9. Representing constraint

Mechanics of Manipulation

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Outline.

Constraint using contact screw and reciprocal product.

Repelling, reciprocal, contrary.

Relation to Reuleaux.

Examples.

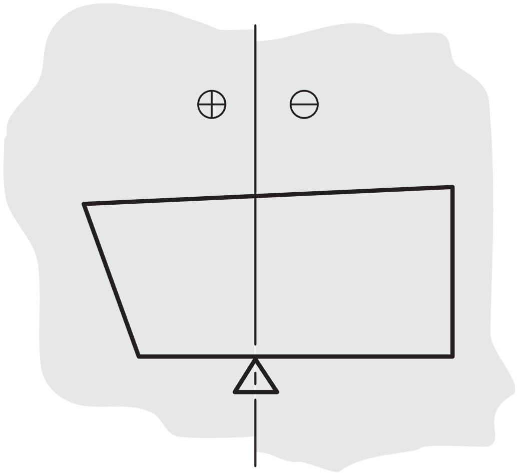
Remember Reuleaux's method?

Perpendicular to constraint divides plane into positive IC's, negative IC's, and IC's of either sign.

Doesn't extend to three dimensions.

Great for humans, bad for computers.

Sometimes equations are better than pictures.



First order model of constraint

Let $\hat{\mathbf{u}}$ be contact normal, inward pointing

Let p be contact point in the constrained body

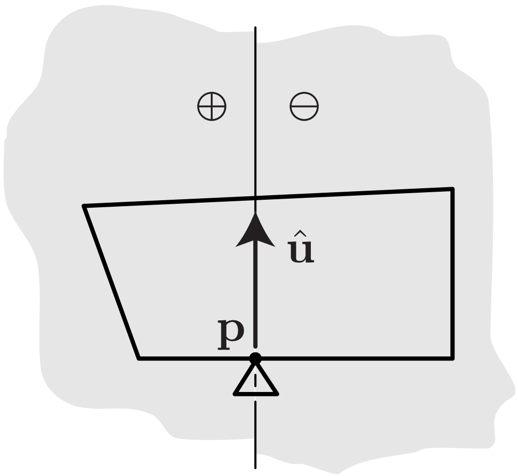
Let $\mathbf{v}_p$ be the velocity of the point p.

Then we write the bilateral velocity constraint as

$$\hat{\mathbf{u}} \cdot \mathbf{v}_p = 0$$

and unilateral velocity constraint as

$$\hat{\mathbf{u}} \cdot \mathbf{v}_p \geq 0$$



Constraint using screw coordinates

Let $(\omega, \mathbf{v}_0)$ be screw coordinates of body velocity

Then velocity of $\mathbf{p}$ is

$$\mathbf{v}_p = \mathbf{v}_0 + \omega \times \mathbf{p}$$

We write the kinematic constraint

$$\hat{\mathbf{u}} \cdot (\mathbf{v}_0 + \omega \times \mathbf{p}) \geq 0$$

Distribute dot product, play with triple product . . .

$$\hat{\mathbf{u}} \cdot \mathbf{v}_0 + (\mathbf{p} \times \hat{\mathbf{u}}) \cdot \omega \geq 0$$

Reciprocal product!

Contact screw

Define **contact screw** to be Plücker coordinates of the contact normal

$$(\mathbf{c}, \mathbf{c}_0) = (\mathbf{u}, \mathbf{p} \times \hat{\mathbf{u}})$$

Write the kinematic constraint as

$$(\mathbf{c}, \mathbf{c}_0) * (\omega, \mathbf{v}_0) \geq 0$$

Reciprocal, contrary, repelling

Definition 3.3: A pair of screws is **reciprocal**, **contrary**, or **repelling**, if their reciprocal product is zero, negative, or positive, respectively.

Bilateral constraint: velocity screw $(\omega, \mathbf{v}_0)$ and contact screw $(\mathbf{c}, \mathbf{c}_0)$ must be *reciprocal*:

$$(\mathbf{c}, \mathbf{c}_0) * (\omega, \mathbf{v}_0) = 0$$

Unilateral constraint: velocity screw $(\omega, \mathbf{v}_0)$ and contact screw $(\mathbf{c}, \mathbf{c}_0)$ must be *reciprocal or repelling*:

$$(\mathbf{c}, \mathbf{c}_0) * (\omega, \mathbf{v}_0) \geq 0$$

Connection to Reuleaux's method

The contact screw $(\mathbf{c}, \mathbf{c}_0)$ is a zero-pitch screw—the Plücker coordinates of the contact normal.

For a planar motion of the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane, the body velocity twist $(\omega, \mathbf{v}_0)$ is also a zero-pitch screw—a pure rotation.

$(\omega, \mathbf{v}_0)$ is the Plücker coordinates of the rotation axis perpendicular to the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane.

Contact normal and directed rotation axis must be reciprocal or repelling. Directed rotation axis must point down ($-$) if it's to the right of the contact normal, and up ($+$) if it's to the left of the contact normal.

Copy the figure from the board

Ex 1: Screw coordinates of planar motions.

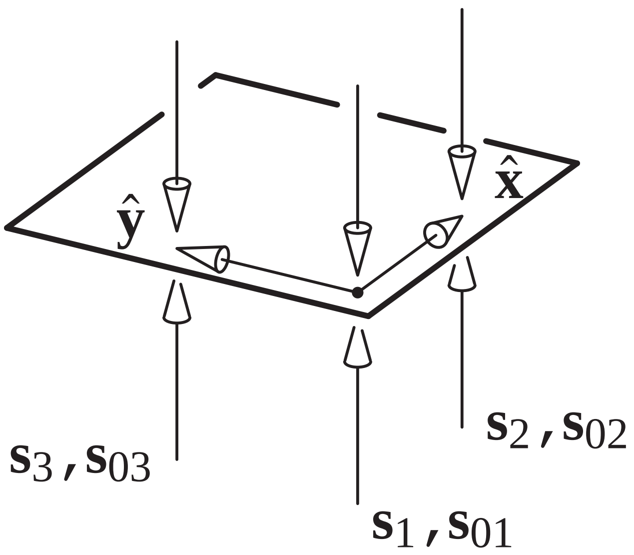
Choose three bilateral constraints aligned with the $\hat{\mathbf{z}}$ axis.

The screw coordinates for the constraints are:

$$(\mathbf{s}_1, \mathbf{s}_{01}) =$$

$$(\mathbf{s}_2, \mathbf{s}_{02}) =$$

$$(\mathbf{s}_3, \mathbf{s}_{03}) =$$



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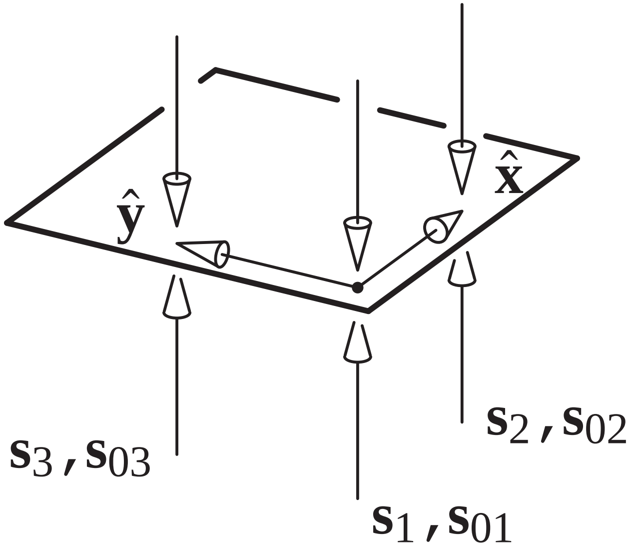
$$(\mathbf{s}_3, \mathbf{s}_{03}) =$$



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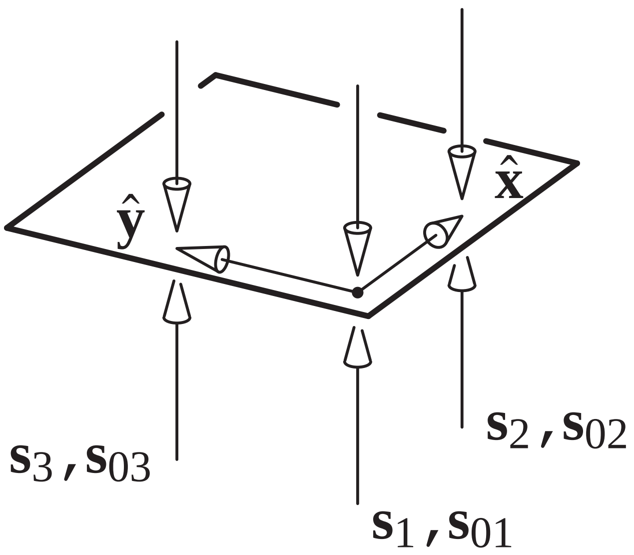
$$(\mathbf{s}_3, \mathbf{s}_{03}) =$$



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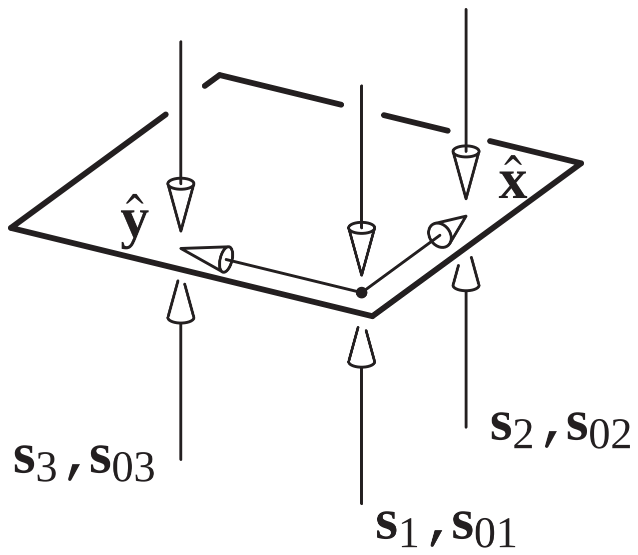

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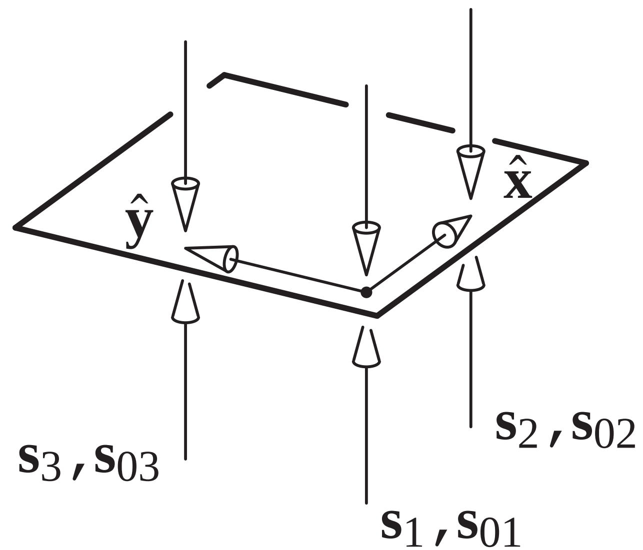


$$(\mathbf{s}_1, \mathbf{s}_{01}) = (0, 0, 1, 0, 0, 0)$$

$$(\mathbf{s}_2, \mathbf{s}_{02}) = (0, 0, 1, 0, -1, 0)$$

$$(\mathbf{s}_3, \mathbf{s}_{03}) = (0, 0, 1, 1, 0, 0)$$

Let the twist be given by

$$(\mathbf{t}, \mathbf{t}_0) = (t_1, t_2, t_3, t_4, t_5, t_6)$$

Ex 1: Form reciprocal products.

The twist must be reciprocal to $(\mathbf{s}_1, \mathbf{s}_{01})$:

$$t_6 = 0$$

... to $(\mathbf{s}_2, \mathbf{s}_{02})$:

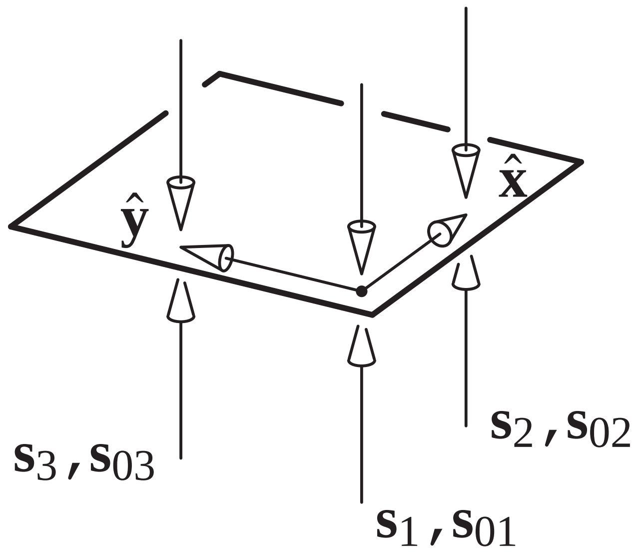
$$t_6 - t_2 = 0$$

... and to $(\mathbf{s}_3, \mathbf{s}_{03})$:

$$t_6 + t_1 = 0$$

Thus the twist must be of the form

$$(\mathbf{t}, \mathbf{t}_0) = (0, 0, t_3, t_4, t_5, 0)$$



Ex 1: Interpreting the answer

The twist must be of the form

$$(\mathbf{t}, \mathbf{t}_0) = (0, 0, t_3, t_4, t_5, 0)$$

To get the pitch:

$$p = \frac{\mathbf{t} \cdot \mathbf{t}_0}{\mathbf{t} \cdot \mathbf{t}} = 0$$

The direction vector $(0, 0, t_3)$ is parallel to $\hat{\mathbf{z}}$

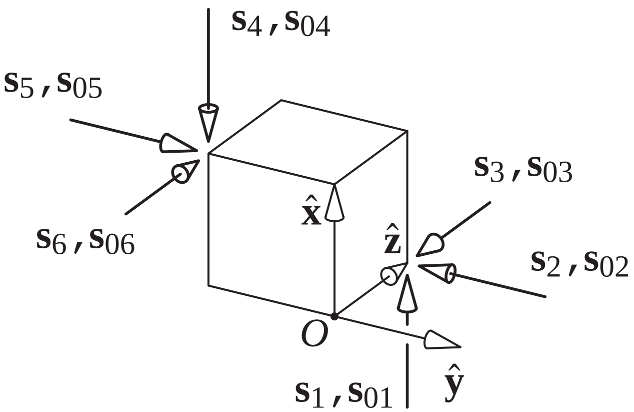
The point closest to the origin is

$$\begin{aligned} \frac{\mathbf{t} \times \mathbf{t}_0}{\mathbf{t} \cdot \mathbf{t}} &= (-t_5 t_3, t_4 t_3, 0)/t_3^2 \\ &= (-t_5/t_3, t_4/t_3) \end{aligned}$$

So the twist represents the rotation center using homogeneous coordinates. As a special case, when $t_3 = 0$, we obtain a pure translational velocity $(t_4, t_5, 0)$.

Ex 2: Squeezing the corners of a cube

We will consider the simpler bilateral problem



$$(\mathbf{s}_1, \mathbf{s}_{01}) =$$

$$(\mathbf{s}_2, \mathbf{s}_{02}) =$$

$$(\mathbf{s}_3, \mathbf{s}_{03}) =$$

$$(\mathbf{s}_4, \mathbf{s}_{04}) =$$

$$(\mathbf{s}_5, \mathbf{s}_{05}) =$$

$$(\mathbf{s}_6, \mathbf{s}_{06}) =$$

Ex 2: Squeezing the corners of a cube

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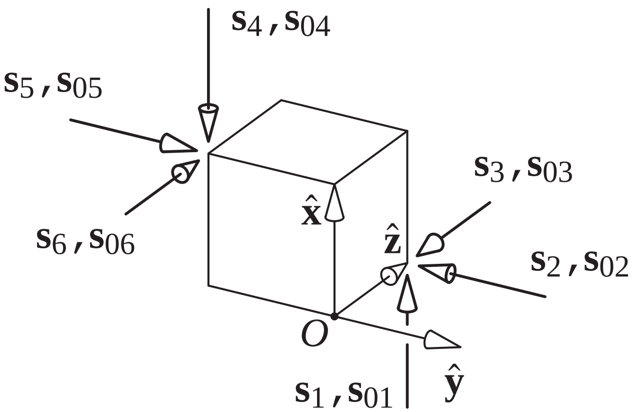


$$(\mathbf{s}_1, \mathbf{s}_{01}) = (1, 0, 0, 0, 1, 0)$$

$$(\mathbf{s}_2, \mathbf{s}_{02}) = (0, -1, 0, 1, 0, 0)$$

$$(\mathbf{s}_3, \mathbf{s}_{03}) = (0, 0, -1, 0, 0, 0)$$

$$(\mathbf{s}_4, \mathbf{s}_{04}) = (-1, 0, 0, 0, 0, -1)$$

$$(\mathbf{s}_5, \mathbf{s}_{05}) = (0, 1, 0, 0, 0, 1)$$

$$(\mathbf{s}_6, \mathbf{s}_{06}) = (0, 0, 1, -1, -1, 0)$$

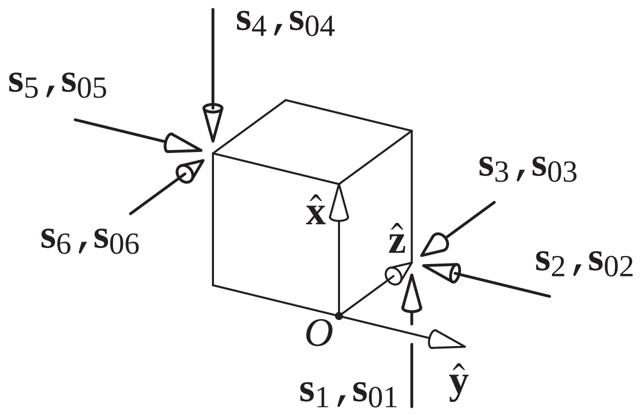
Let $(\mathbf{t}, \mathbf{t}_0)$ be a differential twist. Reciprocal with respect to $(\mathbf{s}_1, \mathbf{s}_{01})$

$$t_4 + t_2 = 0$$

Ex 2: Solving the constraint equations

Reciprocal to all 6 contact screws:

$$\begin{array}{cccccccc} t_4 & & & & +t_2 & & = 0 \\ & -t_5 & & +t_1 & & & = 0 \\ & & -t_6 & & & & = 0 \\ -t_4 & & & & & -t_3 & = 0 \\ & t_5 & & & & +t_3 & = 0 \\ & & t_6 & -t_1 & -t_2 & & = 0 \end{array}$$



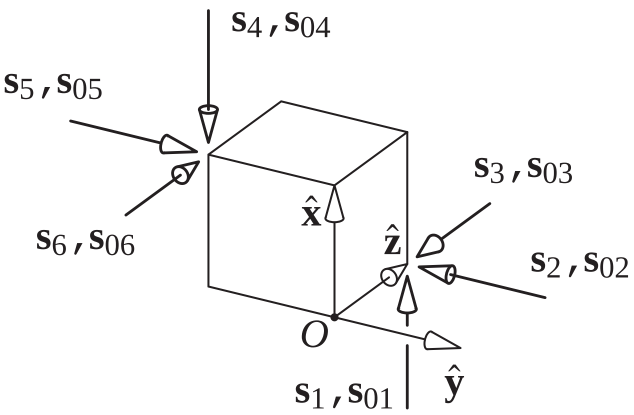
The solutions are of the form

$$(\mathbf{t}, \mathbf{t}_0) = k(1, -1, -1, 1, 1, 0)$$

Ex 2: Interpreting the solution

The solutions are of the form

$$(\mathbf{t}, \mathbf{t}_0) = k(1, -1, -1, 1, 1, 0)$$



Pitch: $\mathbf{t} \cdot \mathbf{t}_0 / \mathbf{t} \cdot \mathbf{t} = 0$.

Point on line closest to origin: $\mathbf{t} \times \mathbf{t}_0 / \mathbf{t} \cdot \mathbf{t} =$

Direction vector: $\mathbf{t} = (1, -1, -1)$.

I.e., as expected, the diagonal of the cube.

Next: Cspace transform and motion planning.