{q}_2}{|\mathbf{q}_2|} \right)$$

$$\frac{\mathbf{p}_2 \times \mathbf{q}_2 - \mathbf{p}_1 \times \mathbf{q}_2}{|\mathbf{q}_2|}$$

$$\frac{\mathbf{q}_{02} - \mathbf{p}_1 \times \mathbf{q}_2}{|\mathbf{q}_2|}$$

Moment about a line

Moment of a line l_2 about a *line* l_1.

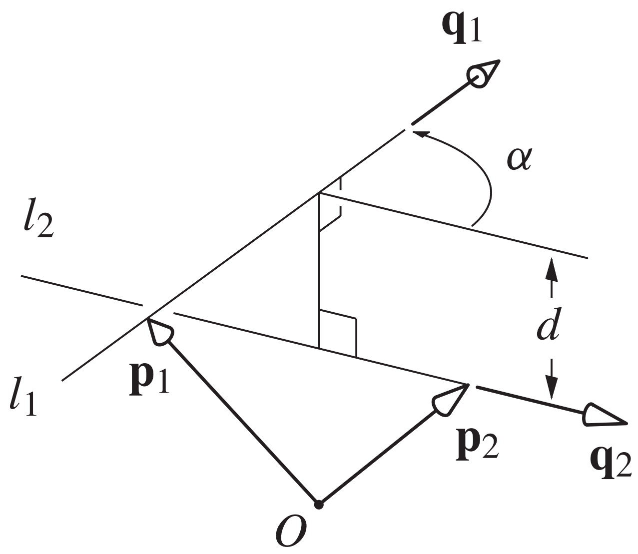
Think of the torque at $\mathbf{p}_1$, and take the component in the $\mathbf{q}_1$ direction:

$$\frac{\mathbf{q}_1}{|\mathbf{q}_1|} \cdot \frac{\mathbf{q}_{02} - \mathbf{p}_1 \times \mathbf{q}_2}{|\mathbf{q}_2|}$$

$$\frac{\mathbf{q}_1 \cdot \mathbf{q}_{02} - \mathbf{q}_1 \cdot \mathbf{p}_1 \times \mathbf{q}_2}{|\mathbf{q}_1 \mathbf{q}_2|}$$

$$\frac{\mathbf{q}_1 \cdot \mathbf{q}_{02} + \mathbf{q}_2 \cdot \mathbf{p}_1 \times \mathbf{q}_1}{|\mathbf{q}_1 \mathbf{q}_2|}$$

$$\frac{\mathbf{q}_1 \cdot \mathbf{q}_{02} + \mathbf{q}_2 \cdot \mathbf{q}_{01}}{|\mathbf{q}_1 \mathbf{q}_2|}$$



It's symmetric. Moment of l_1 about l_2 = moment of l_2 about l_1.

This interesting product has *LOTS* of uses . . .

Reciprocal product / virtual product

Define **reciprocal product**, or **virtual product**:

$$(\mathbf{q}_1, \mathbf{q}_{01}) * (\mathbf{q}_2, \mathbf{q}_{02}) = \mathbf{q}_1 \cdot \mathbf{q}_{02} + \mathbf{q}_2 \cdot \mathbf{q}_{01}$$

For normalized Plücker coordinates, reciprocal product gives moment between the two lines.

More uses for Plücker coordinates

Look at moment geometrically. Distance between the lines times sine of the angle.

$$d \sin\alpha = (\mathbf{q}_1, \mathbf{q}_{01}) * (\mathbf{q}_2, \mathbf{q}_{02}) / |\mathbf{q}_1 \mathbf{q}_2|$$

Note we can also get the angle by

$$\sin\alpha = \mathbf{q}_1 \times \mathbf{q}_2 / |\mathbf{q}_1 \mathbf{q}_2|$$

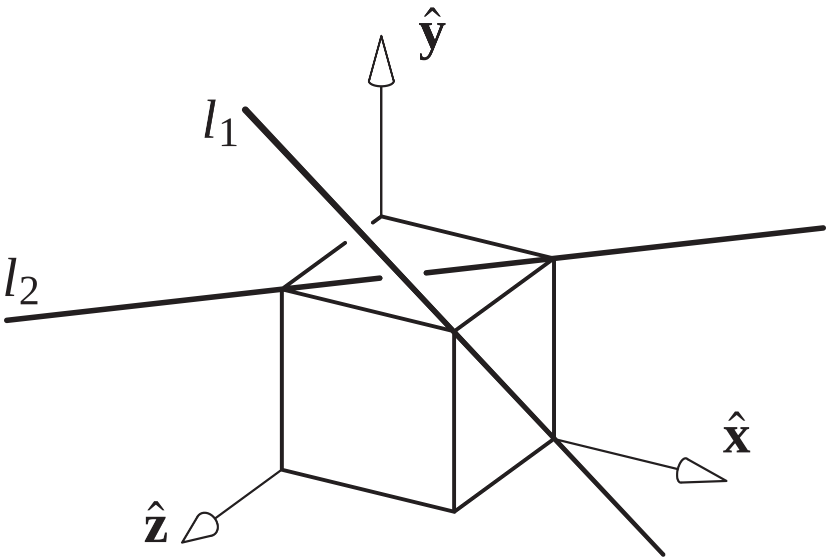
So to get the (signed) distance between two lines:

$$d = \frac{(\mathbf{q}_1, \mathbf{q}_{01}) * (\mathbf{q}_2, \mathbf{q}_{02})}{|\mathbf{q}_2 \times \mathbf{q}_1|}$$

To tell if two lines intersect, check if reciprocal product is zero. (Parallel lines intersect at infinity!)

Example: using Plücker

Find the angle and distance between the two lines:



Screw coordinates

A screw is a line plus a scalar pitch. Seven numbers?

No! We aren't really using those six numbers. Plenty of room to sneak pitch in.

Given a line $(\mathbf{q}, \mathbf{q}_0)$, and pitch p. Define the screw coordinates to be $(\mathbf{s}, \mathbf{s}_0)$, where

$$\mathbf{s} = \mathbf{q}$$
$$\mathbf{s}_0 = \mathbf{q}_0 + p\mathbf{q}$$

Why does this work? Recall $\mathbf{q} \cdot \mathbf{q}_0$ is zero.

To get the pitch back

$$\mathbf{s} \cdot \mathbf{s}_0 = \mathbf{q} \cdot \mathbf{q}_0 + p\mathbf{q} \cdot \mathbf{q}$$
$$p = \frac{\mathbf{s} \cdot \mathbf{s}_0}{\mathbf{s} \cdot \mathbf{s}}$$

Special case screws

Zero pitch: just like Plücker coordinates.

Infinite pitch: $\mathbf{s} = 0$.

Representing a twist

A twist is a screw plus a magnitude. Seven numbers?

No! Remember Plücker coordinates don't use scale. So take Plücker coordinates, normalize them, and scale.

Let θ be the angle of rotation, d the distance of translation, both nonzero.

Let $p = d/\theta$ be the pitch.

$$\left(\theta \, \frac{\mathbf{s}}{|\mathbf{s}|}, \theta \, \frac{\mathbf{s}_0}{|\mathbf{s}|}\right)$$

Substituting the definition of screw coordinates, we obtain

$$\begin{aligned}\left(\theta \, \frac{\mathbf{s}}{|\mathbf{s}|}, \theta \, \frac{\mathbf{s}_0}{|\mathbf{s}|}\right) &= \frac{1}{|\mathbf{q}|} \left(\theta\mathbf{q}, \theta\mathbf{q}_0 + \theta p\mathbf{q}\right) \\ &= \frac{1}{|\mathbf{q}|} \left(\theta\mathbf{q}, \theta\mathbf{q}_0 + d\mathbf{q}\right)\end{aligned}$$

Twists of zero or infinite pitch

Infinite pitch is translation:

$$(\mathbf{s}, \mathbf{s}_0) = \frac{1}{|\mathbf{q}|} (\mathbf{0}, d\mathbf{q})$$

Zero pitch is rotation, identical to scaled Plücker coordinates:

$$\theta(\mathbf{s}, \mathbf{s}_0) = \theta(\mathbf{q}, \mathbf{q}_0).$$

assuming Plücker coordinates were normalized.

Twists of zero or infinite pitch

Here is an interesting and instructive ambiguity. Somebody gives you, for example, a twist with screw coordinates:

$$(0, 0, 0, 1, 0, 0)$$

Is it a zero pitch screw with axis at infinity? I.e. a rotation about an axis at infinity?

Or is it an infinite pitch twist? I.e. a translation in the z direction?

Both!

Consider the extremes

Translation: infinite pitch. $(\mathbf{t}, \mathbf{t}_0) = (0, \mathbf{O}')$. A very nice way to represent a translation.

Rotation through origin: zero pitch.

$$
\begin{aligned}
(\mathbf{t}, \mathbf{t}_0) &= \theta(\frac{\mathbf{s}}{|\mathbf{s}|}, \frac{\mathbf{s}_0}{|\mathbf{s}|}) \\
&= \theta(\frac{\mathbf{q}}{|\mathbf{q}|}, \frac{\mathbf{q}_0}{|\mathbf{q}|}) \\
&= (\theta\hat{\mathbf{n}}, 0)
\end{aligned}
$$

Angle times axis. We didn't cover it, but some people like it. Behaves well at small θ, but doesn't extend to one-to-one smooth map. Obviously.

Differential twists

Consider velocity v along l and angular velocity ω about l.

Let $\mathbf{p}$ be any point on l.

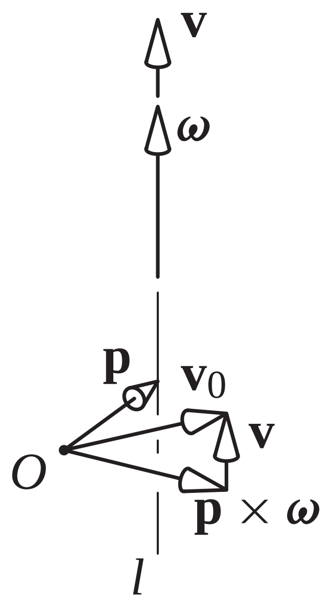
Plücker coordinates of l are

$$(\mathbf{q}, \mathbf{q}_0) = (\omega, \mathbf{p} \times \omega)$$

Pitch is $|\mathbf{v}|/|\omega|$ so screw coordinates are

$$(\mathbf{s}, \mathbf{s}_0) = (\omega, \mathbf{p} \times \omega + \frac{|\mathbf{v}|}{|\omega|}\omega)$$

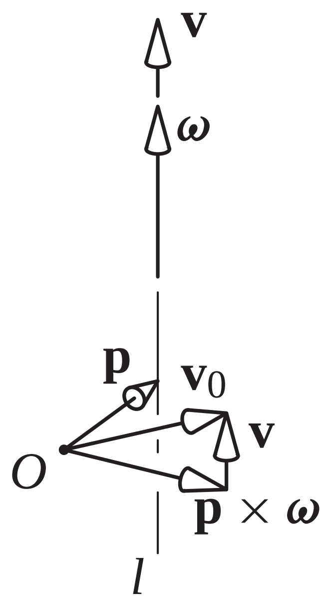
$$(\mathbf{s}, \mathbf{s}_0) = (\omega, \mathbf{p} \times \omega + \mathbf{v})$$



Differential twists

The vector $\mathbf{s}_0$ gives vel of origin $\mathbf{v}_0$

$$(\mathbf{s}, \mathbf{s}_0) = (\omega, \mathbf{v}_0)$$



So, for differential twists, screw coords are close to standard practice.

Important corollary. *Screw coordinates for differential twists form a vector space.* We can add differential twist screw coordinates, and we can multiply them by scalars.