It was natural to put normal contact conditions into the form of a C.P. We now want to extend and include friction if possible.

Complementarity Problems

4/26/03 ①.1)

Let w and z be vectors of length n. Further, let w(z) be a given function. The complementarity problem is: find z satisfying:

$$z \geq 0, \quad w(z) \geq 0, \quad w^T(z)\, z = 0$$

$$\boxed{0 \leq z \perp w(z) \geq 0}$$

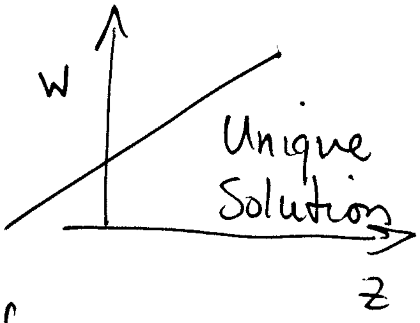
Linear Complementarity Problem (LCP)

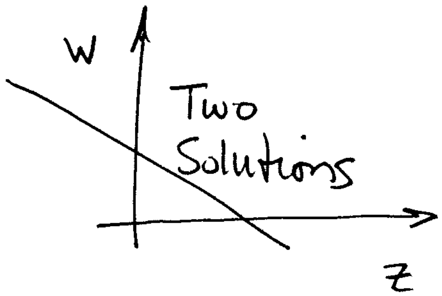
If w(z) is defined as

$$w(z) = Fz + f$$

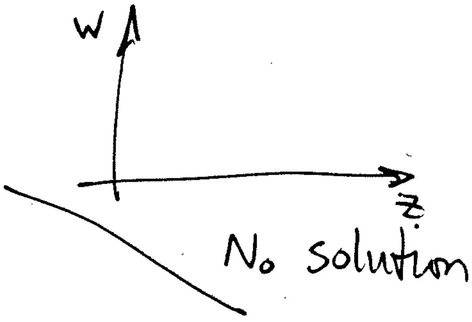
where $F \in \mathbb{R}^{n \times n}$ and $f \in \mathbb{R}^n$ are given constants

LCP of Size 1



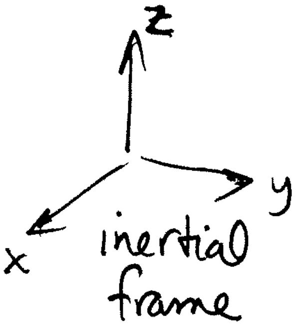


The LCP has a unique solution if F is a P-matrix and Lemke's alg. is guaranteed to find a soln. in finite time.



Dynamics of a Particle

Let $q = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\nu = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$



Newton's Law

$$\sum \text{forces} = F = \frac{d}{dt}(m\nu)$$

$$F = m\dot{\nu} + \nu \cancelto{\text{Assume } 0}{\dot{m}}$$

Equations of Motion in First-Order Form

$$\boxed{\begin{aligned} \dot{\nu} &= F/m \\ \dot{q} &= \nu \end{aligned}}$$

Time Stepping

We want to approx. the solution over the time interval $[a, b]$. (Assume constant time steps.)

$t_l = a + hl$ for $l = 0, 1, \dots, M$ where $h = \frac{b-a}{M}$

3/21/04

Euler's Method

Let $\dot{q} = f(t)$. Taylor expand to approx at $t_{\ell+1}$

$$\frac{1}{h}\left[q(t_{\ell+1}) - q(t_\ell)\right] \cong f(t_\ell)$$

$$q^{\ell+1} \simeq q^{\ell} + f^{\ell} h$$

Explicit if $q^{\ell+1}$ = fcn of q^{ℓ}
Implicit otherwise

f is normally evaluated at t_ℓ, but this is not required.

Apply to our problem

$$\dot{v} = F/m$$

$$\dot{q} = v$$

Use Euler approximations of $\dot{v}$ and $\dot{q}$

$\frac{v^{\ell+1} - v^{\ell}}{h} = F/m$	F^{ℓ}/m ?	$F^{\ell+1}/m$?
$\frac{q^{\ell+1} - q^{\ell}}{h} = v$	v^{ℓ} ?	$v^{\ell+1}$?

3/21/04

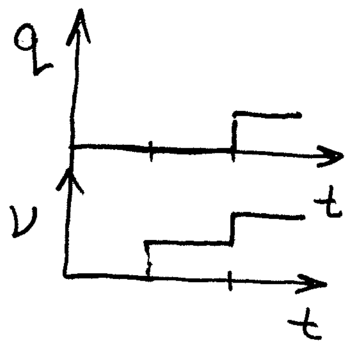
Does it matter where we evaluate F and ν ?

$$\left.\begin{aligned} \nu^{l+1} &= \nu^{l} + h\,F^{l}/m \\ q^{l+1} &= q^{l} + h\,\nu^{l} \end{aligned}\right\} \leftarrow \text{Explicit Method}$$

Everything on RHS is know

l	q^l	ν^l	$\overbrace{F}^{\text{constant}}$	m	h
0	0	0	1	1	1
1	⓪	1	⋮	⋮	⋮
2	1	2			
3	3	3			

delayed response

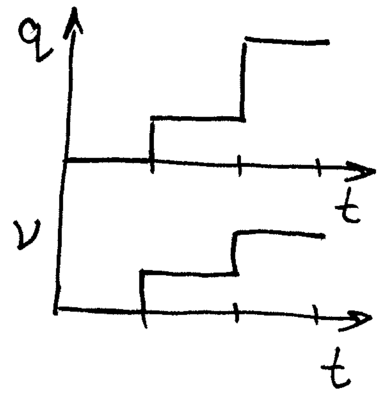


$$\nu^{l+1} = \nu^{l} + hF/m$$

$$q^{l+1} = q^{l} + h\nu^{l+1}$$

l	q^l	ν^l	Constant F, m, h = 1
0	0	0	1 1 1
1	1	1	
2	③	2	
3	6	3	

extra fast response



3/21/04

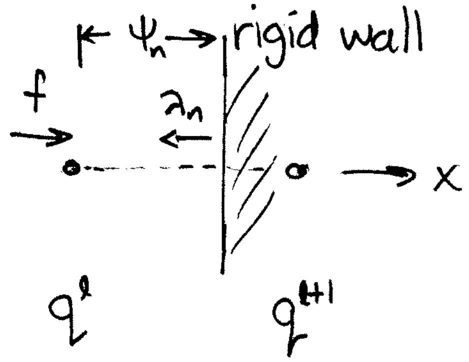
④

Consider Dynamics When Collision is Imminent

Assume 1-dimensional motion.

$$\dot{v} = F/m$$

$$\dot{q} = v$$



Now F = external force + wall force

$$\dot{v} = \frac{1}{m}(f - \lambda_n)$$

$$\dot{q} = v$$

Let Ψ_n = dist to wall.

$$\text{Assume } \left.\begin{array}{l} \lambda_n = 0 \quad \text{if} \quad \Psi_n > 0 \\ \lambda_n \geq 0 \quad \text{if} \quad \Psi_n = 0 \\ \lambda_n > 0 \quad \text{only if} \quad \Psi_n = 0 \end{array}\right\} \Rightarrow \overset{\cancel{0 \leq \lambda_n \perp}}{\boxed{\begin{array}{c} \lambda_n \geq 0 \\ \Psi_n \geq 0 \\ \lambda_n \Psi_n = 0 \end{array}}}$$

3/21/04

So now the dynamics are:

$$\boxed{\begin{aligned} &\dot{v} = \tfrac{1}{m}(f - \lambda_n) \\ &\dot{q} = v \\ \\ &\text{s.t.} \quad 0 \le \lambda_n \perp \psi_n \ge 0 \end{aligned}}$$

$\Leftarrow$ complementarity constraint

Suppose velocity is high enough so that particle will bounce.

Then apply an impact model when particle reaches wall.

Newton's Hypothesis

$$\boxed{v(t_c^+) = -v(t_c^-)\,e}$$

$$\overset{\text{inelastic}}{0} \le e \le \overset{\text{elastic}}{1}$$

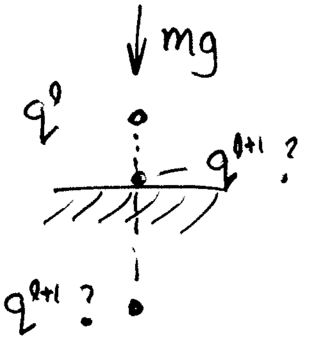
where e is known as the coeff. of rest.

Suppose collision is inelastic or particle will not bounce of by much.

$$\nu^{l+1} = \nu^l + (-g + \overset{?}{\lambda_n}/m)h$$

$$q^{l+1} = q^l + h\,\nu^{?}$$

$$0 \leq \overset{?}{\lambda_n} \perp \overset{?}{\psi_n} \geq 0$$



We want to prevent penetration

Notice that ν^l is not biased by contact force, but ν^{l+1} is !

∴ To prevent q^{l+1} from penetrating, we should use ν^{l+1} in $q^{l+1} = q^l + h\nu^{l+1}$.

Now make complementarity constraint valid.

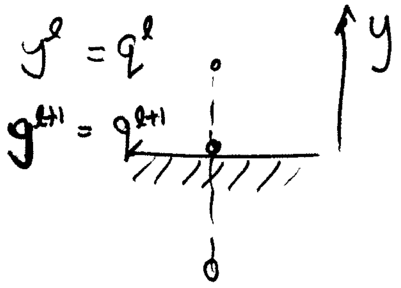
Goal: make system consistent at end of time step.

3/21/04

$$v^{l+1} = v^l + h\left(\frac{\lambda_n^{l+1}}{m} - g^{l+1}\right)$$

$$q^{l+1} = q^l + h v^{l+1}$$

$$0 \le \lambda_n^{l+1} \perp q^{l+1} \ge 0$$



Substituting first two eqs into third

$$0 \le \lambda_n^{l+1} \perp \underbrace{\lambda_n^{l+1}\left(\frac{h^2}{m}\right) + q^l + hv^l - gh^2}_{f(\lambda_n^{l+1})} \ge 0$$

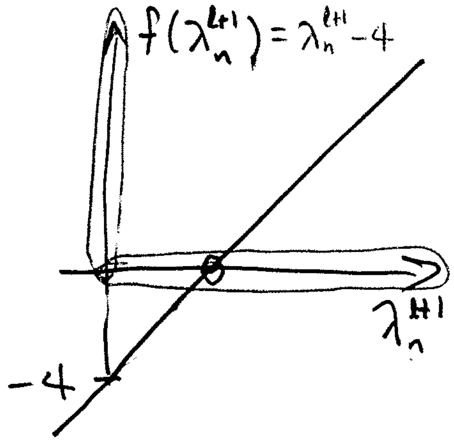
Find Solution, $\Rightarrow \boxed{\begin{array}{c}\lambda_n^{l+1}=0\\ f(\lambda_n^{l+1})\ge 0\end{array}} \underset{\text{OR}}{\Big|} \boxed{\begin{array}{c}f(\lambda_n^{l+1})=0\\ \lambda_n^{l+1}\ge 0\end{array}}$

Let $m = h = g = 1, \quad v^l = -4 \quad q^l = 1$

$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} + \underset{q^l}{1} - \underset{v^l}{4} - \underset{g}{1} \ge 0$$

3/21/04

$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} - 4 \ge 0$$



Unique Solution

$$\boxed{\lambda_n^{l+1} = 4}$$

Interpretation of solution

Substitute Back in :

$$v^{l+1} = -4 + 1(4-1) = -1$$

$$q^{l+1} = 1 + 1(-1) = 0$$

Enough impulse was applied to prevent penetration at end of current time step, BUT NOT ENOUGH TO REMOVE ALL APPROACH VELOCITY!

Next time step

$$0 \le \lambda_n^{l+2} \perp \lambda_n^{l+2} + \overset{q^l=0 \atop v^l}{(-1)} - \overset{g}{1} \ge 0$$

↑ dist covered in h=1 at v^{l+1}

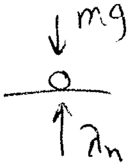
↖ dist that would be covered by grav. accel in h=1.

3/21/04

$$\therefore \quad \lambda_n^{l+2} = 2$$

We see it takes two time steps to fully resolve a collision.

l	q^l	v^l	λ_n^{l}
l	1	−4	0
$l+1$	0	−1	4
$l+2$	0	0	2
$l+3$	0	0	1 — $= mg$
⋮	⋮	⋮	⋮



Why is impulse to stop particle equal to 6 & not 4?

Impulse = Δmomentum

This is because there are also 2 units of gravity impulse over the 2 time steps required to resolve the impulse.

3/22/04

⑩

Suppose wall is moving as $q_w(t)$

$$q - q_w(t) \geq 0$$

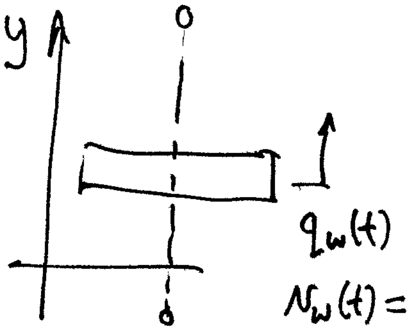
$$\psi_n(q,t) = y - q_w(t) \geq 0$$

Note $y = q$

Discretize

$$\frac{\psi_n^{l+1} - \psi_n^{l}}{h} = \frac{\partial \psi_n}{\partial q}^{\circ} \underbrace{\frac{q^{l+1} - q^{l}}{h}}_{\nu^{l+1}} + \frac{\partial \psi_n}{\partial t}^{\circ}$$

evaluate where?



$N_w(t) = \frac{dq_w}{dt}$

$$\psi_n^{l+1} \simeq \underline{\psi_n^{l}} + \left((1)\nu^{l+1} - N_w(t)\right)h$$

↑ when negative this term acts to stabilize the constraint

↑ if given fcn of time, then could use N_w^{l+1}.

ReWrite LCP

$$\nu^{l+1} = \nu^{l} + h\left(\frac{\lambda_n^{l+1}}{m} - g\right)$$

$$q^{l+1} = q^{l} + h\nu^{l+1}$$

$$0 \leq \lambda_n^{l+1} \perp \psi_n^{l+1} \geq 0$$

3/22/04

Substitute

$$0 \le \lambda_n^{l+1} \perp \underbrace{q^l - q_\omega^l}_{\psi_n^l} + h\left(v^{l+1} - v_\omega\right) \ge 0$$

$$0 \le \lambda_n^{l+1} \perp q^l - q_\omega^l + h\left[v^l + h\left(\frac{\lambda_n^{l+1}}{m} - g\right) - v_\omega\right] \ge 0$$

Let $h = m = g = 1$ $\quad q^l = 1 \quad v^l = -4$

$q_\omega^l = 0 \quad v_\omega^l = 0.5$

$$0 \le \lambda_n^{l+1} \perp 1 - 0 + 1\left[-4 + 1\left(\lambda_n^{l+1}/_1 - 1\right) - \tfrac{1}{2}\right] \ge 0$$

$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} - 4.5 \ge 0$$

$$\underline{\underline{\lambda_n^{l+1} = 4.5}}$$

$v^{l+1} = -4 + \frac{1}{1}4.5 - 1 = -\underline{\underline{0.5}} = v^{l+1}$ ⇐ still has rel. vel. into surface

$q^{l+1} = 1 - 1 \cdot 0.5 = \underline{\underline{+0.5}} = q^{l+1}$ ⇐ on surface

since $q_\omega^{l+1} = q_\omega^l + h\,v_\omega = 0.5$

3/22/04

Next time step

$$0 \le \lambda_n^{l+2} \perp q^{l+1} - q_w^{l+1} + h\left(\nu^{l+1} + \frac{h}{m}\lambda_n^{l+2} - hg - \nu_w\right) \ge 0$$

$$0 \le \lambda_n^{l+2} \perp \cancel{0.5} - \cancel{0.5} + 1\left(-0.5 + \lambda_n^{l+2} - 1 - 0.5\right) \ge 0$$

$$0 \le \lambda_n^{l+2} \perp \lambda_n^{l+2} - 2.0 \ge 0$$

$$\underline{\underline{\lambda_n^{l+2} = 2.0}}$$

$$\nu^{l+2} = -0.5 + 1(2.0 - 1) = \underline{\underline{0.5 = \nu^{l+2}}}$$

$$q^{l+2} = 0.5 + 1 \cdot 0.5 = \underline{\underline{1.0 = q^{l+2}}}$$

Future time steps will have $\lambda_n^{l+j} = 1$ for $j > 2$

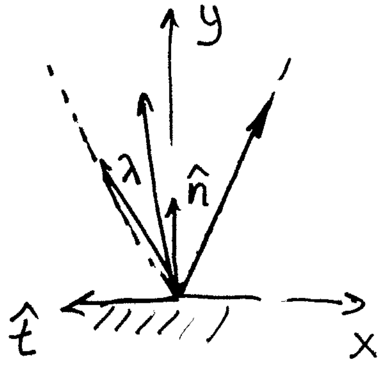
l	Ψ_n^l	ν^l	λ_n^l
l	1	−4	0
$l+1$	0	−0.5	4.5
$l+2$	0	0.5	2
$l+3$	0	0.5	1
	⋮	⋮	⋮

How do we add Add Friction ?

3/22/04

⑬

Coulomb's Law



Let velocity of particle be $\overbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}^{v} = \begin{bmatrix} -v_t \\ v_n \end{bmatrix}$

with position $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Let λ_n be the normal component of contact force, $\lambda_n \geq 0$
λ_t be the tangential " " " " in $\hat{t}$ directi

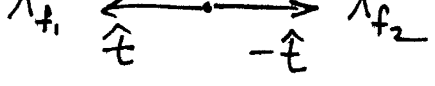
Coulomb's Law is given by:

$$\begin{array}{ccc|c} \lambda_t = -\mu\lambda_n & \text{if} & \cancel{\dot{x}}\ v_t > 0 & \dot{x} < 0 \\ -\mu\lambda_n \leq \lambda_t \leq \mu\lambda_n & \text{if} & v_t = 0 & \dot{x} = 0 \\ \lambda_t = \mu\lambda_n & \text{if} & v_t < 0 & \dot{x} > 0 \end{array}$$

Let's divide friction force into positive and negative parts

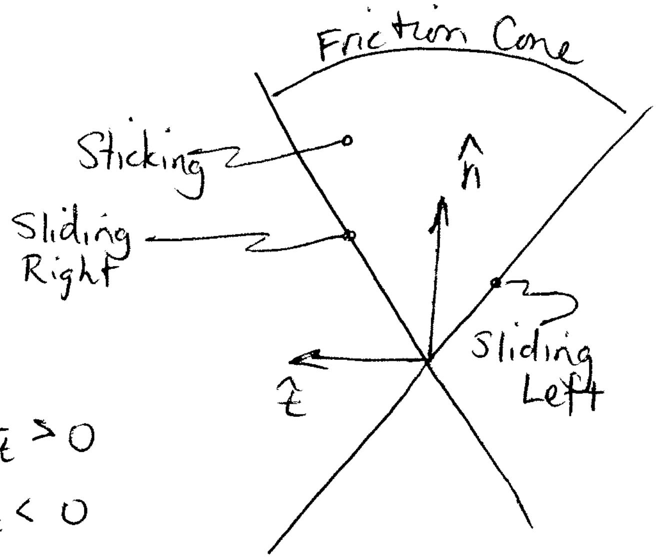
$$\lambda_t = \lambda_{f_1} - \lambda_{f_2}$$

s.t. $\lambda_{f_1}, \lambda_{f_2} \geq 0$



11/12/06

Modeling Friction in Planar Systems



There a 3 physically distinct, important cases to model:

Slide Left $\Rightarrow$ $\lambda_t = -\mu\lambda_n$, $v_t > 0$

Slide Right $\Rightarrow$ $\lambda_t = \mu\lambda_n$, $v_t < 0$

Stick $\Rightarrow$ $-\mu\lambda_n \leq \lambda_t \leq \mu\lambda_n$, $v_t = 0$

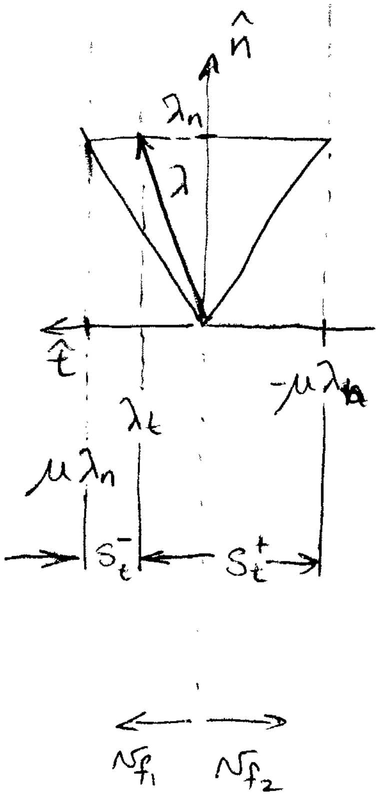
Introduce 2 nonnegative slack variables, s_t^+ and s_t^-

$$s_t^+ = \mu\lambda_n + \lambda_t$$

$$s_t^- = \mu\lambda_n - \lambda_t$$

$s_t^+ = 0 \Rightarrow$ Sliding Left

$s_t^- = 0 \Rightarrow$ Sliding Right



Rewrite v_t as the sum of its nonnegative & nonpositive parts

$$v_t = v_{f_1} - v_{f_2}$$

$$v_{f_1}, v_{f_2} \geq 0$$

11/12/06

Ideally $v_{f_1} \perp v_{f_2}$

or equivalently $|v_t| = v_{f_1} + v_{f_2}$

Friction Complementarity Conditions

$$\boxed{\begin{aligned} 0 \le \mu\lambda_n + \lambda_t \ \perp\ v_{f_1} \ge 0 \\ 0 \le \mu\lambda_n - \lambda_t \ \perp\ v_{f_2} \ge 0 \end{aligned}}$$

4 Cases

$\boxed{\text{Sticking}}$

$\mu\lambda_n + \lambda_t,\ \mu\lambda_n - \lambda_t > 0 \Rightarrow v_{f_1} = v_{f_2} = v_t = 0$

$\boxed{\text{Sliding Right}}$

$\mu\lambda_n + \lambda_t,\ v_{f_2} > 0 \Rightarrow \lambda_t = \mu\lambda_n,\ v_{f_1} = 0$

$\Rightarrow v_t = -v_{f_2} < 0$

$\boxed{\text{Sliding Left}}$

$\mu\lambda_n - \lambda_t,\ v_{f_1} > 0 \Rightarrow \lambda_t = -\mu\lambda_n,\ v_{f_2} = 0$

$\Rightarrow v_t = +v_{f_1} > 0$

$\boxed{\text{Degenerate Sliding}}$

$v_{f_1},\ v_{f_2} > 0 \Rightarrow \lambda_t = -\lambda_n\mu = \mu\lambda_n$

$\Rightarrow \lambda_n = \lambda_t = 0$

An Alternative Formulation

11/13/06

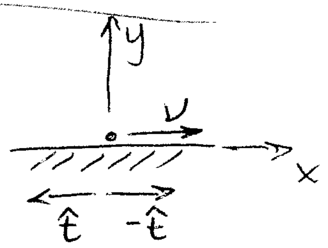
Not as efficient, but extends to 3D problems.

$$\boxed{\begin{array}{l} 0 \leq \lambda_f \perp W_f^T v + E s \geq 0 \\ 0 \leq s \perp \mu \lambda_n - E^T \lambda_f \geq 0 \end{array}}$$

$$W_f^T v = \begin{bmatrix} v_{f_1} \\ v_{f_2} \end{bmatrix} = \text{tangential velocity components}$$

$$E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example:



$$0 \leq \lambda_{f_1} \perp v_{f_1} + s \geq 0 \qquad (1)$$

$$0 \leq \lambda_{f_2} \perp v_{f_2} + s \geq 0 \qquad (2)$$

$$0 \leq s \perp \mu \lambda_n - \lambda_{f_1} - \lambda_{f_2} \geq 0 \qquad (3)$$

$$W_f = [\hat{t} \;\; -\hat{t}]$$

Note $v_{f_2} = -\hat{t}^T v > 0$

$v_{f_2} = \dot{x} > 0$

$v_{f_1} = -v_{f_2} = -\dot{x}$

Consider all 8 cases Systematically

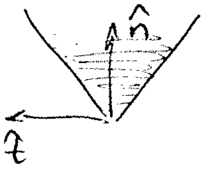
11/13/06.

Case	Signs Left	Sign Right
1	+ + +	0 0 0
2	+ + 0	0 0 +
3	+ 0 +	0 + 0
4	+ 0 0	0 + +
5	0 + +	+ 0 0
6	0 + 0	+ 0 +
7	0 0 +	+ + 0
8	0 0 0	+ + +

Case 1 : $\boxed{\text{Inconsistent}}$

(3) ⇒ $s > 0$, but (1) & (2) ⇒ $s = -v_{f_1} = -v_{f_2}$

Since $v_{f_1} = -v_{f_2}$, $s = v_{f_1} = 0$ q.e.d.



Case 2 : $\boxed{\text{Sticking}}$

$\lambda_{f_1}, \lambda_{f_2} \geq 0$, $v_{f_1} = v_{f_2} = 0$

Case 3 : $\boxed{\text{Right Sliding}}$ $s = v_{f_2}$, $\lambda_{f_1} = \mu \lambda_n$

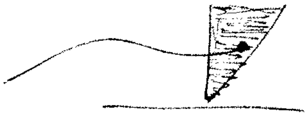
Case 4 : $\boxed{\text{Degenerate Sticking}}$ $\dot{x} = 0$

$\lambda_{f_2} = 0$ $\lambda_{f_1} \geq 0$

Case 5 : $\boxed{\text{Left Sliding}}$ $s = v_{f_1}$, $\lambda_{f_2} = \mu \lambda_n$

Case 6 : $\boxed{\text{Degenerate Sticking}}$

$\dot{x} = 0$, $\lambda_{f_1} = 0$ $\lambda_{f_2} \geq 0$



Case 7 : $\boxed{\text{Degenerate Sliding}}$

$\lambda_{f_1} = \lambda_{f_2} = \lambda_n = 0$, $s > 0$

Case 8 : $\boxed{\text{Degenerate Sticking}}$

$\lambda_{f_1} = \lambda_{f_2} = \lambda_n = 0$, $s = 0$, $v_{f_1} = v_{f_2} = 0$

11/9/06

Time Stepping Subproblem

$$\left.\begin{array}{l}
v^{l+1} = v^l + Fh/m \\[1em]
q^{l+1} = q^l + h v^{l+1} \\[2em]
0 \le \begin{bmatrix} \lambda_n^{l+1} \\ \lambda_f^{l+1} \\ s^{l+1} \end{bmatrix} \perp \begin{bmatrix} \Psi_n(q^{l+1}, t_{l+1}) \\ W_f^T v^{l+1} + E s^{l+1} \\ \mu \lambda_n^{l+1} - E^T \lambda_f^{l+1} \end{bmatrix} \ge 0
\end{array}\right\} \begin{array}{l}\text{Mixed} \\ \text{Nonlinear} \\ \text{Complement.} \\ \text{Problem}\end{array}$$

Where are the nonlinearities?

F(t) could be nonlinear. Could integrate if easy enough

If $\Psi_n(q,t)$ is nonlinear, eg. circular obstacle

If $\hat{t}$ changes over time step, $W_f^T = \begin{bmatrix} \hat{t} \\ -\hat{t} \end{bmatrix}$

LCP's are much easier to solve (use PATH solver),

So linearize

$$\Psi_n^{l+1} = \Psi_n^l + \frac{\partial \Psi_n^l}{\partial q} \Delta q + \frac{\partial \Psi_n^l}{\partial t} \Delta t + \overset{\text{ignore}}{\cancel{H.O.T.}}$$

$$\Delta q = q^{l+1} - q^l = h v^{l+1}$$

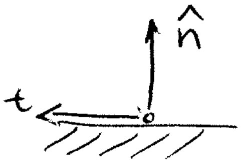
11/9/06

$$p_n^{l+1} = \frac{\psi^{l+1}}{h} \simeq \frac{\psi_n^l}{h} + W_n^T \nu^{l+1} + \frac{\partial \psi_n}{\partial t}$$

where $W_n^T = \hat{n}^T$

Write F in terms of external and contact forces

$$F = \underbrace{W_n \lambda_n}_{\text{normal force}} + \underbrace{W_f \lambda_f}_{\text{friction force}} + \underset{\text{gravity, wind resistance, etc.}}{\underset{\uparrow}{g_{ext}}}$$



Organize Eqs.: Let: $M = \begin{bmatrix} m & \\ & m \end{bmatrix}$, $U = \mu$

$h g_{ext} = p_{ext}$, $h\lambda = p$

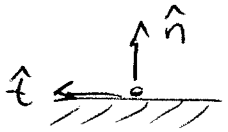
$$\left.\text{just definitions}\right\{ \begin{bmatrix} 0 \\ \hline p_n^{l+1} \\ p_f^{l+1} \\ \sigma^{l+1} \end{bmatrix} = \begin{bmatrix} M & -W_n & -W_f & 0 \\ \hline W_n^T & 0 & 0 & 0 \\ W_f^T & 0 & 0 & E \\ 0 & U & -E^T & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \nu^{l+1} \\ p_n^{l+1} \\ p_f^{l+1} \\ s^{l+1} \end{bmatrix}}_{\text{unknowns}} + \begin{bmatrix} -\ \nu^l - p_{ext}^l \\ \hline \frac{\psi_n^l}{h} + \frac{\partial \psi_n}{\partial t} \\ 0 \\ 0 \end{bmatrix}$$

$$q^{l+1} = q^l + h\nu^{l+1}$$

11/9/06

Define: $p_n^{l+1} = \frac{\Psi_n^l}{h} + W_n^T \nu^{l+1} + \frac{\partial \Psi_n}{\partial t}$

note that $W_n^T = \hat{n}^T$



ReWrite $F = \underbrace{W_n \lambda_n}_{\text{normal force}} + \underbrace{W_f \lambda_f}_{\text{friction force}} + \underbrace{g_{ext}}_{\text{external forces}}$

Let $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, $U = \mu$, $p_{ext} = h g_{ext}$, $p_\alpha = h \lambda_\alpha$

Time Stepping SubProblem – Mixed LCP

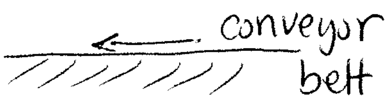
$$\left[\begin{array}{c} 0 \\ \hdashline p_n^{l+1} \\ p_f^{l+1} \\ \sigma^{l+1} \end{array}\right] = \left[\begin{array}{cccc} M & -W_n & -W_f & 0 \\ \hdashline W_n^T & 0 & 0 & 0 \\ W_f^T & 0 & 0 & E \\ 0 & U & -E^T & 0 \end{array}\right] \begin{bmatrix} \nu^{l+1} \\ \lambda_n^{l+1} \\ \lambda_f^{l+1} \\ s^{l+1} \end{bmatrix} + \left[\begin{array}{c} -M\nu^l - p_{ext}^l \\ \hdashline \frac{\Psi_n^l}{h} + \frac{\partial \Psi_n}{\partial t} \\ 0 \\ 0 \end{array}\right]$$

$$0 \leq \begin{bmatrix} p_n^{l+1} \\ p_f^{l+1} \\ \sigma^{l+1} \end{bmatrix} \perp \begin{bmatrix} \lambda_n^{l+1} \\ \lambda_f^{l+1} \\ s^{l+1} \end{bmatrix} \geq 0$$

$$q^{l+1} = q^l + h\nu^{l+1}$$

11/9/06

Another variation. Suppose the contact surface is moving in tangential direction



Define Ψ_f analogous to $\Psi_n \geq 0$

$\Psi_f(q,t)$

only velocity matters

$$\psi_f^{l+1} \simeq \cancel{\psi_f^{l}} + \frac{\partial \psi_f}{\partial q}\Delta q + \frac{\partial \psi_f}{\partial t}\Delta t$$

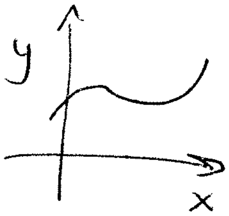
rel. tang. velocity $= \cancel{\partial t}\, W_f^T \nu^{l+1} + \frac{\partial \psi_f}{\partial t}$

Change to LCP is in only the constant vector.

It becomes
$$\begin{bmatrix} -M\nu^{l} - p_{ext}^{l} \\ \frac{\psi_n^{l}}{h} + \frac{\partial \psi_n}{\partial t} \\ \frac{\partial \psi_f}{\partial t} \\ 0 \end{bmatrix}$$

One more variation : Equality Constraints

$\Theta(q,t) = 0$ e.g. particle moves on a wire



$$\Theta^{l+1} \simeq \Theta^{l} + \frac{\partial \Theta}{\partial q} \Delta q + \frac{\partial \Theta}{\partial t} \Delta t$$

$$\frac{\Theta^{l}}{h} + W_b^T \nu^{l+1} + \frac{\partial \Theta}{\partial t} \overset{\sim}{=} 0$$

New Matrix & Vector of Mixed LCP

$$\left[\begin{array}{c} 0 \\ 0 \\ \hline \rho_n^{l+1} \\ \rho_f^{l+1} \\ \sigma^{l+1} \end{array}\right] = \left[\begin{array}{ccccc} M & -W_b & -W_n & -W_f & 0 \\ W_b^T & 0 & & & 0 \\ \hline W_n^T & & & & 0 \\ W_f^T & & & & E \\ 0 & 0 & U & -E^T & 0 \end{array}\right] \left[\begin{array}{c} \nu^{l+1} \\ p_b^{l+1} \\ p_n^{l+1} \\ p_f^{l+1} \\ s^{l+1} \end{array}\right] + \left[\begin{array}{c} -M\nu^{l} - p_{ext}^{l} \\ \frac{\Theta^{l}}{h} + \frac{\partial \Theta^{l}}{\partial t} \\ \hline \frac{\Psi_n^{l}}{h} + \frac{\partial \Psi_n}{\partial t} \\ \frac{\partial \Psi_f}{\partial t} \\ 0 \end{array}\right]$$

$$0 \leq \begin{bmatrix} \rho_n^{l+1} \\ \rho_f^{l+1} \\ \sigma^{l+1} \end{bmatrix} \perp \begin{bmatrix} p_n^{l+1} \\ p_f^{l+1} \\ s^{l+1} \end{bmatrix} \geq 0$$

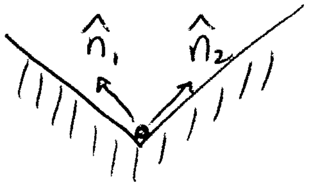
$$q^{l+1} = q^{l} + h\nu^{l+1}$$

11/9/06 ㉓

What chagges for multiple unilateral contacts

$$\Psi_n = [\psi_{1n} \quad \psi_{2n} \quad \cdots]^T$$

$$W_n = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \cdots \end{bmatrix}$$



$$W_f = \begin{bmatrix} \hat{t}_1 & -\hat{t}_1 & \hat{t}_2 & -\hat{t}_2 & \cdots \end{bmatrix}$$

$$E = \left[\begin{array}{c:c} 1 & 0 \\ 1 & 0 \\ \hdashline 0 & 1 \\ 0 & 1 \end{array}\right]_{\ddots}$$

block diagonal

$$U = \begin{bmatrix} \mu_1 & \\ & \mu_2 \end{bmatrix}_{\ddots}$$

$\dfrac{\partial \psi_n}{\partial t}$ is $(n_c \times 1)$ $\quad$ $\dfrac{\partial \psi_f}{\partial t}$ is $(2n_c \times 1)$

What about more equality constraints?

$$\Theta = \begin{bmatrix} \Theta_1 \\ \vdots \\ \Theta \end{bmatrix}$$

11/9/06

㉔

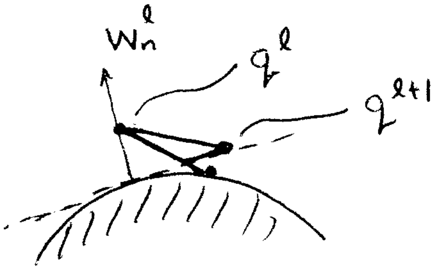
Solution existence

Can Prove soln existence (or not) by eliminating $\nu^{\ell+1}$ & $p_b^{\ell+1}$

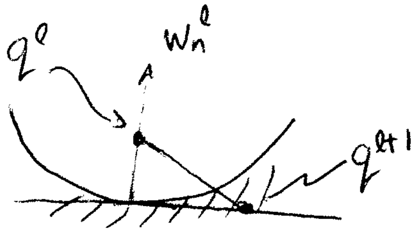
If can eliminate, and $\frac{\psi_n^\ell}{h} \geq 0$, then solution exists and can be found in finite time by Lemke's algorithm.

Errors

ψ_n nonlinear

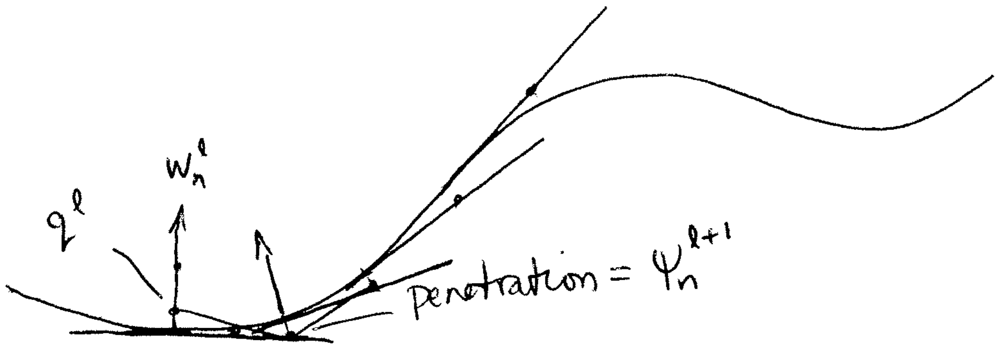


contact force will exist



penetration exists

Constraint Stabilization



$\frac{\psi_n^{l}}{h}$ = outward normal component of velocity needed to eliminate penetration

contains ↑

$p_n \perp p_n$

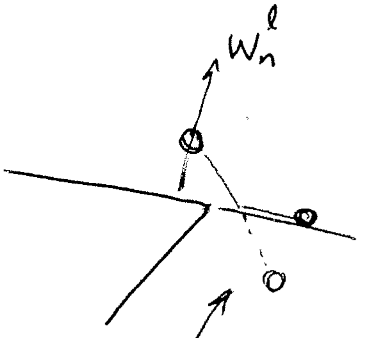
$\therefore \frac{\psi_n^l}{h}$ requires p_n^{l+1} to be large enough to eliminate penetration.

if

Not physically realistic impulse.

11/9/06

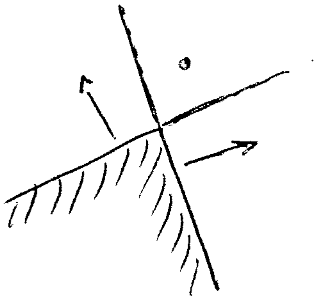
Error due to Polygonalization (and explicit integration method)



Should have moved here

If we used W_n^{l+1} we could avoid this.

Alternative: exact representation of polygonal free space.



$$\psi_{1n}^{l+1} \cong \psi_{1n}^{l} + W_{in}^{T} \nu^{l+1} + \frac{\partial \psi_{1n}}{\partial t} \geq 0$$

OR

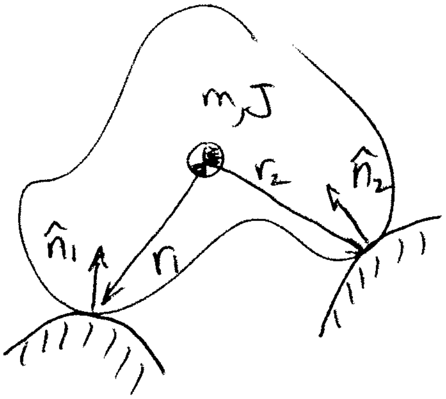
$$\psi_{2n}^{l+1} \cong \psi_{2n}^{l} + W_{2n}^{T} \nu^{l+1} + \frac{\partial \psi_{2n}}{\partial t} \geq 0$$

Egan, Berard, Trinkle, Tech report

11/9/06

Extension to Planar Rigid Bodies

$$M\dot{\nu} = W_n \lambda_n + W_f \lambda_f + g_{ext}$$



$$M = \text{diag}(m, m, J)_{(3\times3)}$$

$$W_n = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \cdots \\ (r_1 \times \hat{n}_1)_z & (r_2 \times \hat{n}_2)_z & \cdots \end{bmatrix}_{(3\times n_c)}$$

$$W_f = \begin{bmatrix} \hat{t}_1 & -\hat{t}_1 & \cdots \\ (r_1 \times \hat{t}_1)_z & -(r_1 \times \hat{t}_1)_z & \cdots \end{bmatrix}_{(3\times 2n_c)}$$

$\underset{(3\times1)}{p_{ext}} = h g_{ext}$ includes moment component

$$E = \underset{\text{diag}}{\overset{\text{block}}{}}\left(\cdots \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdots\right)$$

$$U = \text{diag}(\mu_1, \mu_2, \ldots)$$

$$\Psi_n \quad {}_{(n_c \times 1)}$$

$$\Psi_f \quad {}_{(2n_c \times 1)}$$

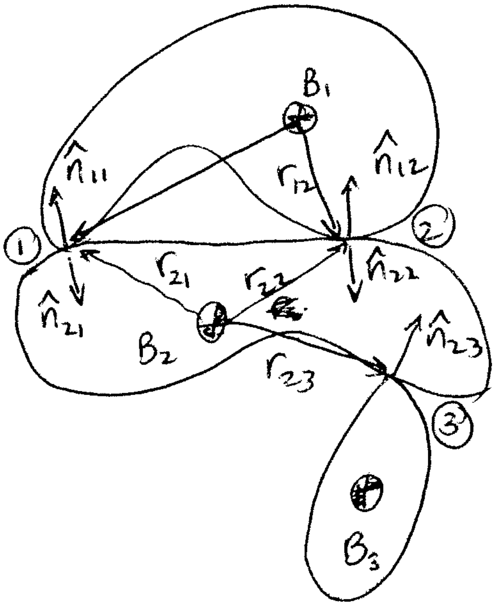
Multiple Planar Rigid Bodies

11/9/06

㉘

$$M = \text{blockdiag}\left(\begin{bmatrix} m_1 & & \\ & m_1 & \\ & & J_1 \end{bmatrix} \cdots\right)$$

$$W_n = \begin{array}{cl}
\left[\begin{array}{c:c:c}
\hat{n}_{11} & \hat{n}_{12} & \\
r_{11}\times\hat{n}_{11} & r_{12}\times\hat{n}_{12} & 0 \\ \hdashline
\hat{n}_{21} & \hat{n}_{22} & \hat{n}_{23} \\
r_{21}\times\hat{n}_{21} & r_{22}\times\hat{n}_{22} & r_{23}\times\hat{n}_{23} \\ \hdashline
 & & \hat{n}_{33} \\
0 & 0 & r_{33}\times\hat{n}_{33}
\end{array}\right]
&
\begin{array}{l} \Big\} B_1 \\ \\ \Big\} B_2 \\ \\ \Big\} B_3 \end{array}
\\
\begin{array}{ccc} \underbrace{\qquad\quad}_{\text{cont 1}} & \underbrace{\qquad\quad}_{\text{cont 2}} & \underbrace{\qquad\quad}_{\text{cont 3}} \end{array} &
\end{array}$$



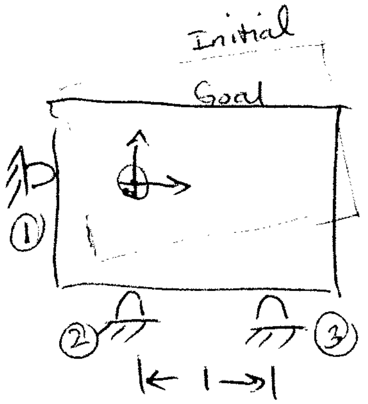
W_f analogous

P_{ext} $(3n_b \times 1)$

An application that's not just simulation. 11/9/06

Frictionless Parts Seating

Determine impulse to apply to cause contact at all three points.



$$\begin{bmatrix} 0 \\ \rho_n^{l+1} \end{bmatrix} = \begin{bmatrix} M & -W_n \\ W_n^T & 0 \end{bmatrix} \begin{bmatrix} \nu^{l+1} \\ p_n^{l+1} \end{bmatrix} + \begin{bmatrix} -M\nu^l - p_{ext}^l \\ \frac{\psi_n^l}{h} + \frac{\partial \psi_n}{\partial t} \end{bmatrix}$$

Eliminate $\nu^{l+1} = \nu^l$

$$\rho_n^{l+1} = W_n^T M^{-1} W_n \; p_n^{l+1} + W^T M^{-1} (\nu^l + p_{ext}^l) + \frac{\psi_n^l}{h} + \frac{\partial \psi_n}{\partial t}$$

Assume $M = I$, $\nu^l = 0$, $\frac{\partial \psi_n}{\partial t} = 0$, $h = \underline{1}$

Note that $W_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad W_n^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \det(W_n) = 1$

Close contacts $\Rightarrow$ $\underline{\rho_n^{l+1} = 0}$

11/9/06

Simplify

$$p_n^{l+1} = \boxed{-W_n^{-1} p_{ext} - W_n^{-1} W_n^{-T} \psi_n^l \geq 0}$$

Assume sensor can measure gaps.

Then all is known except p_{ext}

Inequality represents a polytope in p_{ext} space.

Multiply by

$$p_{ext} \leq \begin{bmatrix} -\psi_{1n}^l \\ -\psi_{2n}^l \\ \psi_{2n}^l - \psi_{3n}^l \end{bmatrix}$$

Since $\psi_n^l > 0$, p_{ext} has strictly negative x & y components

if $\psi_{2n}^l > \psi_{3n}^l$, $(p_{ext})_3$ is ≤ positive number

force helps twist block

$\psi_{3n}^l > \psi_{2n}^l$, (p_{ext}) ~~has~~ is strictly ~~neg~~ ~~min~~ ~~≥~~ ~~component~~

What if ψ_{in}^l not known accurately?

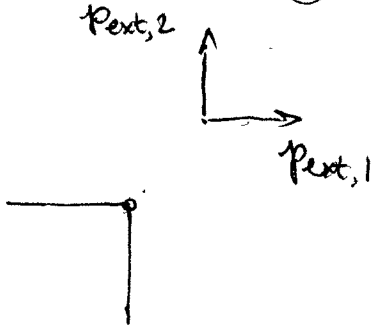
What about pt into corner problem?

3/27/04

(31)

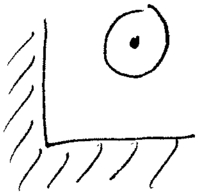
$$W_n = M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

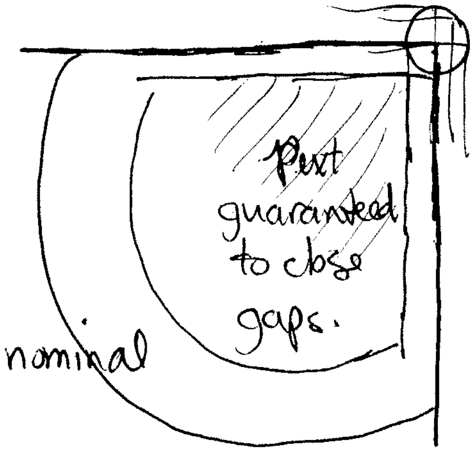
then $p_{ext} \leq \begin{bmatrix} -\psi^l_{1n}/h \\ -\psi^l_{2n}/h \end{bmatrix}$



Uncertainty.

If q^l uncertain, then ψ^l_n is uncertain



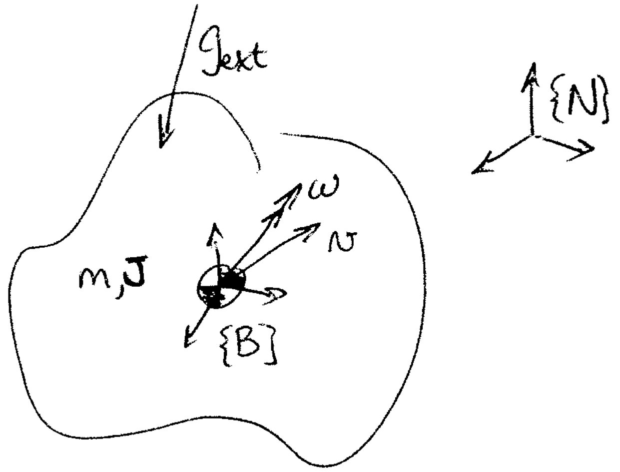


11/13/06

Generalize to Spatial Case

Significant Changes:

- Rotation Kinematics
- Nonlinear Friction Constraint
- New term in dynamics
- Matrix dimensions



Generalized velocity

$$\nu = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \quad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Configuration

$$q = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \begin{matrix} \text{unit} \\ \text{quaternion} \\ \text{a.k.a.} \\ \text{Euler parameters} \end{matrix} \qquad e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

Rotational Kinematics $\quad \dot{q} = G\,\nu$

11/13/06

$$G = \left[\begin{array}{c|c} I_{(3\times3)} & \\ \hline & {}^{B}B(q)_{(4\times3)} \end{array}\right]_{(7\times6)}$$

$I_{(3\times3)} = 3\times3$ identity matrix $\Rightarrow$ $\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}$

$${}^{B}B = \frac{1}{2}\begin{bmatrix}-e_1 & -e_2 & -e_3\\ e_0 & -e_3 & e_2\\ e_1 & e_0 & -e_1\\ -e_2 & e_3 & e_0\end{bmatrix} \Rightarrow \begin{bmatrix}\dot{e}_0\\ \dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\end{bmatrix} = {}^{B}B(q)\,{}^{B}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

very important

Properties: $G^T G = I_{(6\times6)}$

$$G G^T \dot{q} = \dot{q}$$

Also need rotation matrix

$${}^{N}_{B}R(q) = \left[\underbrace{\begin{matrix}1-2(e_2^2+e_3^2)\\ 2(e_1e_2+e_0e_3)\\ 2(e_1e_3-e_0e_2)\end{matrix}}_{{}^{N}x_{B}} \;\vdots\; \begin{matrix}2(e_1e_2-e_0e_3)\\ 1-2(e_1^2+e_3^2)\\ 2(e_2e_3+e_0e_1)\end{matrix} \;\vdots\; \begin{matrix}2(e_1e_3+e_0e_2)\\ 2(e_2e_3-e_0e_1)\\ 1-2(e_2^2+e_1^2)\end{matrix}\right]$$

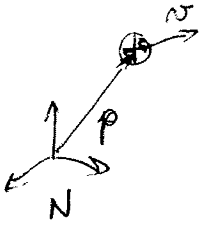
Change in Dynamic Eqs.

11/13/06

③

Sum of forces $\sum_i f_i = F$

Sum of Moments $\sum_i r_i \times f_i + n_i = N$



Newton : $F = \frac{d}{dt}(mv) \overset{m = const}{\Longrightarrow} \boxed{F = m\dot{v}}$

Euler : $N = \frac{d}{dt}(J\omega)$

because $J\omega$ is the angular momentum of a rotating body, its derivative has two parts

$$\frac{d}{dt}(J\omega) = \boxed{J\dot{\omega} + \omega \times J\omega = N}$$

Prepare for integration / simulation - put in first-order form.

$$\dot{v} = F/m$$

$$\dot{\omega} = J^{-1}(N - \omega \times J\omega)$$

$$\dot{x} = v$$

$$\dot{e} = B\omega$$

11/13/06

J = Inertia Matrix

J is 3×3, P.D., & symmetric.

represents mass distribution

In frame B, J is constant = ^BJ

Frames of representation of dynamic eqs.

$^N\dot{v} = {}^NF/m$

$^N\dot{\omega} = {}^NJ^{-1}({}^NN - {}^N\omega \times {}^NJ\,{}^N\omega)$

$^N\dot{x} = {}^Nv$

$^N\dot{e} = {}^NB\,{}^N\omega$

~~Define ^NJ from ^BJ:~~

~~$^B\dot{\omega} = {}^B_NR\,{}^N\dot{\omega}$~~

11/13/06
⑤

Define ${}^N J$ from ${}^B J$

$${}^N_B R \left({}^B J \, {}^B\dot{\omega} = {}^B N - {}^B\omega \times {}^B J \, {}^B\omega \right)$$

$$\underbrace{{}^N_B R \, {}^B J \, {}^B_N R}_{{}^N J} \, {}^N\dot{\omega} = \underbrace{{}^N_B R \, {}^B N}_{{}^N N} - \underbrace{{}^N_B R \, {}^B\omega}_{{}^N\omega} \times \underbrace{{}^N_B R \, {}^B J \, {}^B_N R}_{{}^N J} \, {}^N\omega$$

$${}^N J = {}^N_B R \, {}^B J \, {}^B_N R$$

${}^N J$ is P.D. & Symmetric.

$$g_{ext} = \begin{bmatrix} \text{gravity,} \\ \text{drag,} \\ \text{etc} \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega \times J\omega \end{bmatrix}$$

Complementarity Conditions and the following

$$\dot{\nu} = M^{-1} \left(W_n \lambda_n + W_f \lambda_f + g_{ext} \right) + W_b \lambda_b$$

$$\dot{q} = G \nu \quad , \quad e_0^2 + e_1^2 + e_2^2 + e_3^2 - 1 = 0 = \Theta(q)$$

where $M = \left[\begin{array}{c|c} mI & 0 \\ \hline 0 & J \end{array}\right]$ $\quad G = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & B \end{array}\right]$

3D Dynamics - Contact Friction

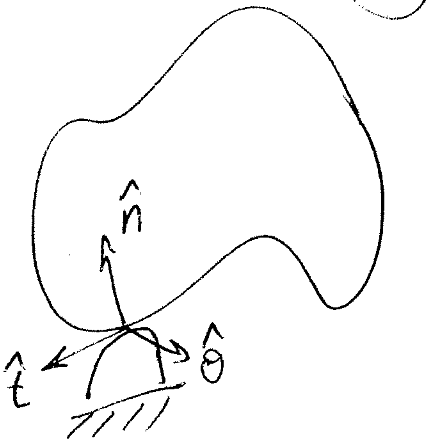
3/29/04

36

$$f = \hat{n}\lambda_n + \hat{t}\lambda_t + \hat{\theta}\lambda_o$$

~~θ~~

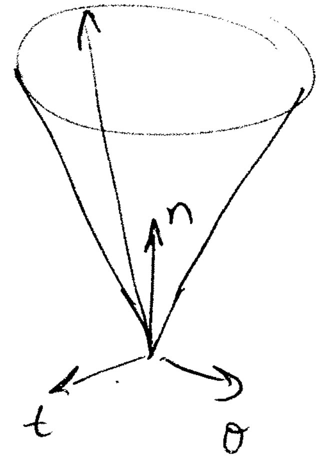
$$r \times f = r \times \hat{n}\lambda_n + r \times \hat{t}\lambda_t + r \times \hat{\theta}\lambda_o$$



Friction Model - ~~When Slid~~

Friction acts to maximize rate at which energy is dissipated

Friction force lies with a cone.



Sliding or Rolling → $\lambda_t^2 + \lambda_o^2 \le \mu^2 \lambda_n^2$

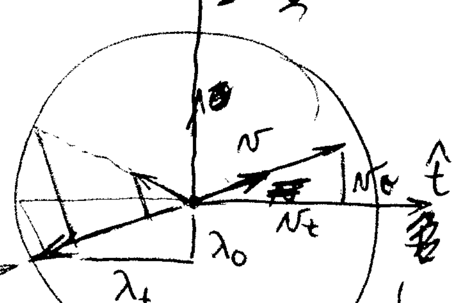
Sliding $(v_t, v_o) \neq 0$ $\quad (\lambda_t, \lambda_o) \in \operatorname{argmax}\{-v_t\lambda_t' - v_o\lambda_o' : \lambda_t'^2 + \lambda_o'^2 \le \mu^2\lambda_n^2\}$

where $v_t = W_t^T \nu$, $v_o = W_o^T \nu$

~~$(\lambda_t, \lambda_o) \in \operatorname{argmax}\{-W_t$~~

~~$(\lambda_t, \lambda_o) \in \operatorname{argmax}\{-\nu(W_t + W_o)$~~

$$(\lambda_t, \lambda_o) \in \operatorname{argmax}\{-\nu^T [W_t \; W_o] \begin{bmatrix} \lambda_t' \\ \lambda_o' \end{bmatrix} : (\lambda_t', \lambda_o') \in F_i\}$$



maximal power dissipation when opposite to v.

Friction Limit Surface

When sliding we can solve for λ_t, λ_o :

$$\lambda_t = \frac{-\mu \lambda_n \nu_t}{\sqrt{\nu_t^2 + \nu_o^2}} \qquad \lambda_o = \frac{-\mu \lambda_n \nu_o}{\sqrt{\nu_t^2 + \nu_o^2}}$$

NONLINEAR CONSTRAINTS

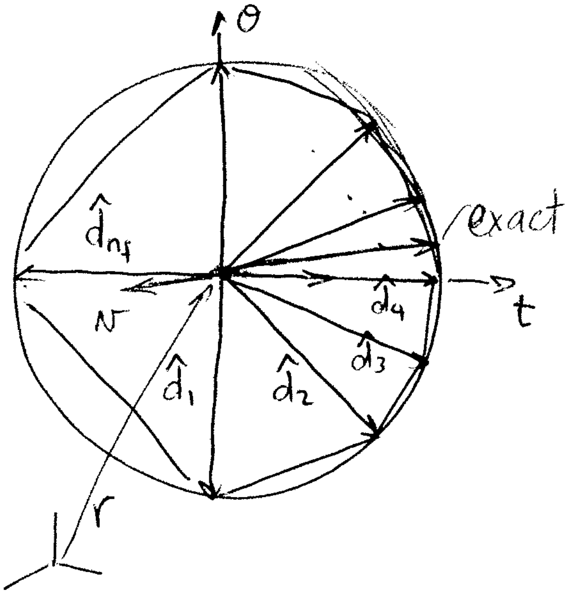
If we knew the approximate sliding direction, then we could linearize with Taylor series

But we don't!

And $\sqrt{\nu_t^2 + \nu_o^2}$ can go to zero!

Skip to Page (7.1)

Approximate Friction Limit Surface as a Polygon



Friction force :

$$\hat{d}_1 \lambda_{1f} + \hat{d}_2 \lambda_{2f} + \cdots + \hat{d}_{n_f} \lambda_{n_f f}$$

$$\lambda_{if} \geq 0 \quad \forall\ i$$

Friction moment :

$$r \times \hat{d}_1 \lambda_{1f} + \cdots + r \times \hat{d}_{n_f} \lambda_{n_f f}$$

Friction Wrench

$$W_f \lambda_f \qquad W_f = \underset{(6 \times n_f)}{\begin{bmatrix} \hat{d}_1 & \cdots & \hat{d}_{n_f} \\ r \times \hat{d}_1 & & r \times \hat{d}_{n_f} \end{bmatrix}} \qquad \lambda_f = \underset{(n_f \times 1)}{\begin{bmatrix} \lambda_{1f} \\ \lambda_{2f} \\ \vdots \\ \lambda_{n_f f} \end{bmatrix}} \geq 0$$

11/13/06

⑦.①

Instantaneous Dynamics

$$\dot{\nu} = M^{-1}(W_n \lambda_n + W_t \lambda_t + W_o \lambda_o + W_b \lambda_b + gext)$$

$$\dot{q} = G \nu$$

$$\Theta = 0 \quad , \quad \lambda_b \text{ free}$$

$$0 \leq \lambda_n \perp \psi_n \geq 0$$

$$(\lambda_t, \lambda_o) \in \operatorname{argmax} \left\{ -\nu^T [W_t \ W_o] \begin{bmatrix} \lambda_t' \\ \lambda_o' \end{bmatrix} : (\lambda_t', \lambda_o') \in \mathcal{F} \right\}$$

where $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_{n_c}$

Sum all contact forces

11/13/06

$$W_n \lambda_n + W_f \lambda_f$$

$$\lambda_n, \lambda_f \geq 0$$

$$\lambda_n = \begin{bmatrix} \lambda_{1n} \\ \lambda_{2n} \\ \vdots \\ \lambda_{n_c n} \end{bmatrix} \quad \lambda_f = \begin{bmatrix} \lambda_{1f} \\ \lambda_{2f} \\ \vdots \\ \lambda_{n_c f} \end{bmatrix}$$

where $\lambda_{jf} = [\lambda_{jf_1} \; \cdots \; \lambda_{jf_{nd}}]$

Over small time step

$$W_n p_n + W_f p_f$$

$$p_n, p_f \geq 0 \quad , \quad p_\alpha = h \lambda_\alpha$$

How do we write constraints to pick best friction force?

$$0 \leq p_f^{l+1} \perp W_f^T \nu^{l+1} + E s^{l+1} \geq 0$$

$$0 \leq s^{l+1} \perp U p_n^{l+1} - E^T p_f^{l+1} \geq 0$$

$$U = \mathrm{diag}(\mu_1, \cdots, \mu_{n_c}) \qquad E = \mathrm{BlkDiag}(e_1, e_2, \cdots e_{n_c})$$

$$\text{where } e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{(n_d \times 1)}$$

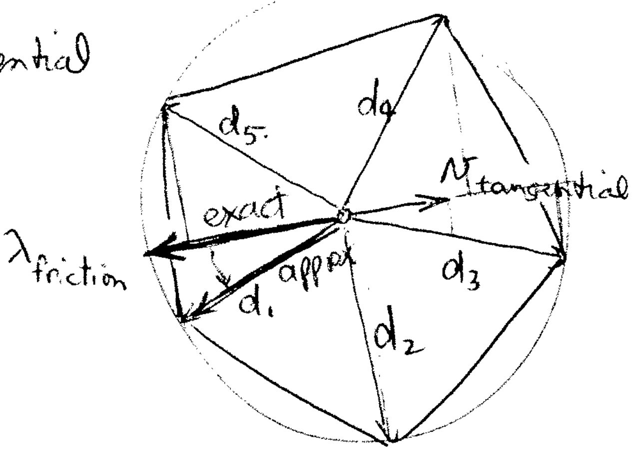
Nondegenerate Solutions of the LCP

Assume $(W_t + W_o)\nu = \nu_{\text{tangential}}$

Note that

$\nu_{\text{tang}} \cdot d_1$

dissipates most energy.



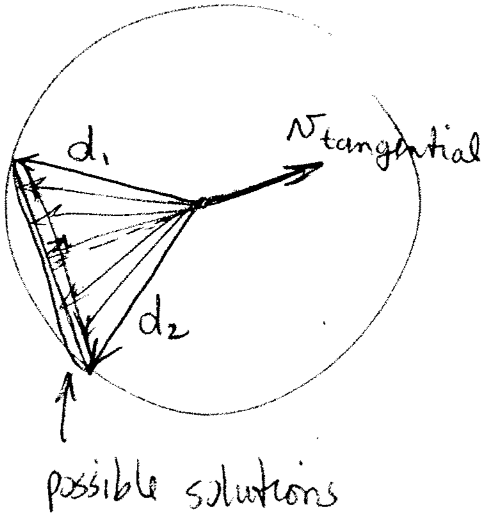
$\therefore \lambda_{f_1} \geqq 0, \quad \lambda_{f_2} = \lambda_{f_3} = \lambda_{f_4} = \lambda_{f_5} = 0$

Some solutions

find $\nu_{\text{tangential}}$

such that

$$W_f^T \nu = \begin{bmatrix} \left.\begin{matrix}\min \\ \min\end{matrix}\right\} \\ \text{larger} \\ \vdots \\ \text{larger} \end{bmatrix}$$



Then $\lambda_{f_1}, \lambda_{f_2} \geq 0, \quad \lambda_{f_j} = 0 \quad \forall j \neq 1, 2$

11/13/06

Time Stepping LCP

Same as page ㉒ 11/9/06

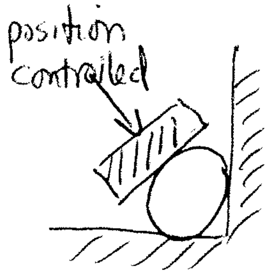
Note that for every body we have

$$\Theta_i = e_{i0}^2 + e_{i1}^2 + e_{i2}^2 + e_{i3}^2 - 1 = 0$$

11/13/06

⑪

LCP Solution non-existence



$\psi_n^{l+1} \geq 0$ is infeasible

i.e. block moves toward corner by some finite amount over timestep h.

Disk is larger than space between

$\psi_n^{l+1} \not\geq 0$.

An Example: Box on Floor

3/29/04

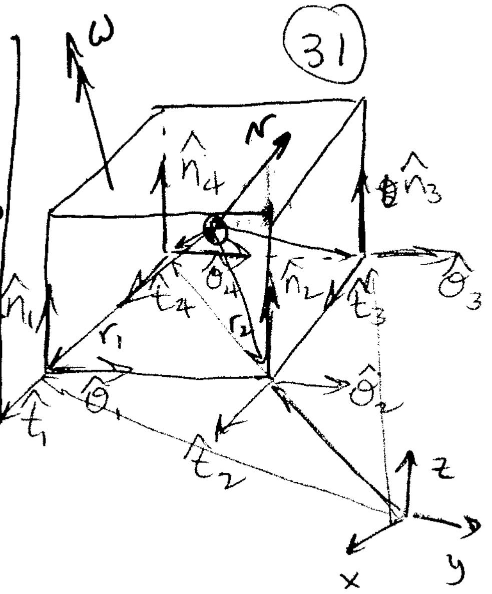
(31)

Assume Body-Fixed Frame is principal axes

$^B J = \mathrm{diag}(1, 2, 3)$

$m = 1$

$$\nu = \begin{bmatrix} \nu_x \\ \nu_y \\ \nu_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad q = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

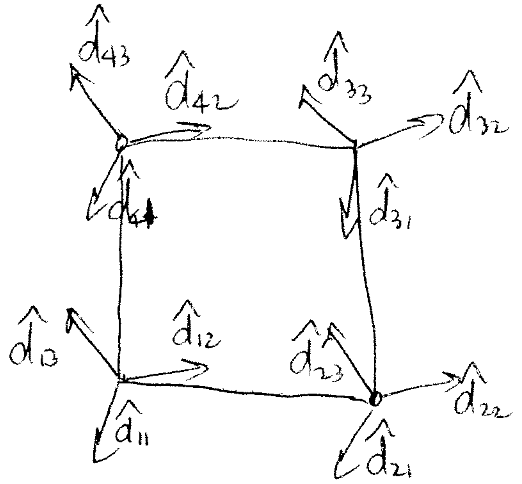


$$^N W_n = \begin{bmatrix} ^N\hat{n}_1 & \cdots & ^N\hat{n}_4 \\ ^N r_1 \times {}^N\hat{n}_1 & & ^N r_4 \times {}^N\hat{n}_4 \end{bmatrix}$$

r's from c.g. to contact point.

~~$^N W_f = \begin{bmatrix} ^N W_t & -W_t \end{bmatrix}$~~

$$^N W_f = \begin{bmatrix} ^N\hat{d}_{11} & \hat{d}_{12} & \hat{d}_{13} & \hat{d}_{21} & \cdots \\ ^N r_1 \times {}^N\hat{d}_{11} & r_1 \times \hat{d}_{12} & r_1 \times \hat{d}_{13} & r_2 \times \hat{d}_{21} & \cdots \end{bmatrix}$$



$$M = \left[\begin{array}{c|c} \begin{matrix} m & & 0 \\ & m & \\ 0 & & m \end{matrix} & O \\ \hline O & ^N I \end{array}\right] \qquad p_{ext} = \left[\begin{array}{c} 0 \\ 0 \\ -mg \\ \hline h({}^N\omega \times {}^N I \, {}^N\omega) \end{array}\right]$$

~~$W_E =$~~ $\Theta^1 = e_0^2 + e_1^2 + e_2^2 + e_3^2 - 1 = 0 \qquad [0 \quad 0 \quad 0 \quad 2e_0 \quad 2e_1 \quad 2e_2 \quad 2e$

$$W_E^T = \frac{\partial \Theta}{\partial q} G(q) \qquad W_E = G^T(q) \left(\frac{\partial \Theta}{\partial q}\right)^T \quad \text{where} \left(\frac{\partial \Theta}{\partial q}\right)^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2e_0 \\ 2e_1 \\ 2e_2 \\ 2e_3 \end{bmatrix}$$

3/29/04

$$U = \begin{bmatrix} \mu_1 & & & \\ & \mu_2 & & 0 \\ & 0 & \mu_3 & \\ & & & \mu_4 \end{bmatrix}_{(n_c \times n_c)}$$

$$E^T = \begin{bmatrix} 1\;1\;1 & & & \\ & 1\;1\;1 & & \\ & & 1\;1\;1 & \\ & & & 1\;1\;1 \end{bmatrix}_{(n_c \times (n_c \cdot n_f))}$$

$$\frac{\partial \Theta}{\partial t} = 0$$

$\psi_n^l / h \Longleftrightarrow \psi_n^l$ = gap at current time

$$\psi_n^l = \begin{bmatrix} 0.001 \\ -0.0014 \\ -0.010 \\ -0.0004 \end{bmatrix}$$

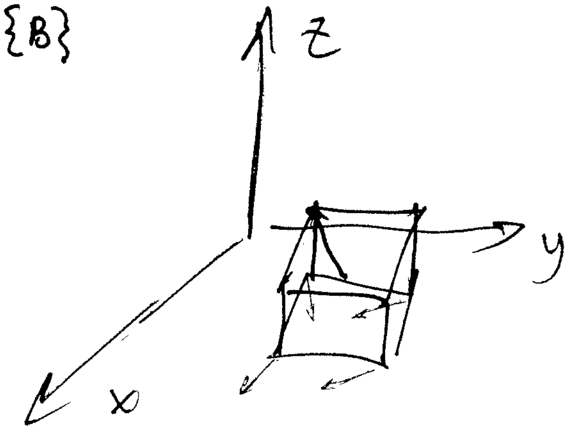
e.g.

Assume ~~unit cube~~. Look at nonpenetration for one corner.

$$\psi_n = \left[\begin{array}{c|c} {}^N_B R(q) & \begin{matrix} q_1 \\ q_2 \\ q_3 \end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right]_3 \left[\begin{array}{c} 1/2 \\ 1/2 \\ 1/2 \\ \hline 1 \end{array}\right] \geq 0$$

corner in $\{B\}$

It is ok to Keep all 8 constraints active at all times.

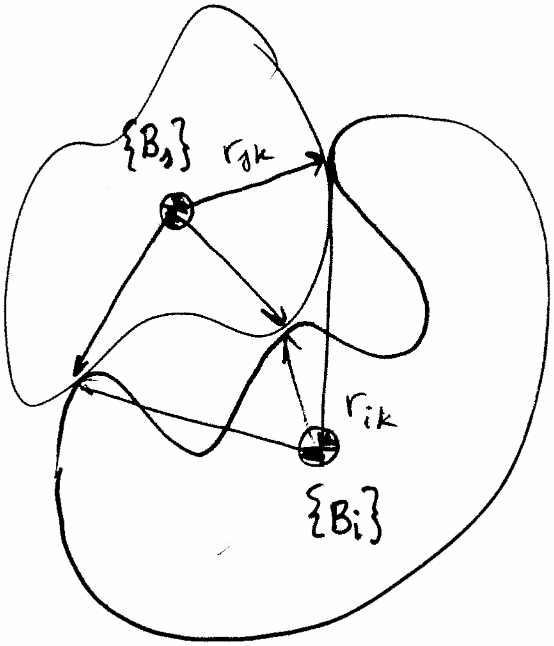


3/29/04

More Bodies

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_{nb} \end{bmatrix} \quad \nu = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_{nb} \end{bmatrix} \quad p_{ext} = \begin{bmatrix} p_{1,ext} \\ p_{2,ext} \\ \vdots \\ p_{nb,ext} \end{bmatrix}$$

$$(7nb \times 1) \qquad (6n_b \times 1) \qquad (6nb \times 1)$$

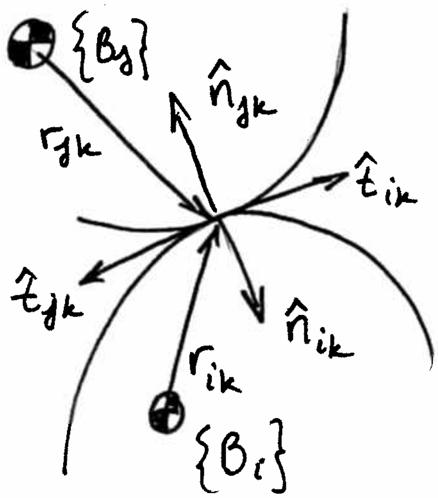


$$M = \operatorname{diag}(M_1, \ldots, M_{nb})$$

$$W_n = \left[\begin{array}{ccc:c:c} & & & & \\ \hdashline & & & W_{ik} & \\ \hdashline & & & & \\ \hdashline & & & W_{jk} & \\ \hdashline & & & & \\ 1 & 2 & \cdots & k & \cdots\cdots n_c \end{array}\right] \begin{array}{l} \\ \}\, \text{body } i \\ \\ \}\, \text{body } j \\ \\ (6nb \times n_c) \end{array}$$

$(6n_b \times n_c)$

$\longleftarrow$ contacts $\longrightarrow$

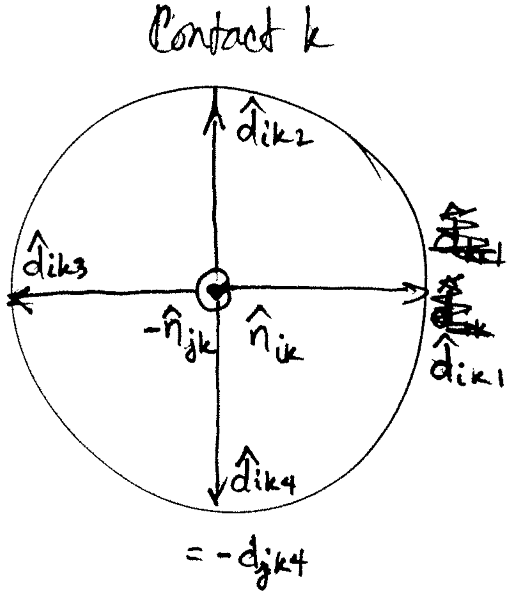


where $${}^N W_{ik} = \begin{bmatrix} {}^N\hat{n}_{ik} \\ {}^N r_{ik} \times {}^N\hat{n}_{ik} \end{bmatrix} \Rightarrow$$

$${}^N W_{jk} = \begin{bmatrix} -{}^N\hat{n}_{ik} \\ {}^N r_{jk} \times (-{}^N\hat{n}_{ik}) \end{bmatrix} = \begin{bmatrix} {}^N\hat{n}_{jk} \\ {}^N r_{jk} \times {}^N\hat{n}_{jk} \end{bmatrix}$$

3/29/04

$$W_f = \begin{array}{cc} \begin{array}{ccc} \quad & \text{Contact } k & \quad \end{array} & \\ \left[\begin{array}{c|c|c} & & \\ \hline & W_{ik,f} & \\ \hline & & \\ \hline & W_{jk,f} & \\ \hline & & \end{array}\right] & \begin{array}{l} \\ \text{Body } i \\ \\ \text{Body } j \\ \\ \end{array} \end{array}$$



$$W_{ik,f} = \begin{bmatrix} \hat{d}_{ik,1} & \cdots & \hat{d}_{ik,n_f} \\ r_{ik} \times \hat{d}_{ik,1} & \cdots & r_{ik} \times \hat{d}_{ik,n_f} \end{bmatrix}$$

Similar for $W_{jk,f}$.

Convenient to let ${}^N\hat{d}_{ik,1} = -{}^N\hat{d}_{jk,1}$, etc.

Then it is easy to use one set of contact ~~force~~ impulse parameters $p_{kn}, p_{kf1}, p_{kf2}, \ldots$

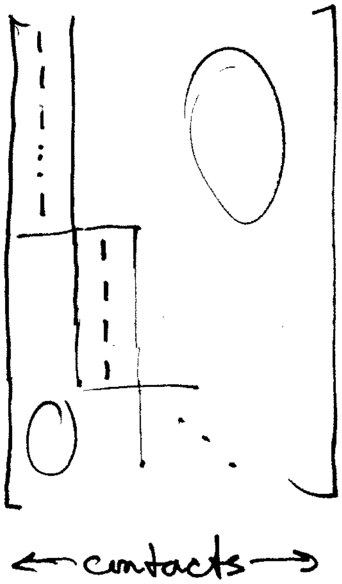
$$\Theta(q) = \begin{bmatrix} e_{10}^2 + e_{11}^2 + e_{12}^2 + e_{13}^2 - 1 \\ e_{20}^2 + e_{21}^2 + e_{22}^2 + e_{23}^2 - 1 \\ \vdots \\ \end{bmatrix}$$

$$\frac{\partial \Theta}{\partial q} = \left[\begin{array}{ccccccc|ccccccc|c} 0 & 0 & 0 & 2e_{10} & 2e_{11} & 2e_{12} & 2e_{13} & & & & & & & & \cdots \\ \hline 0 & 0 & 0 & & & & & 0 & 0 & 0 & 2e_{20} & 2e_{21} & 2e_{22} & 2e_{23} & \\ \hline & & & & 0 & & & & & & & & & & \ddots \end{array}\right]$$

3/29/04

$E =$

$(n_c \times (n_c \cdot n_f))$



$$p_f = \begin{bmatrix} \vdots \\ p_{kf1} \\ p_{kf2} \\ \hline \vdots \\ p_{(k+1)f1} \\ p_{(k+1)f2} \\ p_{(k+1)f3} \\ \vdots \\ \hline \vdots \end{bmatrix} \updownarrow \text{contacts}$$

Final Size of LCP $(7n_b + n_c(2+n_f))$

Could eliminate $7n_b$ variables to make problem smaller, but then the LCP matrix becomes dense and solver converges more slowly.