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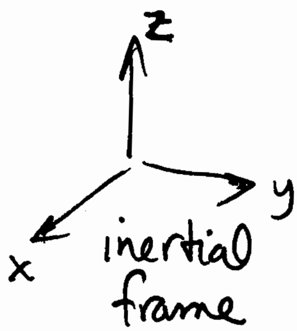
Dynamics of a Particle

Let $q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad v = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$



Newton's Law

$$\sum \text{forces} = F = \frac{d}{dt}(mv)$$

Assume 0

$$F = m\dot{v} + \cancel{v\dot{m}}$$

Equations of Motion in First-Order Form

$$\boxed{\begin{aligned} \dot{v} &= F/m \\ \dot{q} &= v \end{aligned}}$$

Time Stepping

We want to approx. the solution over the time interval $[a, b]$. (Assume constant time steps.)

$t_l = a + hl$ for $l = 0, 1, \ldots, M$ where $h = \dfrac{b-a}{M}$

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Euler's Method

Let $\vec{q} = f(t)$. Taylor expand to approx at $t_{\ell+1}$

$$q(t_{\ell+1}) = q(t_\ell + h) \cong \frac{df}{dt} h + q(t_\ell)$$

$$q^{\ell+1} \simeq q^\ell + \frac{\partial f}{dt} h$$

Explicit if $q^{\ell+1} = $ fcn of q^ℓ
Implicit otherwise

$\frac{df}{dt}$ normally evaluated at t_ℓ, but this is not required.

Apply to our problem

$$\dot{v} = F/m$$

$$\dot{q} = v$$

Use Euler approximations of $\dot{v}$ and $\dot{q}$

$\frac{v^{\ell+1} - v^\ell}{h} = F/m$		F^ℓ/m ?	$F^{\ell+1}/m$?
$\frac{q^{\ell+1} - q^\ell}{h} = v$		v^ℓ ?	$v^{\ell+1}$?

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Does it matter where we evaluate F and ν ?

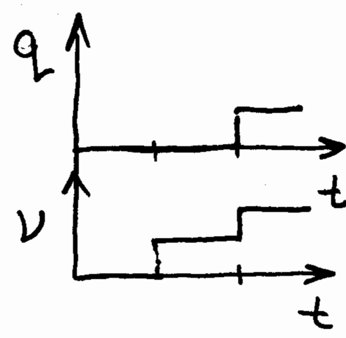
$$\left.\begin{aligned} \nu^{l+1} &= \nu^{l} + h\,F^{l}/m \\ q^{l+1} &= q^{l} + h\,\nu^{l} \end{aligned}\right\}$$

← Explicit Method

Everything on RHS is known

l	q^l	ν^l	F (constant)	m (constant)	h (constant)
0	0	0	1	1	1
1	0	1	⋮	⋮	⋮
2	1	2	⋮	⋮	⋮

delayed response

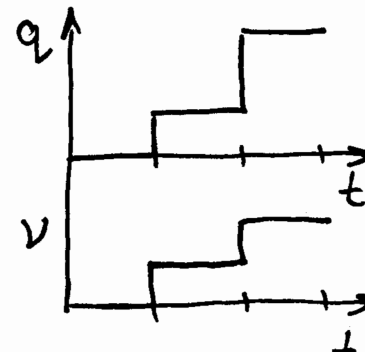


$$\nu^{l+1} = \nu^{l} + hF/m$$

$$q^{l+1} = q^{l} + h\nu^{l+1}$$

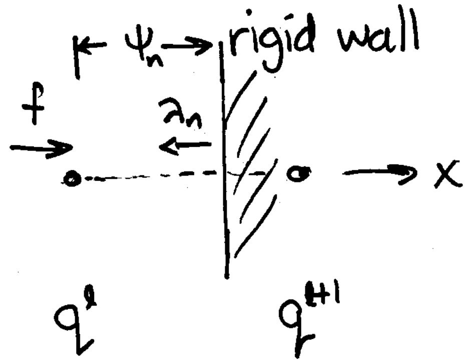
l	q^l	ν^l	Constant F, m, h = 1		
0	0	0	1	1	1
1	1	1			
2	3	2			

extra fast response



Consider Dynamics When Collision is Imminent

Assume 1-dimensional motion.



$$\dot{\nu} = F/m$$

$$\dot{q} = \nu$$

Now F = external force + wall force

$$\dot{\nu} = \frac{1}{m}(f - \lambda_n)$$

$$\dot{q} = \nu$$

Let ψ_n = dist to wall.

$$\left.\begin{array}{l} \text{Assume } \lambda_n = 0 \text{ if } \psi_n > 0 \\ \lambda_n \geq 0 \text{ if } \psi_n = 0 \\ \lambda_n > 0 \text{ only if } \psi_n = 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{c} \lambda_n \geq 0 \\ \psi_n \geq 0 \\ \lambda_n \psi_n = 0 \end{array}}$$

$\cancel{0 \leq \lambda_n \perp}$

So now the dynamics are:

$$\boxed{\begin{aligned} \dot{v} &= \tfrac{1}{m}(f - \lambda_n) \\ \dot{q} &= v \\ \text{s.t.} \quad & 0 \le \lambda_n \perp \Psi_n \ge 0 \end{aligned}}$$

$\Leftarrow$ complementarity constraint

Suppose velocity is high enough so that particle will bounce.

Then apply an impact model when particle reaches wall.

Newton's Hypothesis

$$\boxed{v(t_c^+) = -v(t_c^-)e}$$

inelastic elastic

$0 \le e \le 1$

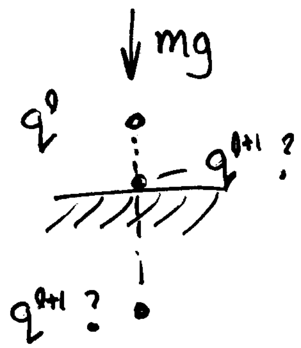
where e is known as the coeff. of rest.

Suppose collision is inelastic or particle will not bounce of by much.

$$\nu^{l+1} = \nu^l + (-g + \overset{?}{\lambda_n}/m)h$$

$$q^{l+1} = q^l + h\nu^?$$

$$0 \le \overset{?}{\lambda_n} \perp \overset{?}{\psi_n} \ge 0$$



We want to prevent penetration

Notice that ν^l is not biased by contact force, but ν^{l+1} is !

$\therefore$ To prevent q^{l+1} from penetrating, we should use ν^{l+1} in $q^{l+1} = q^l + h\nu^{l+1}$.

Now make complementarity constraint valid.

Goal: make system consistent at end of time step.

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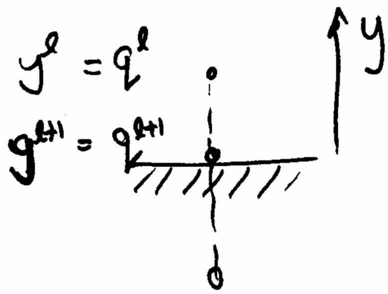
$$v^{l+1} = v^l + h\left(\frac{\lambda_n^{l+1}}{m} - g^{l+1}\right)$$

$$q^{l+1} = q^l + h v^{l+1}$$

$$0 \le \lambda_n^{l+1} \perp q^{l+1} \ge 0$$



Substituting first two eqs into third

$$0 \le \lambda_n^{l+1} \perp \underbrace{\lambda_n^{l+1}\left(\frac{h^2}{m}\right) + q^l + h v^l - g h^2}_{f(\lambda_n^{l+1})} \ge 0$$

Find Solution, $\Rightarrow$ $\boxed{\begin{matrix} \lambda_n^{l+1} = 0 \\ f(\lambda_n^{l+1}) \ge 0 \end{matrix}}$ OR $\boxed{\begin{matrix} f(\lambda_n^{l+1}) = 0 \\ \lambda_n^{l+1} \ge 0 \end{matrix}}$

Let $m = h = g = 1$, $v^l = -4$ $q^l = 1$

$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} + \underset{q^l}{1} - \underset{v^l}{4} - \underset{g}{1} \ge 0$$

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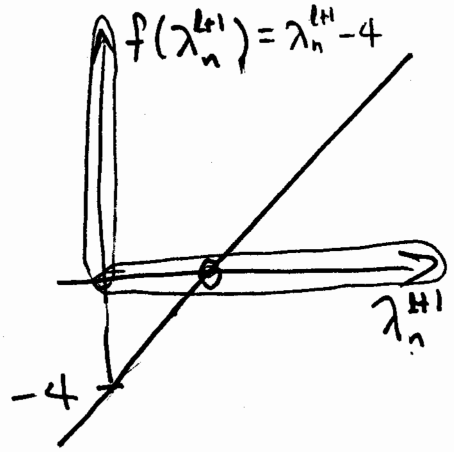
$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} - 4 \ge 0$$



Unique Solution

$$\boxed{\lambda_n^{l+1} = 4}$$

<u>Interpretation of solution</u>

Substitute Back in:

$$\nu^{l+1} = -4 + 1(4-1) = -1$$

$$q^{l+1} = 1 + 1(-1) = 0$$

Enough impulse was applied to prevent penetration at end of current time step, BUT <u>NOT ENOUGH TO REMOVE ALL APPROACH VELOCITY</u>!

Next time step

$$0 \le \lambda_n^{l+2} \perp \lambda_n^{l+2} + \overset{q^l=0}{\overset{\nu^l}{(-1)}} - \overset{g}{1} \ge 0$$

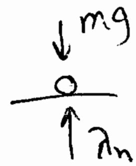
dist covered in $h=1$ at ν^{l+1}

dist that would be covered by grav. accel in $h=1$.

$\therefore \quad \lambda_n^{l+2} = 2$

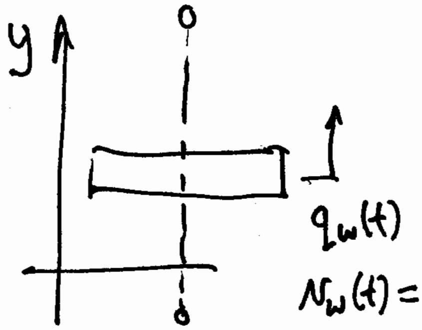
We see it takes two time steps to fully resolve a collision.

l	q^l	v^l	λ_n^l
l	1	-4	0
$l+1$	0	-1	4
$l+2$	0	0	2
$l+3$	0	0	1 — $= mg$
$\vdots$	$\vdots$	$\vdots$	$\vdots$



Suppose wall is moving as $q_w(t)$

$$\psi_n(q,t) = y - q_w(t) \geq 0$$



$\nu_w(t) = \frac{dq_w}{dt}$

Discretize

evaluate where?

$$\frac{\psi_n^{l+1} - \psi_n^l}{h} = \frac{\partial \psi_n}{\partial q} \underbrace{\frac{q^{l+1} - q^l}{h}}_{\nu^{l+1}} + \frac{\partial \psi_n}{\partial t}$$

$$\psi_n^{l+1} \simeq \underline{\psi_n^l} + \left((1)\nu^{l+1} - \nu_w(t)\right)h$$

when negative this term acts to stabilize the constraint

if given fcn of time, then could use ν_w^{l+1}.

ReWrite LCP

$$\nu^{l+1} = \nu^l + h\left(\frac{\lambda_n^{l+1}}{m} - g\right)$$

$$q^{l+1} = q^l + h\nu^{l+1}$$

$$0 \leq \lambda_n^{l+1} \perp \psi_n^{l+1} \geq 0$$

Substitute

$$0 \le \lambda_n^{l+1} \perp \underbrace{q^l - q_\omega^l}_{\psi_n^l} + h(v^{l+1} - v_\omega) \ge 0$$

$$0 \le \lambda_n^{l+1} \perp q^l - q_\omega^l + h\left[v^l + h\left(\frac{\lambda_n^{l+1}}{m} - g\right) - v_\omega\right] \ge 0$$

Let $h = m = g = 1$ $\qquad q^l = 1 \quad v^l = -4$

$\qquad q_\omega^l = 0 \quad v_\omega^l = 0.5$

$$0 \le \lambda_n^{l+1} \perp 1 - 0 + 1\left[-4 + 1\left(\lambda_{n/1} - 1\right) - \tfrac{1}{2}\right] \ge 0$$

$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} - 4.5 \ge 0$$

$$\underline{\underline{\lambda_n^{l+1} = 4.5}}$$

$v^{l+1} = -4 + \frac{1}{1} 4.5 - 1 = \underline{\underline{-0.5 = v^{l+1}}}$ $\Leftarrow$ still has rel. vel. into surf

$q^{l+1} = 1 - 1 \cdot 0.5 = \underline{\underline{+0.5 = q^{l+1}}}$ $\Leftarrow$ on surface

since $q_\omega^{l+1} = q_\omega^l + h\, v_\omega = 0.5$

Next time step

$$0 \le \lambda_n^{l+2} \perp q^{l+1} - q_w^{l+1} + h\left(v^{l+1} + \frac{h}{m}\lambda_n^{l+2} - hg - v_w\right) \ge 0$$

$$0 \le \lambda_n^{l+2} \perp \cancel{0.5} - \cancel{0.5} + 1\left(-0.5 + \lambda_n^{l+2} - 1 - 0.5\right) \ge 0$$

$$0 \le \lambda_n^{l+2} \perp \lambda_n^{l+2} - 2.0 \ge 0$$

$$\underline{\underline{\lambda_n^{l+2} = 2.0}}$$

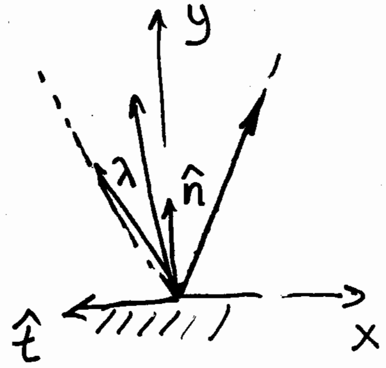
$$v^{l+2} = -0.5 + 1(2.0 - 1) = \underline{\underline{0.5 = v^{l+2}}}$$

$$q^{l+2} = 0.5 + 1 \cdot 0.5 = \underline{\underline{1.0 = q^{l+2}}}$$

Future time steps will have $\lambda_n^{l+j} = 1$ for $j > 2$

l	ψ_n^l	v^l	λ_n^l
l	1	-4	0
$l+1$	0	-0.5	4.5
$l+2$	0	0.5	2
$l+3$	0	0.5	1
	⋮	⋮	⋮

How do we add Add Friction?



Coulomb's Law

Let velocity of particle be $\overbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}}^{v} = \begin{bmatrix}-v_t\\ v_n\end{bmatrix}$

with position $\begin{bmatrix}0\\ 0\end{bmatrix}$

Let λ_n be the normal component of contact force, $\lambda_n \geq 0$

λ_t be the tangential " " " " in $\hat{t}$ direc

Coulomb's Law is given by:

$\lambda_t = -\mu\lambda_n$	if $v_t > 0$	$\dot{x} < 0$
$-\mu\lambda_n \leq \lambda_t \leq \mu\lambda_n$	if $v_t = 0$	$\dot{x} = 0$
$\lambda_t = \mu\lambda_n$	if $v_t < 0$	$\dot{x} > 0$

Let's divide friction force into positive and negative parts

$$\lambda_t = \lambda_{f_1} - \lambda_{f_2}$$

$$\lambda_{f_1} \xleftarrow{\hat{t}} \cdot \xrightarrow{-\hat{t}} \lambda_{f_2}$$

s.t. $\lambda_{f_1}, \lambda_{f_2} \geq 0$

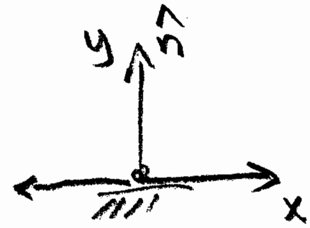
We need something to differentiate between sliding and rolling.

Introduce nonnegative slack variable s that represents sliding speed.

$$s \geq 0$$

$$v = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \qquad v_{f_1} = v_t = -\dot{x}$$

$$v_{f_2} = -v_t = \dot{x}$$



Note that this is a wrench matrix

$$\overset{W_f^T}{\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}} \overset{v}{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}} + \begin{bmatrix} s \\ s \end{bmatrix} \geq 0$$

$$\left.\begin{array}{ll} -\dot{x} + s \geq 0 & s \geq \dot{x} \\ \dot{x} + s \geq 0 & s \geq -\dot{x} \end{array}\right\} \quad s \geq |\dot{x}|$$

if $|\dot{x}| > 0$, then $s > 0$

Depending on which constraint is tight, we can eliminate one friction force.

$$0 \leq \begin{bmatrix} \lambda_{f_1} \\ \lambda_{f_2} \end{bmatrix} \perp \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} s \\ s \end{bmatrix} \geq 0$$

If both are loose we eliminate both. But can't do that.

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Sliding Cases

If $\dot{x} > 0$, then some friction must be active

$$0 \le \overleftarrow{\lambda_{f_1}} \perp -\dot{x} + s \ge 0$$

$$0 \le \underrightarrow{\lambda_{f_2}} \perp \dot{x} + s \ge 0$$

$\dot{x} + s > 0 \;\therefore\; \lambda_{f_2} = 0$

(sliding right $\Rightarrow$ friction force component toward right is zero

~~≡~~ Coulombs law says

$$\lambda_{f_1} = \mu \lambda_n \implies \cancel{\lambda_{f_1}} -\dot{x} + s = 0$$

$$\implies s = \dot{x}$$

If $\dot{x} < 0$,

$$0 \le \lambda_{f_1} \perp -\dot{x} + s \ge 0 \implies s - \dot{x} > 0 \;\therefore\; \lambda_{f_1} =$$

$$0 \le \lambda_{f_2} \perp \dot{x} + s \ge 0$$

$$\therefore \lambda_{f_2} = \mu \lambda_n \implies s = -\dot{x}$$

But Nothing Here Ensures $\lambda_{f_i} = \mu \lambda_n$!

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(16)

Since One of $\lambda_{f_1}, \lambda_{f_2}$ is zero, try:

$$0 \le s \perp \mu\lambda_n - \lambda_{f_1} - \lambda_{f_2} \ge 0$$

$$0 \le \begin{bmatrix} \lambda_{f_1} \\ \lambda_{f_2} \end{bmatrix} \perp \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} s \\ s \end{bmatrix} \ge 0$$

Case: Rolling / Sticking $(\dot{x} = 0)$

$$0 \le \lambda_{f_1} \perp -\cancel{\dot{x}} + s \ge 0 \quad (1)$$

redundant

$$0 \le \lambda_{f_2} \perp +\cancel{\dot{x}} + s \ge 0 \quad (2)$$

$$0 \le s \perp \mu\lambda_n - \lambda_{f_1} - \lambda_{f_2} \ge 0 \quad (3)$$

If $s > 0$, then $\lambda_{f_1} = \lambda_{f_2} = 0$ (from (1), (2))

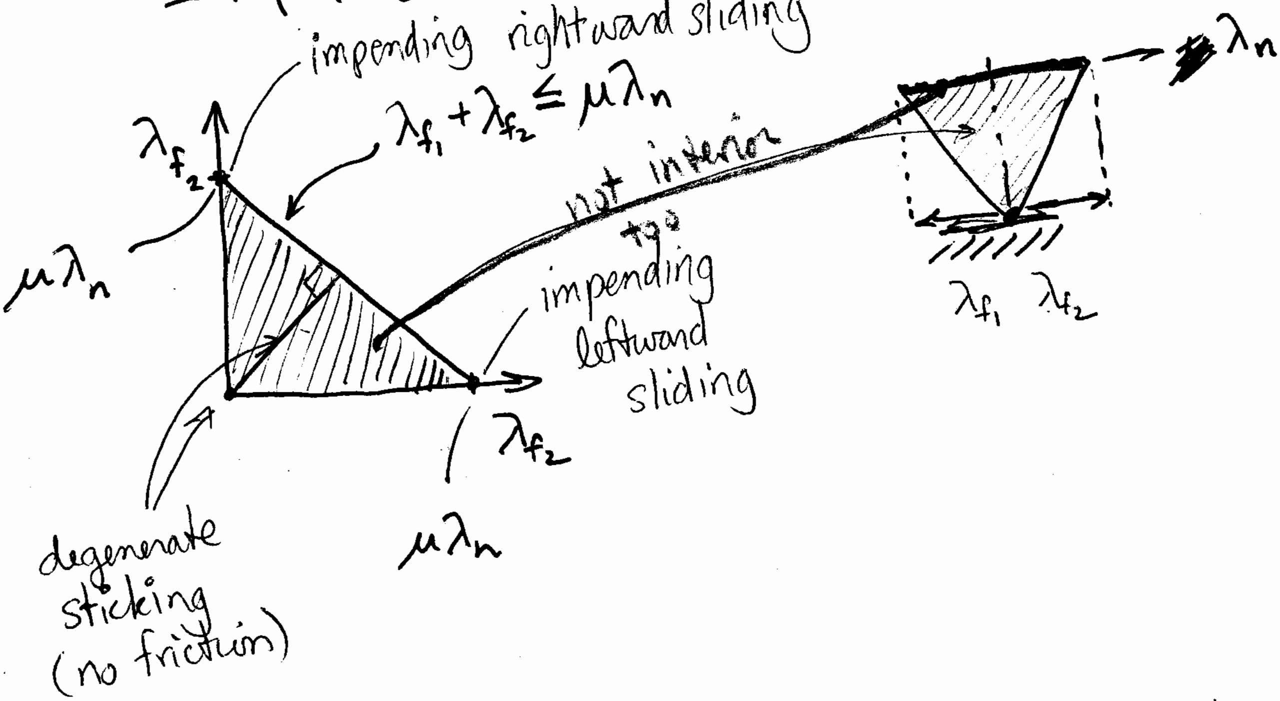
from (3) $\mu\lambda_n \equiv 0$

Not consistent! ~~but s not interpretted as sliding velocity~~

If $s = 0$, then from (1) & (2), $\lambda_{f_1}, \lambda_{f_2} \ge 0$

from (3) $\mu\lambda_n - \lambda_{f_1} - \lambda_{f_2} \ge 0$

Interpret second case



Summary: If $\dot{x} = 0$, then $s = 0$ is only consistent case, and friction force must be within cone.

Case Sliding $\dot{x} \neq 0$,

$$\boxed{\begin{aligned} &0 \le \lambda_{f_1} \perp -\dot{x} + s \ge 0 && (1) \\ &0 \le \lambda_{f_2} \perp \dot{x} + s \ge 0 && (2) \\ &0 \le s \perp \mu\lambda_n - \lambda_{f_1} - \lambda_{f_2} \ge 0 && (3) \end{aligned}}$$

Since $\begin{matrix} s = |\dot{x}| \\ s \ge \dot{x} \\ s \ge -\dot{x} \end{matrix}$, $\boxed{s > 0}$

Skip

Assume $\dot{x} > 0$

$(3) \Rightarrow \quad \mu\lambda_n = \lambda_{f_1} + \lambda_{f_2}$ which is > 0

$\therefore$ at least one of λ_{f_i} must be > 0

Of $-\dot{x}+s$ and $\dot{x}+s$, only the first can $= 0$.

$\therefore \quad \dot{x} = s$ and $\underline{\underline{\lambda_{f_1} \geq 0}}$ $\Leftarrow$ eq. (1)

Eq(2) $\Rightarrow$ since $\dot{x}+s > 0$, $\underline{\underline{\lambda_{f_2} = 0}}$

$\therefore \quad \boxed{\mu\lambda_n = \lambda_{f_1}}$

Consistent w/ Friction Law!

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Equality Constraints

$\Theta_i(q,t) = 0 \qquad i = 1, 2, \ldots$

Why joints & quaternion length constr.

Taylor Expand... (drop i subscript)

$\Theta(q+\Delta q, t+h) = \Theta^{l+1}$

$\Theta(q,t) = \Theta^{l}$

$\Theta^{l+1} = \Theta^{l} + \frac{\partial \Theta}{\partial q}\Delta q + \frac{\partial \Theta}{\partial t} h$

should be zero but is not quite

$\Theta^{l+1} = \Theta^{l} + \frac{\partial \Theta}{\partial q}(q^{l+1} - q^{l}) + \frac{\partial \Theta}{\partial t} h$

But Θ^{l+1} should equal $\Theta^{l} = 0$

But Θ^{l} will not be exactly 0, so keep it

$$0 = \frac{\partial \Theta}{\partial q} v^{l+1} + \frac{\partial \Theta}{\partial t} + \Theta^{l}/h$$

kinematic

constraint stabilization

It was natural to put normal contact conditions into the form of a C.P. We now want to extend and include friction if possible. 4/26/03

Complementarity Problems

(1.1)

Let w and z be vectors of length n. Further, let $w(z)$ be a given function. The complementarity problem is: find z satisfying:

$$z \geq 0, \quad w(z) \geq 0, \quad w^T(z)\, z = 0$$

$$\boxed{0 \leq z \perp w(z) \geq 0}$$

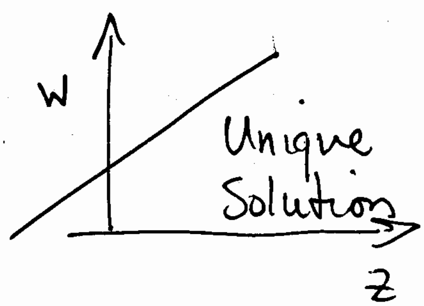
Linear Complementarity Problem (LCP)

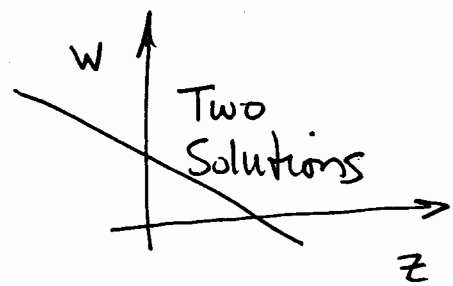
If $w(z)$ is defined as

$$w(z) = Fz + f$$

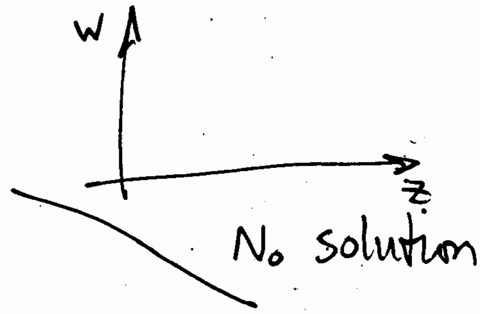
where $F \in \mathbb{R}^{n \times n}$ and $f \in \mathbb{R}^n$ are given constants

LCP of Size 1





The LCP has a unique solution if F is a P-matrix and Lemke's alg. is guaranteed to find a soln. in finite time.



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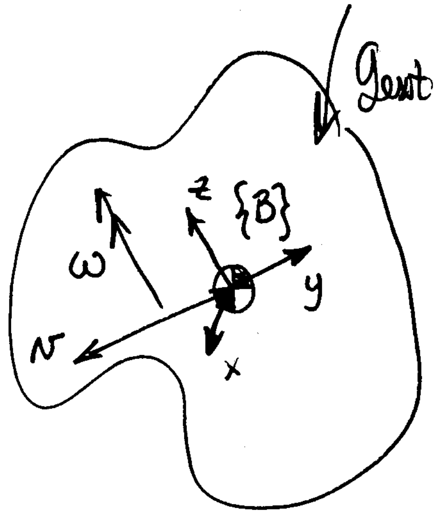
Generalize to Spatial Case with varying contact types



ton: $\sum_i f_i = m \mathbb{I}_{(3\times3)} \mathbf{a}$

er: $\sum n_i = \sum_i r_i \times f_i$

$$= \mathbf{J}\alpha + \omega \times \mathbf{J}\omega$$

where $a = \dot{\nu}$, $\nu = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \hdashline \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$, $q = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}$ $\quad e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$

$$\omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$M\dot{\nu} = \begin{bmatrix} \sum f_i \\ \hdashline \sum r_i \times f_i \end{bmatrix} + \begin{bmatrix} 0_{3\times1} \\ \hdashline -\omega \times J\omega \end{bmatrix}$$

where $M = \mathrm{diag}(m\mathbb{I}_{3\times3}, \mathbf{I})$

M is P.D. & symmetric

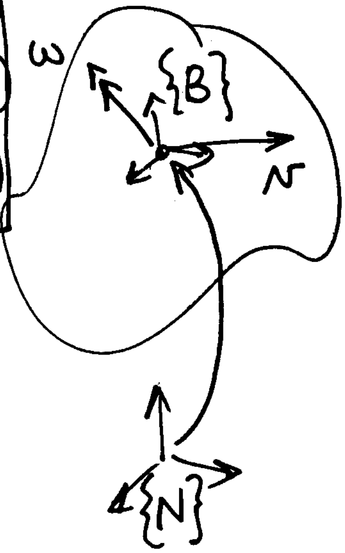
Rotational Kinematics

$$\dot{q} = G(q)\,\nu$$

where $G = \begin{bmatrix} I_{(3\times3)} & 0 \\ 0 & B(q)_{(4\times3)} \end{bmatrix}_{(7\times6)}$. $\quad G^T G = I_{(6\times6)}$

$$B(q) = \begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_3 & e_0 & -e_1 \\ -e_2 & e_1 & e_0 \end{bmatrix} \frac{1}{2} \qquad \begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = B(q) \begin{bmatrix} {}^B\omega_x \\ {}^B\omega_y \\ {}^B\omega_z \end{bmatrix}$$

$${}^N_B R(q) = \begin{bmatrix} 1-2(e_2^2+e_3^2) & 2(e_1e_2-e_0e_3) & 2(e_1e_3+e_2e_0) \\ 2(e_1e_2+e_3e_0) & 1-2(e_1^2+e_3^2) & 2(e_2e_3-e_1e_0) \\ 2(e_1e_3-e_0e_2) & 2(e_2e_3+e_1e_0) & 1-2(e_2^2+e_1^2) \end{bmatrix}$$



$${}^B\omega = {}^B_N R\, {}^N\omega$$

$$\dot{e} = B\, {}^B\omega$$

RBD w/ no Contact or Ext. Wrenches

3/22/04

20.1

$$\boxed{\begin{gathered} M\dot{\nu} = \begin{bmatrix} 0_{(3\times1)} \\ -\omega\times J\omega \end{bmatrix}_{(6\times1)} + \begin{bmatrix} F \\ N \end{bmatrix} \\[1em] \dot{q} = G(q)\,\nu \\[1em] e_0^2+e_1^2+e_2^2+e_3^2-1=0 \end{gathered}}$$

Break up equations to use ODE45 (or similar).

$$\cancel{m\,{}^N\dot{\nu} = {}^N F}$$

$$\cancel{{}^BJ\,{}^B\dot{\omega} = {}^BN - {}^B\omega\times{}^BJ\,{}^B\omega}$$

$$\cancel{{}^N\dot{x} = {}^N\nu}$$

$$\cancel{\dot{e} = B\,{}^B\omega}$$

$$\dot{\nu} = F/m$$

$${}^BJ\,\dot{\omega} = N - \omega\times{}^BJ\omega \quad\Rightarrow$$

$$\dot{x} = \nu$$

$$\dot{e} = B\omega$$

$$\boxed{\begin{gathered} {}^N\dot{\nu} = {}^NF/m \\ \cancel{{}^B\dot{\omega} = {}^BJ_B^{-1}\left({}^BN - {}^B\omega\times{}^BJ\,{}^B\omega\right)} \\ {}^N\dot{\omega} = {}^N_BR\,{}^BJ^{-1}\left({}^BN - {}^B\omega\times{}^BJ\,{}^B\omega\right) \\ {}^N\dot{x} = {}^N\nu \\ \dot{e} = B\,{}^B\omega \end{gathered}}$$