Second-Order Stability Cells of a Frictionless Rigid Body Grasped by Rigid Fingers

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Abstract

The most secure type of grasp of a frictionless woorkpiece is the form-closure grasp. However, task constraints may make achieving form-closure impossible or undesirable. In this case, one needs to employ a force-closure [1] grasp. In this paper, we study the subclass of force-closure grasps known as second-order stable grasps, which typically have a small number of contacts. We derive conditions for second-order stability and represent second-order stability cells as conjunctions of equations and inequalities in the configuration variables of the system. These cells are the subsets of the system's configuration space for which the frictionless workpiece is second-order stable. We also determine the minimum and maximum numbers of contacts necessary for second-order stability. Our results are applied to a simple planar whole-arm manipulation system to generate one of its second-order stability cells.

1 Introduction

Consider a multi-arm robotic system or hand manipulating a frictionless workpiece with contacts allowed anywhere on any link. This type of manipulation has been referred to as whole-arm manipulation. The task of planning the joint trajectories required to reposition and reorient the workpiece within the whole-arm manipulator is referred to as the *whole-arm manipulation planning problem* [17].

Little progress has been made toward a general, systematic approach to the whole-arm manipulation planning problem, primarily due to the difficulties associated with predicting the motion of bodies with multiple concurrent contacts. For example, both dynamic [9] and quasistatic [19] multibody models often yield nonunique predictions of the system motion which can be resolved by accurately modeling the compliance and friction properties at the contacts [14]. Also, the kinematic constraints associated with each rolling contact are nonholonomic and thus further complicate planning.

In this paper, we assume that the three-dimensional workpiece is frictionless, the fingers are rigid, and that inertial effects are negligible. This leads to a holonomic model that predicts motions uniquely. While the frictionless assumption is not appropriate in many potential grasping and whole-arm manipulation applications, we have chosen it for two reasons. First, an autonomous robotic system must be able to function under all environmental conditions. For example, if a robot is to clean-up a contaminated worksite, it must not fail because it needs to move slippery objects. Second, we have analytically and experimentally verified that under certain conditions, the motion predicted by the frictionless, quasistatic model is a good approximation to the motion predicted by the analogous quasistatic and dynamic models with friction [15].

In particular, this paper is devoted to frictionless grasps which have second-order stability. Second-order stable grasps are characterized by a small number of linearly independent contacts: less than six in the spatial case and less than three in the planar case. Thus second-order stable grasps do not have enough contacts to achieve form closure [7] or first-order stability [16]. Second-order stability exists when point representing the configuration of the workpiece is at the bottom of a potential energy well that is smooth (specifically quadratic) in at least one direction. This is in contrast to first-order stable grasps for which the configuration point is at the bottom of a well that is not smooth in any direction.

Second-order stable grasps are important, because the more secure form-closure and first-order stable grasps cannot always be achieved or may not always be optimal in the context of the given task. For example in grasp acquisition, the number of contacts between the workpiece and the hand increases monotonically from zero, but form-closure requires seven contacts and first-order stability requires six. In addition, second-order stable grasps represent an alternative type of grasp to be exploited by whole-arm manipulation planners, thereby expanding the range of manipulation tasks that they can plan.

The contributions of this paper are: (1) a formal definition of second-order stable grasps, (2) the formulation of second-order stability cells (SS-cells) as conjunctions of equations and inequalities in the C-

[1] We use Reuleaux's definition of force-closure [12].

space variables in a form suitable for the approximate cell-decomposition of C-space, (3) the establishment of the minimum and maximum numbers of linearly independent contacts that planar and spatial second-order stable grasps can have, and (4), the introduction of a graphical test for second-order stability for grasps of polygons in the plane.

2 Background

Previous studies of second-order stable grasps have been limited to cases in which the system was compliant. Salisbury studied active stiffness control of a grasped workpiece [13]. He noted that for grasp stability, the control gains and the grasp geometry had to be selected to ensure that the grasp stiffness matrix would be positive definite. This condition guaranteed that the grasping fingers would generate restoring forces in response to all deflections of the grasped object. Similar uses of second-order stability in grasping can be found in [1, 6, 10].

Previous studies of rigid body grasping have not considered second-order stability. Instead, researchers have concentrated on form-closure grasps, because they are the most resistant to external disturbing forces [7, 13, 11]. However, it is clear that such grasps are less mobile and therefore less useful in dexterous manipulation, since then the goal is grasp reorientation, not a strong static grasp.

The work presented here is distinct in that we consider second-order stability when all bodies are rigid and all fingers are rigidly positioned. Thus the result of our analysis manifests some characteristic features of both previous analyses. Specifically, a compliant grasp has second-order stability if the Hessian matrix of the potential energy stored in the deflected fingers (*i.e.*, the grasp stiffness matrix) is positive definite. Similarly, a rigid grasp has second order stability if the Hessian of the potential energy of the workpiece projected onto the space of kinematically admissible workpiece configurations is positive definite.

3 System Model

We study three-dimensional systems analogous to the two-dimensional one shown in Figure 1. This system consists of two serial-link manipulators (bodies 2 and 3, and bodies 4, 5, and 6), two fixed bodies (bodies 0 and 1), and two workpieces (bodies 7 and 8). We refer to the collection of bodies 0-6 as the manipulator, since they are the subset of bodies whose positions are actively controlled to manipulate the workpieces. The manipulator and the workpiece(s) taken together are referred to as the system. In this paper, we limit our analysis to systems with one workpiece and assume that all joints are rigidly position-controlled.

The configuration of the system is completely specified by two vectors: $\mathbf{q}$, of length n_q, the number of degrees of freedom of the workpiece, and $\boldsymbol{\theta}$, of length n_θ, the number of joints. $\mathbf{q}$ represents the configuration of workpieces with respect to the universal frame, U, and $\boldsymbol{\theta}$ contains the positions of the (revolute and prismatic) joints. The C-space $\mathcal{X}$, of the system is the space of dimension $n_q + n_\theta$ containing all possible values of $\mathbf{q}$ and $\boldsymbol{\theta}$.

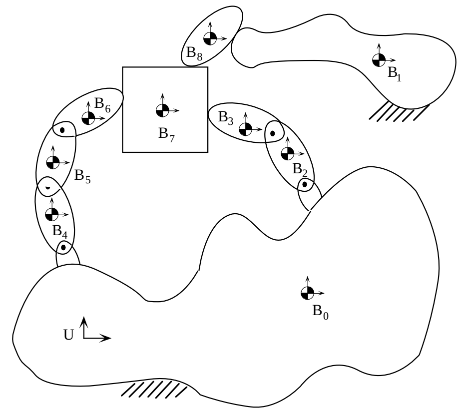

Figure 1: A General Planar System

In our analysis, we utilize subsets of C-space corresponding to various topologically equivalent contact states called "contact formations" [2]. A contact formation (CF) is a set of elemental contacts, where an elemental contact is one between a pair of features on two bodies. Examples include, edge-edge and face-vertex contacts. The existence of the i^{th} elemental contact causes the distance function corresponding to the two contacting features, the geometric C-function, $f_{geo,i}(\mathbf{q}, \boldsymbol{\theta})$, to be zero [17]. The maintenance of a CF causes a vector of C-functions, $\mathbf{f}_{geo}$, to be zero, thereby defining a "surface" in C-space. The portion of each of these surfaces for which bodies do not interpenetrate is referred to as a CF-cell [17] and is denoted by $\mathcal{CF}$.

4 Stability Conditions

Let us assume that the fingers' controllers are stiff enough to neglect joint position errors. In this case, the potential energy of the system for a given manipulator configuration is dominated by the height of the center of mass of the workpiece. Therefore, for a given configuration, $\boldsymbol{\theta}$, of the manipulator, the stable configurations of the workpiece are the local minima of the following nonlinear optimization problem [16]:

$$\underset{\mathbf{q}}{Minimize} \quad V(\mathbf{q}) \tag{1}$$

$$Subject\ to: \quad \mathbf{q} \in \mathcal{C}_{valid} \tag{2}$$

where V is the potential energy of the workpiece and $\mathcal{C}_{valid}$ is the union of free space and contact space [8]. Note that a configuration, $\mathbf{q}$, is an element of $\mathcal{C}_{valid}$, if, in that configuration, the workpiece does not overlap any link of the manipulator; contact is allowed.

The necessary conditions for $\mathbf{q}$ to be a local minimum can be shown to be [5]:

$$\mathbf{f}_{geo}(\mathbf{q}^*, \boldsymbol{\theta}) \geq \mathbf{0} \quad (3)$$

$$\mathbf{W}_n(\mathbf{q}^*, \boldsymbol{\theta})\mathbf{c}_n{}^* = -\mathbf{g}_{obj}(\mathbf{q}^*) \quad (4)$$

$$\mathbf{c}_n{}^* \geq \mathbf{0} \quad (5)$$

$$\boldsymbol{\lambda}^* \geq \mathbf{0} \quad (6)$$

where $\mathbf{c}_n \in \mathcal{R}^{n_c}$ is the wrench intensity vector, whose elements are the contact force magnitudes which play the role of the Lagrange multipliers, an asterisk indicates that the superscripted quantity takes on its value at a local minimum, $\mathbf{g}_{obj}$ is the gravitational wrench acting on the workpiece, $\mathbf{W}_n$ is the wrench matrix, and $\boldsymbol{\lambda}$ is the vector of eigenvalues of the projected Hessian matrix, $\mathbf{P}$:

$$\mathbf{P} = \mathbf{Z}^T\mathbf{H}\mathbf{Z}. \quad (7)$$

Here $\mathbf{Z}$ is any matrix whose columns form a basis for the left null space of the $n_q \times n_c$ Jacobian matrix, $\frac{\partial \mathbf{f}_{geo}}{\partial \mathbf{q}}$, and $\mathbf{H}$ is given as:

$$\mathbf{H} = -\sum_{i=1}^{n_c} \frac{\partial^2 f_{geo,i}}{\partial \mathbf{q}^T \partial \mathbf{q}} c_{n_i}, \quad (8)$$

and $f_{geo,i}$ is the i^{th} element of the vector function $\mathbf{f}_{geo}$.

Sufficient conditions for workpiece stability given the current CF are as follows [5]:

$$\mathbf{f}_{geo}(\mathbf{q}^*) = \mathbf{0} \quad (9)$$

$$\mathbf{W}_n(\mathbf{q}^*, \boldsymbol{\theta})\mathbf{c}_n{}^* = -\mathbf{g}_{obj}(\mathbf{q}^*) \quad (10)$$

$$\mathbf{c}_n{}^* \geq \mathbf{0} \quad (11)$$

$$\boldsymbol{\lambda}_+^* > \mathbf{0}. \quad (12)$$

where the vector $\boldsymbol{\lambda}_+$ is the vector of eigenvalues of a new projected Hessian matrix, $\mathbf{P}_+$, defined as follows:

$$\mathbf{P}_+ = \mathbf{Z}_+^T\mathbf{H}\mathbf{Z}_+. \quad (13)$$

Here $\mathbf{Z}_+$ is a matrix whose columns form a basis for the left null space of $\mathbf{W}_{n+}$, which is formed from $\mathbf{W}_n$ by removing the columns corresponding to the zero-valued elements of $\mathbf{c}_n$. Note that equation (9) is written as an equality, because it defines the "surface" in C-space where all currently considered contacts are maintained. Equations (10)-(13) define the subset of the surface for which the workpiece is stable. If an element of $\mathbf{c}_n$ were to become negative, that would imply a loss of contact, which in turn implies that one of the elements of $\mathbf{f}_{geo}$ has become positive.

The physical interpretation of the sufficient conditions is that the contacts with zero wrench intensities can do no work on the workpiece and therefore cannot affect its stability. Equivalently, kinematically admissible infinitesimal displacements of the workpiece increase its potential energy. Displacements that maintain all contacts increase the potential energy at a rate proportional to the square of the elements of the displacements, while those that break at least one active contact increase the potential energy in direct proportion to the elements.

Thus we define a workpiece configuration to be second-order stable if each kinematically admissible perturbation that maintains all the contacts increases the potential energy of the workpiece at a rate proportional to the square of elements of the perturbation.

Conditions for Second-Order Stability

$$\mathbf{f}_{geo} = \mathbf{0} \quad (14)$$

$$\mathbf{W}_n\mathbf{c}_n = -\mathbf{g}_{obj} \quad (15)$$

$$\mathbf{c}_n \geq \mathbf{0} \quad (16)$$

$$\boldsymbol{\lambda}_+ > \mathbf{0} \quad (17)$$

Henceforth, we consider only stable configurations, so we drop the asterisk superscripts.

5 Second-Order Stability Cells

The workpiece can have second-order stability only if the projected Hessian, $\mathbf{P}_+$, is positive definite. This in turn, implies that the rank of $\mathbf{W}_{n+}$ must be less than n_q. Therefore, the number of contacts with nonzero contact forces, n_{c+}, must be less than n_q, but the total number of contacts, n_c, is unconstrained. Thus we must consider all of the following three cases:

- *Case 1:* $Rank(\mathbf{W}_n) = n_c$; $\quad 1 \leq n_c < n_q$.
- *Case 2:* $Rank(\mathbf{W}_n) = n_q$; $\quad n_q \leq n_c \leq n_q + n_\theta$.
- *Case 3:* $Rank(\mathbf{W}_n) = r$; $\quad r < min\{n_c, n_q\}$; $1 \leq n_c \leq n_q + n_\theta$.

Note that we do not consider cases for which there are more than $n_q + n_\theta$ contacts, because such cases only occur under special geometric circumstances [17]. Also, because of space limitations, we will only consider the first case here. Details of the second and third cases can be found in [18].

5.1 Case 1 SS-cells

Since $\mathbf{W}_n$ is full rank, we can write the workpiece equilibrium equation as follows:

$$\begin{bmatrix} \mathbf{W}_I(\mathbf{q}, \boldsymbol{\theta}) \\ \mathbf{W}_{III}(\mathbf{q}, \boldsymbol{\theta}) \end{bmatrix} \mathbf{c}_n = \begin{bmatrix} -\mathbf{g}_I \\ -\mathbf{g}_{III} \end{bmatrix} \quad (18)$$

where the matrix, $\mathbf{W}_I \in \mathcal{R}^{n_c \times n_c}$ is nonsingular, $\mathbf{W}_{III} \in \mathcal{R}^{(n_q - n_c) \times n_c}$ contains the rows of $\mathbf{W}_n$ not in $\mathbf{W}_I$, $\mathbf{g}_I \in \mathcal{R}^{n_c}$, and $\mathbf{g}_{III} \in \mathcal{R}^{n_q - n_c}$ are the corresponding partitions of the external wrench, $\mathbf{g}_{obj}$. Solving for $\mathbf{c}_n$ yields:

$$\mathbf{c}_n = -Adj(\mathbf{W}_I)\mathbf{g}_I / Det(\mathbf{W}_I) \quad (19)$$

where Adj and Det are the matrix adjoint and determinant operators. Partitioning $\mathbf{c}_n$ into vectors of zero-valued and nonzero-valued elements, gives the vector-valued physical C-functions, $\mathbf{f}_{phy1}$ and $\mathbf{h}_{phy1}$ as:

$$\mathbf{f}_{phy1}(\mathbf{q}\ ,\boldsymbol{\theta}\ ) = \mathbf{c}_{n0} = -\mathbf{E}_0 Adj(\mathbf{W}_I)\mathbf{g}_I/Det(\mathbf{W}_I) = \mathbf{0} \quad (20)$$

$$\mathbf{h}_{phy1}(\mathbf{q}\ ,\boldsymbol{\theta}\ ) = \mathbf{c}_{n+} = -\mathbf{E}_+ Adj(\mathbf{W}_I)\mathbf{g}_I/Det(\mathbf{W}_I) > \mathbf{0} \quad (21)$$

where $\mathbf{E}_0$ and $\mathbf{E}_+$ are matrices composed of zeros and ones that select the elements of $\mathbf{c}_n$ which are equal to or greater than zero, respectively. Note that all elements of $\mathbf{c}_n$ are uniquely determined by the configuration of the system. Therefore all configurations corresponding to Case 1 can be represented uniquely in C-space. This implies that second-order stable configurations can be maintained during manipulation without active compliant control.

The lower partition of equation (18) must also be included in the set of equations defining an SS-cells. Thus we define an additional vector, $\mathbf{f}_{phy3}$, of physical C-functions:

$$\mathbf{f}_{phy3}(\mathbf{q}\ ,\boldsymbol{\theta}\ ) = \mathbf{g}_{III} - \mathbf{W}_{III} Adj(\mathbf{W}_I)\mathbf{g}_I/Det(\mathbf{W}_I) = \mathbf{0}. \quad (22)$$

The constraint that the eigenvalues of $\mathbf{P}_+$ be strictly positive yields more physical C-functions:

$$\mathbf{h}_{phy2} = \boldsymbol{\lambda}_+ > \mathbf{0} \quad (23)$$

where the vector $\boldsymbol{\lambda}_+$ is the set of solutions to the equation $Det(\lambda_+ \mathbf{I} - \mathbf{P}_+) = \mathbf{0}$, which is a polynomial in the scalar λ_+ with coefficients that are functions of the components of $\mathbf{q}$ and $\boldsymbol{\theta}$. The degree of this characteristic polynomial is equal to the dimension of $\mathbf{P}_+$, which, for Case 1, is equal to $6 - n_c$ ($3 - n_c$ in the plane).

As will be shown below, a polyhedral workpiece with only two contacts cannot have second-order stability; at least one eigenvalue will be nonpositive. Thus there must be between three and five contacts, which implies that the degree of the characteristic polynomial between three and one (for a polygon in the plane, the degree must be one). Therefore when the eigenvalues are distinct, they can be written as explicit functions of the elements of $\mathbf{q}$ and $\boldsymbol{\theta}$ using the cubic formula yielding the following vector of physical C-functions:

$$\mathbf{h}_{phy2}(\mathbf{q}\ ,\boldsymbol{\theta}\ ) = \boldsymbol{\lambda}_+(\mathbf{q}\ ,\boldsymbol{\theta}\ ) > \mathbf{0}. \quad (24)$$

We can now define a Case-1 SS-cell as follows:

$$\mathcal{SS} = \mathcal{CF} \cap \mathcal{P} \quad (25)$$

where $\mathcal{CF}$ is the CF-cell in question and the set $\mathcal{P}$ is defined as

$$\mathcal{P} = \{(\mathbf{q}\ ,\boldsymbol{\theta}\ ) \in \mathcal{X} \mid \mathbf{f}_{phy} = \mathbf{0} \bigwedge \mathbf{h}_{phy} > \mathbf{0}\}, \quad (26)$$

where $\mathbf{f}_{phy}$ is formed by the vertical concatenation of $\mathbf{f}_{phy1}$ and $\mathbf{f}_{phy3}$, and $\mathbf{h}_{phy}$ is the concatenation of $\mathbf{h}_{phy1}$ and $\mathbf{h}_{phy2}$.

In the typical case, the dimension of an SS-cell is the difference between the dimension of C-space and the number of equations defining the "surface" from which the SS-cell was "carved" (by applying the inequalities, $\mathbf{h}_{phy} > \mathbf{0}$). The numbers of equations defining the surface is the sum of the lengths of $\mathbf{f}_{geo}$, $\mathbf{f}_{phy1}$, and $\mathbf{f}_{phy3}$; n_c, l (usually $l = 0$), and $n_q - n_c$, respectively, where l is the number of contacts with zero-valued contact forces. The elements of $\mathbf{f}_{geo}$ are included here even though they do not appear in the definition of $\mathcal{SS}_1$, because they are used in the definition of $\mathcal{CF}$ [17]. Since SS-cells are defined in the $(n_q + n_\theta)$-dimensional C-space, $\mathcal{X}$, we see that the dimension of Case-1 SS-cells is $n_\theta - l$. Interestingly, the dimensions of typical Case-2 and Case-3 SS-cells are also $n_\theta - l$ [18].

The most important implication of SS-cells normally having dimension n_θ (recall that l is usually zero) is that when position-controlling the manipulator, the motion of the workpiece can be determined unquely even though the kinematic constraints alone are insufficient to predict the motion.

5.2 Example: SS-cell

The above analysis was applied to the system shown in Figure 2. A portion of the two-contact CF-cell with vertices a and c contacting the left and right fingers, respectively, is stable. This portion is shown projected onto joint space. The configuration of the system shown on the right side of the Figure emphasizes that we have not enforced the constraints that the contacts lie on the fingers. Note that this SS-cell is connected to the three-contact FS-cell discussed in [17] along a portion of Curve 1. Along this curve, the determinant of the 3×3 wrench matrix, $\mathbf{W}_n$, of the three-contact CF, is zero, which turns out to be quite significant in the algebraic geometry of CF-cells [3].

5.3 Minimum Numbers of Contacts

It is known that the minimum number of contacts required for first-order stability is n_q (six and three in the spatial and planar cases, respectively). Here we show that two contacts are necessary for second-order stability of a polygon in the plane and more than two are necessary for a polyhedron in three- space.

Lemma 1 *For a polygon restricted to planar motion and acted upon by gravity, at least two frictionless contacts are necessary to achieve second-order stability.*

Proof: Consider a polygon at rest and acted upon by gravity. If there are no contacts, the polygon will accelerate away from its initial configuration. Therefore at least one contact is needed to stabilize the polygon. However, we will now show that one contact is also insufficient. Note that by one contact we mean that one C-function, $f_{geo,i}$, is zero. Thus we must

Figure 2: SS-cell of a Two-Contact CF

consider only two cases: one type-A contact and one type-B contact. Contact between a convex and a concave vertex is considered two contacts, because two C-functions are zero.

Case A: The situation with one type-A contact is shown in Figure 3. Clearly, since equilibrium is a necessary condition for second-order stability, the contact normal must be opposite to the direction of gravity. The eigenvalues of $\mathbf{P}_+$ are given by: $\lambda_+ = \frac{h_{33} \pm \sqrt{h_{33}^2+4}}{2}$, where h_{33} is the bottom, right-hand element of the matrix $\mathbf{H}$. Clearly, one eigenvalue is guaranteed to be nonpositive, so the polygon is unstable. For the situation shown in Figure 3, h_{33} is nonzero. Physically, the negative eigenvalue corresponds to translation of the workpiece to the right combined with clockwise rotation about the original point of contact.

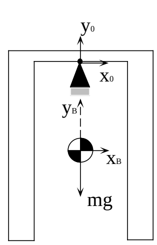

Figure 3: One Type-A Contact

Case B: The situation with one type-B contact is shown in Figure 4. The eigenvalues are 0 and d, with 0 corresponding to horizontal translation. .... □.

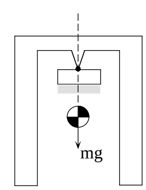

Figure 4: An Unstable Equilibrium Configuration with One Type-B Contact

Lemma 2 *For a polygon restricted to planar motion and acted upon by gravitational, two frictionless contacts are not sufficient for second-order stability.*

Proof: An object in two-point contact with a horizontal table top may be in equilibrium, but will has a zero eigenvalue corresponding to horizontal translation. □.

Theorem 1 *For a polygon restricted to planar motion and acted upon by a gravitational force, necessary and sufficient conditions to achieve second-order stability are:*

- *Both contact normals have components which oppose gravity (i.e., the y-components of the contact normals must be positive.*
- *The components of the contact normals perpendicular to gravity must oppose each other (i.e., the x-components of the contact normals must have opposite signs.*

- *The contact normals must intersect at a point on the line of action of the gravitational force.*
- *The center of mass of the workpiece must lie below the critical point, $\mathbf{y}_{crit}$, which is a point on the line of action of the graviational force a distance d_{crit} above the intersection of the two contact normals:*

$$d_{crit} = \frac{(j_1 n_{2x}|\mathbf{r}_1| - j_2 n_{1x}|\mathbf{r}_2|)}{(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)_z} \qquad (27)$$

where j_i is -1 if contact i is of type A and +1 if it is of type B, n_{ix} is the horizontal component of the normal at contact i, $|r_i|$ is the distance from the intersection of the contact normals to contact i, and $(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)_z$ is the out-of-plane component of the cross product of the contact normals in the order shown (see Figure 5).

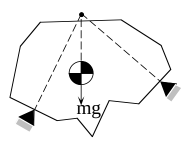

Figure 5: A Stable Equilibrium Configuration with Two Contacts

Proof: The first three necessary and sufficient conditions result directly from solving the equilibrium equation (16), subject to the requirement that the contact forces be compressive, inequality (17).

The fourth condition was derived using the definitions of the C-functions $f_{geo,i}$ given in [4]. Choosing the coordinate frames of the world and workpiece to coincide with their origins at the point of intersection of the contact normals, the left null space of the Jacobian matrix, $\frac{\partial \mathbf{f}_{geo}}{\partial \mathbf{q}}$, (the wrench matrix) yields $\mathbf{Z}^T = [0\ 0\ 1]$. This expresses the fact that the point on the workpiece coincident with the intersection of the contact normals can only rotate instantaneously. The matrix $\mathbf{P}$ then becomes the scalar

$$\mathbf{P} = -(\frac{\partial^2 f_{geo,1}}{\partial q_3^2} c_{1,n} + \frac{\partial^2 f_{geo,2}}{\partial q_3^2} c_{2,n}) \qquad (28)$$

where q_3 is the orientation of the workpiece.

Given that the workpiece is in equilibrium, the wrench intensities at the contacts are given by: $c_{1,n} = -n_{2x} mg/(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)_z$ and $c_{2,n} = n_{1x} mg/(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)_z$, where $n_{1,x}$ and $n_{2,x}$ are the x-components of the contact normals. Substituting in the required second partial derivatives of the C-functions yields:

$$\mathbf{P} = -mg y_{cg} + \frac{mg(j_1 n_{2x}|\mathbf{r}_1| - j_2 n_{1x}|\mathbf{r}_2|)}{(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)_z} \qquad (29)$$

where y_{cg} is the y-component of the center of mass of the workpiece relative to the intersection point of the contact normals. Noting that for stability, $\mathbf{P}$ must be strictly positive and that mg is positive leads to the last condition given in the Theorem. .... □.

The most interesting implication of Theorem 1 is that while the instantaneous center of rotation of the workpiece is the point of intersection of the contact normals, that point's instantaneous acceleration is not purely angular. Note that the center of mass can lie above the intersection of the contact normals when both contacts are of type A.

Corollary 1 *For a polyhedral workpiece acted upon by a gravitational force, at least three frictionless contacts are necessary to achieve second-order stability.*

Proof: A necessary condition for a polyhedron to be in equilibrium with one frictionless contact is that the contact lie on the line of action of the gravitational force with its normal directly opposite the direction of gravity. Given this situation, consider the motion of the polyhedron in a plane containing the gravitational force. It is clear from the proof of Lemma 1, that the polyhedron cannot be stable.

A necessary condition for a polyhedron to be in equilibrium with two frictionless contacts is that they lie in a plane containing the gravitational force. This being so, perturbations of the workpiece out of this plane cannot be resisted by restoring forces. ..... □.

6 Conclusion and Future Research

We have introduced and formalized the concepts of *second-order stability* and *second-order stability cells.* Second-order stability cells are subsets of the C-space of a manipulation system for which a frictionless workpiece is stable with three to five frictionless contacts (two for planar systems). Second-order stable grasp are less stable than form-closure grasps, but can be useful in manipulation tasks before form-closure can be attained or when the task requires mobility rather than security (as is the case is dexterous manipulation). In the planar case, we have derived simple stability condition that can be tested graphically.

In future work, we plan to study two problems. First, the mating of slippery parts using second-order stable grasps. One can see the potential usefulness of this idea as follows. Assume that one part is fixtured and a second part is transported with positional errors to its mating location. When the parts first make contact, it will be possible for the assembly task to fail through jamming, because the additional contact can result in a form-closure grasp of the workpiece. However, if the part is carried to its mating position using a second-order stable grasp, then the first additional contact cannot lead to form-closure, because there will not be enough contacts. This suggests the problem of determining mating trajectories such that unplanned premature contacts result in error-reducing motions, like those achieved by Whitney's remote center of compliance device.

The second problem left to future work is to delineate the domain of applicability of our quasistatic, frictionless, rigid body model. As stated above, under certain conditions for planar systems, the motion predicted by our frictionless quasistatic model is a good approximation of the motion predicted by the quasistatic frictional model. We have also observed in some situations, that the motion predicted by a dynamic rigid body model with friction corresponds to a curve in C-space very close to the second-order stability cell. We plan to analytically characterize these situations for planar and spatial systems.

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