Mason Lecture 9 9/28/06 (1)

Screw Coords for Differential Twists

Recall def. of Screw

θ = angle of rotation

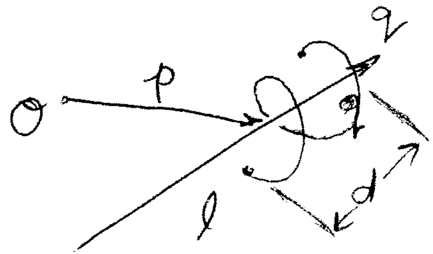
d = length of ~~dis~~ translation

l = line of screw axis

\rho $= \rho =$ pitch $= \dfrac{d}{\theta}$

$$\text{screw} \triangleq \left(\theta \frac{s}{\|s\|}, \theta \frac{s_0}{\|s\|} \right)$$

$$= \frac{1}{\|q\|} \left(\theta q, \theta q_0 + d q \right)$$

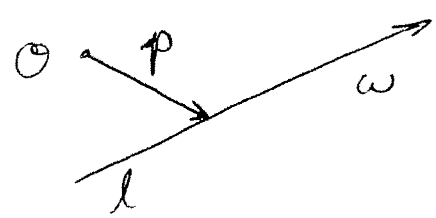


Consider differential motion occuring over differential time, dt

$\theta = \omega \, dt$, where ω is the angular velocity

Plücker coords $(q, q_0) = (\omega, p \times \omega)$

$$\text{pitch} = \frac{\|v\|}{\|\omega\|}$$



~~Screw coords for diff twist~~

~~$(s, s_0) = (\omega, p \times \omega)$~~

Screw Coords for diff. twist: ω plays role of q.

$$(s, s_0) = \left(\omega,\ p \times \omega + \frac{\|v\|}{\|\omega\|}\omega\right)$$

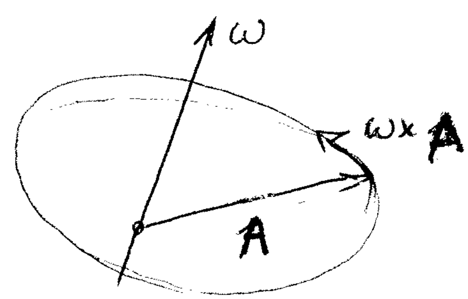
note that v is $\parallel$ to ω (by construction)

$$\therefore \quad v = \frac{\omega}{\|\omega\|}$$

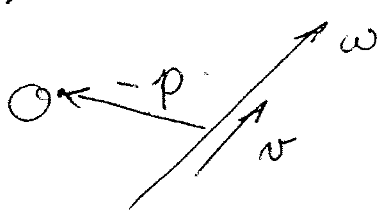
$$(s, s_0) = (\omega,\ p \times \omega + v)$$

What is the physical interp?

given a vector A,
$\omega \times A$ = velocity
or time rate of
change of A



What is the twist? → $(\omega,\ \omega \times (-p) + v)$



$\omega \times -p + v$
= velocity of point attached to moving body that is instantaneously coincident with the origin.

Lecture 9 begins here

Kinematic Constraints – First Order

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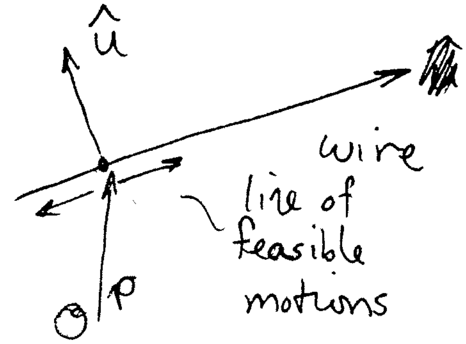
Let v_p be a point of interest

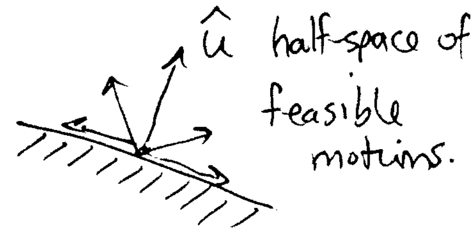
Let $\hat{u}$ be a direction, then

$$\hat{u} \cdot v_p = 0$$

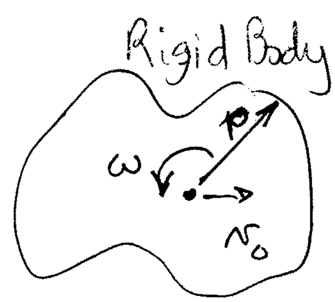
constrains the point to move along the ~~direction~~



$$\hat{u} \cdot v_p \geq 0$$



Recall $v_p = v_o + \omega \times p$



$$\hat{u} \cdot v_p =$$

$$= \hat{u} \cdot v_o + (\omega \times p) \cdot \hat{u}$$

$$= \hat{u} \cdot v_o + (p \times \hat{u}) \cdot \omega$$

v_o = velocity of origin

Looks like reciprocal product defined for Plücker coordinates

$$(q, q_o) * (p, p_o) = q \cdot p_o + p \cdot q_o$$

Define contact screw

$$(c, c_o) = (\hat{u}, p \times \hat{u})$$

Consideration of constraint gives rise to a "contact screw"

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2

Kinematic Constraint can now be written as:

$$(c, c_0) * (\omega, v_0) = c \cdot v_0 + c_0 \cdot \omega \overset{>0}{\underset{<0}{=}} 0$$

Called virtual product, because contact screw can be

Definition: Two screws are <u>reciprocal</u>, <u>contrary</u>, or <u>repelling</u> if the reciprocal product is $=0$, <0, or >0.

Bilateral constraints must be reciprocal

Unilateral constraints are reciprocal or repelling

Note:

Every point on a rigid body has same ω

Diff twist represents velocity of pt at origin

In (c, c_0), c represents direction of constraint

c_0 is moment of constraint about origin.

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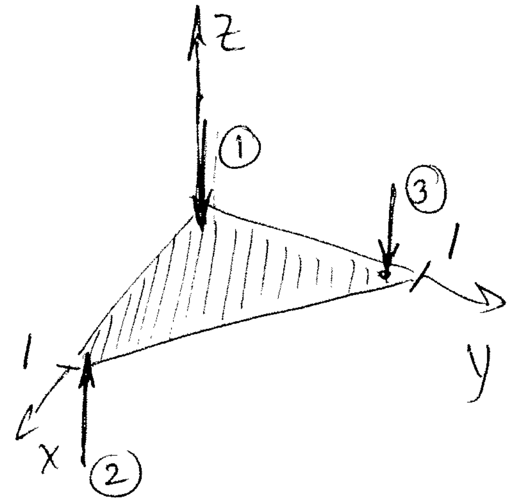
1

Example: Contact constraints for planar motion



Let contact points be at:

$r_1 = (0, 0, 0)$

$r_2 = (1, 0, 0)$

$r_3 = (0, 1, 0)$

Contact screws, (c_i, c_{oi}) $i = 1, 2, 3$

1 $(\underbrace{0\ 0\ -1}_{s}; \underbrace{0\ 0\ 0}_{r \times s = s_o})$

2 $(0\ 0\ 1; 0\ -1\ 0)$

3 $(0\ 0\ -1; -1\ 0\ 0)$

Recall that moment about origin = moment about each of the axes.
Moment between ∥ lines s=0 Moment ∥ intersecting " "

How can we determine possible motion of "grasped" triangle?

Let $(t, t_o) = (\omega, v_o)$ be the diff. twist of the Δ.

Assume bilateral contacts.

$$(t, t_o) * (c_i, c_{oi}) = 0 \quad \forall\ i = 1,2,3.$$

$$t \cdot c_{oi} + t_o \cdot c_i = 0 \qquad i = 1,2,3$$

$$c_i \cdot v_o + c_{oi} \cdot \omega = 0$$

reverse to match book examples

$$\left[\begin{array}{c:c} c_1 & c_{o1} \\ c_2 & c_{o2} \\ c_3 & c_{o3} \end{array}\right] \begin{bmatrix} v_o \\ \hdashline \omega \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & \underbrace{-1 \quad -1 \quad 0}_{\text{Det} = 1} & & & 0 \end{bmatrix} \begin{bmatrix} v_{ox} \\ v_{oy} \\ v_{oz} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

① $\quad -v_{oz} = 0$

② $\quad v_{oz} - \omega_y = 0 \Rightarrow \omega_y = 0$

③ $\quad -v_{oz} - \omega_x = 0 \Rightarrow \omega_x = 0$

Note that v_{ox}, v_{oy}, ω_z are unconstrained! Planar motion.

1.) Twist axis direction

$(t, t_0) = (\omega, v_0)$

$$\boxed{t = \omega = \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}}$$

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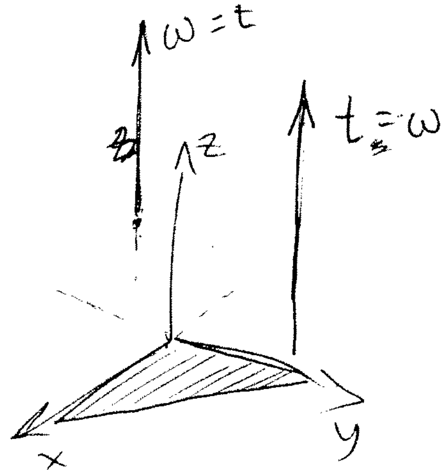
2.1

2.) Find twist ~~axis~~ pitch

$$(t, t_0) = (\omega, v_0)$$

$$= \left(\begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix}, \begin{bmatrix} v_{0x} \\ v_{0y} \\ 0 \end{bmatrix} \right)$$



$$\boxed{\rho = \frac{t \cdot t_0}{t \cdot t} = 0}$$

3.) Find Closest point on twist axis to origin (Instantaneous center of rotation)

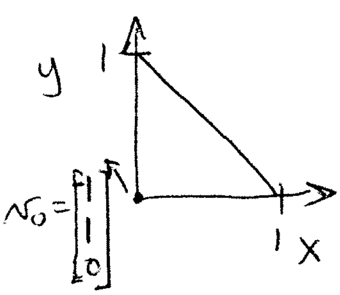
$$\frac{t \times t_0}{t \cdot t} = \frac{\omega \times v_0}{\omega \cdot \omega} = \begin{bmatrix} -v_{0y}/\omega_z \\ v_{0x}/\omega_z \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & \omega_z \\ v_{0x} & v_{0y} & 0 \end{vmatrix}$$

$$\omega_z \begin{bmatrix} -v_{0y} \\ v_{0x} \\ 0 \end{bmatrix}$$

Let $\omega_z = 1/2$

Let $v_0 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$



$v_0 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$(-2, -2)$ $\bullet$ $1/2 = \omega_z$ twist axis

Suppose we use a different origin?

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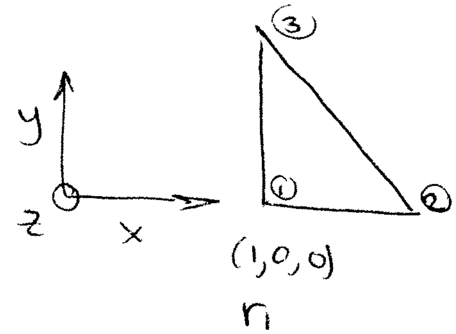
2.1

Suppose we use a different origin

$r_1 = (1,0,0)$

$r_2 = (2,0,0)$

$r_3 = (1,1,0)$



<u>Contact screws</u>

S_1, S_2, S_3 are not changed

$r \times S_i$ is changed by

$r_{i,new} = r_{i,old} + \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\therefore S_{i,new} = S_{i\,old} + \underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix} \times S_{i\,old}}_{\Delta S_i}$

$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0 & 0 & -1, & 0 & 1 & 0 \\ 0 & 0 & 1, & 0 & -2 & 0 \\ 0 & 0 & -1, & -1 & 1 & 0 \end{pmatrix} \begin{bmatrix} N_{ox} \\ N_{oy} \\ N_{oz} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Determinant still $\neq 0$
$= 1$

$\Delta S_{1,3} = \begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\Delta S_2 = -\Delta S_3$

Check for freedoms.

Same result as before, $N_{oz} = \omega_x = \omega_y = 0$

Freedoms: N_{ox}, N_{oy}, ω_z

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(2.3)

1. Check twist axis direction

$$\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix} \qquad \text{same as before}$$

2. Find pitch

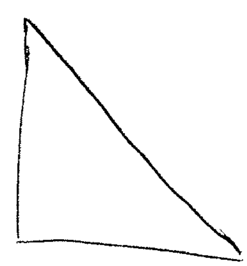
$$\rho = \frac{t \cdot t_0}{t \cdot t} = \frac{\begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix} \cdot \begin{pmatrix} N_{0x} \\ N_{0y} \\ 0 \end{pmatrix}}{t \cdot t} = 0$$

Same as before.

3. Find closest point on twist axis to origin

Only one difference from before

Since origin was changed,
(N_{0x}, N_{0y})
specifies velocity of



Further constraint in the plane

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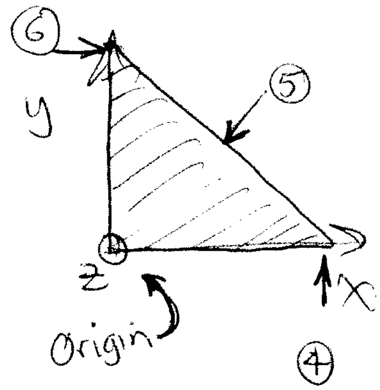
3

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_{ox} \\ v_{oy} \\ v_{oz} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

already eliminated by contacts 1, 2, 3.



Ask: will triangle be able to move?

Can this move?

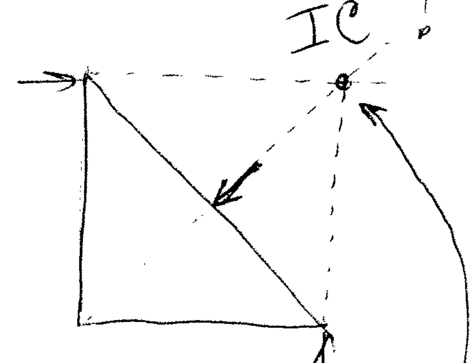
Assume bilateral constraints

$$\underbrace{\begin{bmatrix} 0 & 1 & 1 \\ -\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 1 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} v_{ox} \\ v_{oy} \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathrm{Det}(A) = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$$

Null Space is 1D

$$\begin{bmatrix} v_{ox} \\ v_{oy} \\ \omega_z \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \lambda \in \mathbb{R}$$



location of center of rotation $\frac{t \times t_0}{t \cdot t} = \frac{\omega \times v_0}{\omega \cdot \omega} = \frac{1}{\omega_z} \begin{bmatrix} -v_{oy} \\ v_{ox} \\ 0 \end{bmatrix}$

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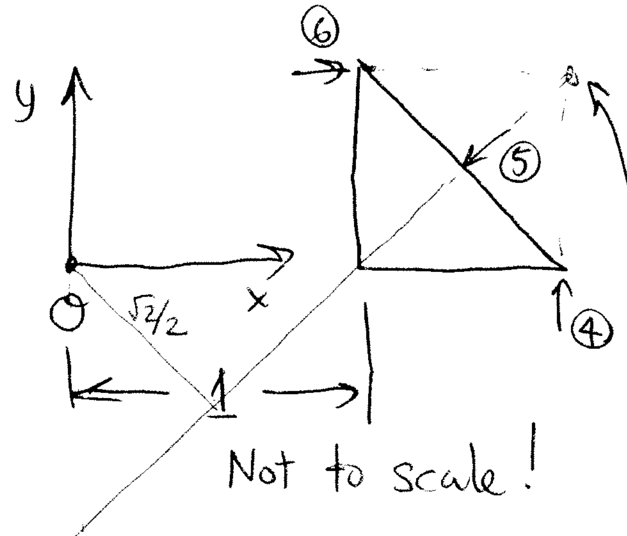
3.1

Suppose we move frame of representation?



Constraints

$$\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} N_{ox} \\ N_{oy} \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

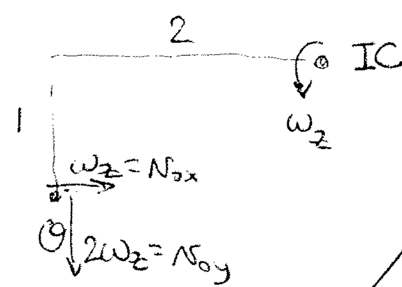
$$\begin{bmatrix} N_{ox} \\ N_{oy} \\ \omega_z \end{bmatrix} = \text{Null Sp}(A)\,\gamma$$

arbitrary scalar

Matlab gives

$$\begin{bmatrix} N_{ox} \\ N_{oy} \\ \omega_z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \gamma$$

Intuitive Check



$\therefore$ Center of rotation is $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ same point

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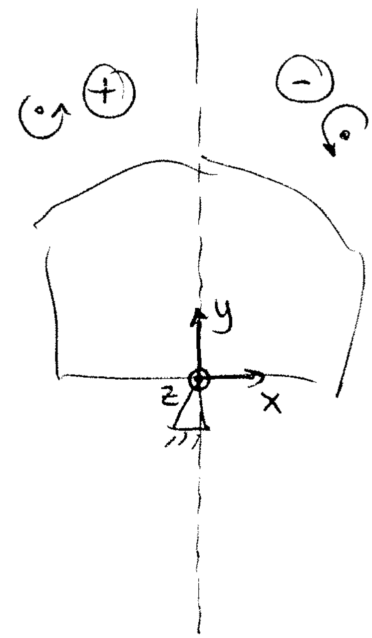
Revisit Reuleaux's Method



$\boxed{\text{Repelling screws}}$

$$(c, c_0) * (\omega, v_0) \geq 0$$

$$(0\ 1\ \cancel{0}\ ,\ \cancel{0}\ \cancel{0}\ 0) \begin{bmatrix} v_{0x} \\ v_{0y} \\ \cancel{v_{0z}} \\ \cancel{\omega_x} \\ \cancel{\omega_y} \\ \omega_z \end{bmatrix} \geq 0$$

$$\Rightarrow v_{0y} \geq 0$$

$\Rightarrow v_{0x}, \omega_z$ arbitrary

What are IC locations such that $v_{0y} \geq 0$?

$$\cancel{\text{who}} \quad \frac{t \times t_0}{t \cdot t} = \begin{bmatrix} -v_{0y}/\omega_z \\ +v_{0x}/\omega_z \\ 0 \end{bmatrix}$$

Must have $v_{0y} \geq 0$

Case 1: ~~choose~~ $\omega_z > 0,\ v_{0y} > 0$

$$\Rightarrow p_x < 0$$

Case 2: $\omega_z < 0,\ v_{0y} > 0$

$$\Rightarrow p_x > 0$$

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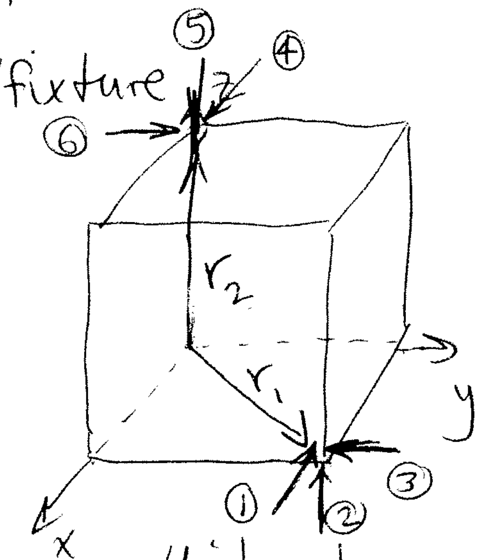
1

Contact constraint example: grasping

Let (S_i, S_{oi}) be the contact screw for the ith contact in a grasp/fixture
(c_i, c_{oi})

Let (t, t_o) be the differential twist of the object.
(ω, v_o)



Unit cube

Determine instantaneous motions allowed by grasp.

$r_1 = (1, 1, 0)$

$r_2 = (0, 0, 1)$

$s_1 = (-1\ 0\ 0) \qquad s_{o1} = (0\ 0\ 1)$

$s_2 = (0\ 0\ 1) \qquad s_{o2} = (1\ {-1}\ 0)$

$s_3 = (0\ {-1}\ 0) \qquad s_{o3} = (0\ 0\ {-1})$

$s_4 = (1\ 0\ 0) \qquad s_{o4} = (0\ 1\ 0)$

$s_5 = (0\ 0\ {-1}) \qquad s_{o5} = (0\ 0\ 0)$

$s_6 = (0\ 1\ 0) \qquad s_{o6} = (-1\ 0\ 0)$

Note: moment about origin = (moment about x-axis, y-axis, z-axis)
has 3 components

Moment between || lines = 0

Moment between intersecting lines = 0

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2

Assume constraints are bilateral

Find legal differential twists if any exist.

$$
\begin{array}{c}
(1)\\ (2)\\ (3)\\ (4)\\ (5)\\ (6)
\end{array}
\underbrace{\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & 1 & 1 & 0\\
0 & -1 & 0 & 0 & 0 & -1\\
1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & 0
\end{bmatrix}}_{A}
\begin{bmatrix} v_{ox}\\ v_{oy}\\ v_{oz}\\ \omega_x\\ \omega_y\\ \omega_z \end{bmatrix}
=
\begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix}
$$

(5) $\Rightarrow$ $v_{oz} = 0$

Null space has 1 dimension

$$
\mathrm{Null}(A) = \begin{bmatrix} -1\\ 1\\ 0\\ \hline 1\\ 1\\ -1 \end{bmatrix}
\begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} v_o = t_o \Rightarrow \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \omega = t \end{array}
$$

corner of cube at origin cannot move up or down.

Note, if origin had been chosen to lie on axis of rotation, then first 3 elements would be zero

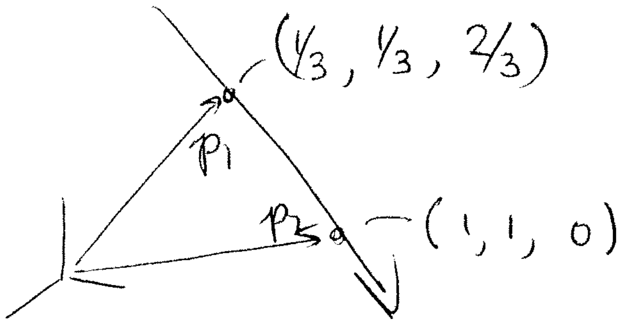
axis of rotation is along line ∥ to cube diagonal between grasp points.

Point closest to origin

$$
\frac{t \times t_o}{t \cdot t} \qquad \frac{1}{3}\begin{vmatrix} i & j & k\\ 1 & 1 & -1\\ -1 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 1\\ 1\\ 2 \end{bmatrix} = \begin{bmatrix} 1/3\\ 1/3\\ 2/3 \end{bmatrix}
$$

~~Points~~ Point seems wrong.

Test to see if it is on axis.



direction $(1, 1, -1)$

$$\left.\begin{array}{rl} & \quad\;\; i \qquad j \qquad k \\ p_2 - p_1 = & \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{2}{3}\right) \\ \text{cross with} & \left(1 \quad 1 \quad -1\right) \end{array}\right\} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Yes! It is on the line.

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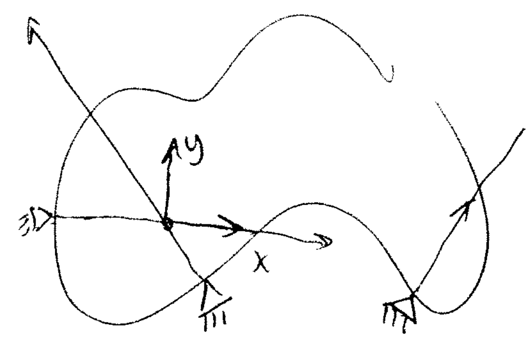
A final point about screw systems.

Choose ~~for~~ your coordinate frame to make your own life easy.

Results ~~are I of c,0~~

(c, c_0) & (t, t_0) are independent of coordinate frame location & orientation!



$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ .8 & .6 & 2 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

λ_i