On the Stability and Instantaneous Velocity of Grasped Frictionless Objects

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Abstract—**Grasp and manipulation planning of slippery objects often relies on the "form closure" grasp, which is stable regardless of the external force applied to the object. Despite its importance, an efficient quantitative test for form closure valid for any number of contact points has not been available. The primary contribution of this paper is the introduction of such a test formulated as a linear program, the optimal objective value of which provides a measure of how far a grasp is from losing form closure. When the grasp does not have form closure, manipulation planning requires a means to predict the object's stability and instantaneous velocity, given the joint velocities of the hand. The "classical" approach to computing these quantities is to solve the systems of kinematic inequalities corresponding to all possible combinations of separating or sliding at the contacts. All combinations resulting in the interpenetration of bodies or the infeasibility of the equilibrium equations are rejected. The remaining combination (sometimes there are more than one) is consistent with all the constraints and is used to compute the velocity of the manipulated object and the contact forces, which indicate whether or not the object is stable. Our secondary contribution is the formulation of a linear program whose solution yields the same information as the classical approach. The benefit of this formulation is that explicit testing of all possible combinations of contact interactions is usually avoided by the algorithm used to solve the linear program.**

I. Introduction

In grasp and manipulation planning, the two most important classes of grasps are known as "form closure" and "force closure" grasps. These terms are borrowed from the field of machine design in which they have been in use since 1875 when Reuleaux [23] studied the mechanics of some "early machines." One machine was the water wheel, whose axel was usually laid in a groove of semicircular cross section. The proper operation required the weight of the wheel to maintain or "close" the contact between the groove and the axel. Thus, the terminology "force closure" came to describe contacts whose maintenance depended on an externally applied force. If, instead, the contact was maintained by virtue of the geometry of the contacting elements (as would be the case of an axel in a cylindrical hole), then the term "form closure" was adopted. This terminology is still in use today in the mechanisms research community (see [11]) and was first introduced into the robotics research community by Salisbury [24]. Since then, motivated by the mathematical interpretation of "closure," some authors (notably, Nguyen [18], Mishra *et al.* [17], and Li [12]) have chosen to use "force closure" to mean what Reuleaux and Salisbury meant by form closure. In this paper, we follow the precedent set by Reuleaux and Salisbury by adopting the following definitions.

Definition: Form Closure: A fixed set of contacts on a rigid body is said to exhibit *form closure* if the body's equilibrium is maintained despite the application of any possible externally applied wrench (force and moment). Equivalently, the contacts prevent all motions of the body, including infinitesimal motions.

Definition: Force Closure: A fixed set of contacts on a rigid body is said to exhibit *force closure* if the maintenance of the body's equilibrium requires the application of an externally applied wrench. Equivalently, the contacts do not prevent all infinitesimal motions of the body.

Note that, in the remainder of this paper, we will use the words "body" and "object" interchangeably.

While the open literature abounds with papers on grasping and grasp planning (see [21] for a good bibliography of grasping literature published before 1988), an efficient quantitative test for form closure valid for any number of contact points is not available. Reuleaux [23] studied the form-closure problem for rigid lamina restricted to move in a plane. He showed that at least four higher-pair (point) contacts were required to prevent all motion of the lamina. He also provided a graphical technique to test a set of four contacts for form closure. These ideas were used by Nguyen to develop algorithms to synthesize form-closure grasps of given rigid lamina and were extended for use with three-dimensional objects [18]. The conditions for form closure of an arbitrary three-dimensional rigid body were first given in 1900 by Somoff [26], who established that a minimum of seven point contacts was necessary. Much later, Lakshminarayana [11] described an approach to synthesizing form-closure grasps of three-dimensional frictionless objects, and he gave an insightful physical interpretation of the associated equations.

Mishra *et al.* [17] were the first to prove the existence of a small upper bound on the number of discrete points needed for a form-closure grasp of a frictionless object. They showed that if the object was "nonexceptional" (i.e., the object's surface was not one of revolution), then $2n_q$ contact points were sufficient to balance all possible external wrenches, where n_q is the number of degrees of freedom of motion of

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the uncontacted object. This bound, however, seemed loose to Markenschoff *et al.* who succeeded in "closing the gap" [15]. They proved the stronger result that any nonexceptional frictionless object can be grasped in form closure with only seven contact points. They stated that their proof, which was based on infinitesimal perturbations of the contact points away from the maximal inscribed sphere, can be used as the basis for algorithms for synthesizing form-closure grasps, and it would seem reasonable, too, that form-closure tests could also be developed, but no algorithms were presented. For the purpose of grasp synthesis, Nguyen [18] and Mishra *et al.* [17] developed grasp tests that indicated only the existence or nonexistence of form closure. However, the binary nature of the tests motivated Kirkpatrick *et al.* [10] to formulate a quantitative test for "positive grips" with form closure based on Steinitz's theorems. Unfortunately, these results are restricted to frictionless grasps of polyhedra with at least $2n_q$ contacts occurring only at "nonsingular" points on the object's surface, where singular points are those for which the surface normal is ill-defined. These restrictions are seen as significant drawbacks since in dexterous manipulation it is common (and occasionally desirable) for fewer contacts to occur and for some of them to be on vertices of the object.

A. Contributions

The primary contribution of this paper is the formulation of a quantitative test, based on purely geometric information, for detecting form closure. This test takes the form of a linear program and produces a crude measure (qualitatively similar to Kirkpatrick's) of how "far" a grasp is from losing form closure. In contrast to Kirkpatrick's test, our test is valid for frictionless grasps with any number of contacts as long as their locations and normal directions are known. The problem of contacts occurring at nondifferentiable surface points is not a consideration here, because a unique, computable normal is available in all but the ephemeral and pathological cases of a convex vertex in contact with either a convex edge (spatial case only) or another convex vertex. The test is also valid for frictional grasps, but due to its dependence on purely geometric quantities, does not quantify the stabilizing friction effects. However, the test can be modified to quantify the friction effects [31].

The maintenance of form closure during manipulation requires compliant control of the hand. In this situation, computing the velocity of the object is straightforward since the applicable kinematic constraints must be chosen to perform the compliant motion. To maintain force closure, compliant motion is not required, so the applicable kinematic constraints are not known. As a result, the determination of the object's motion due to finger motions typically proceeds via the "classical" approach in which each possible set of applicable kinematic constraints is hypothesized and then tested for consistency with the equilibrium relationships. The secondary contribution of this paper is the derivation of a linear program that can be used to predict the instantaneous velocity of a frictionless object more efficiently than solution by the classical approach. Based on the linear program, we prove sufficient conditions for characterizing the object's stability and for the uniqueness of the object's instantaneous velocity and the contact forces during manipulation.

B. Paper Layout

This paper is organized as follows. In Section II, we present our assumptions and derive a linear program that can be used to predict the quasi-static motion of a grasped frictionless object. In Section III, we consider the stability of form- and force-closure grasps and develop our quantitative form-closure test. We also identify a subclass of force-closure grasps called *strong force-closure* grasps and develop a test for their recognition. Section IV is devoted to indeterminacies in the linear program derived in Section II; we present several theoretical results pertaining to object stability and to the existence and uniqueness of the contact forces and the object's velocity.

II. Quasi-Static Frictionless Mechanics

In this section, we derive a linear program that can be used to predict the instantaneous quasi-static motion of a frictionless object, where by "quasi-static" we imply that dynamic effects are negligible [16]. Our derivation depends on the following list of assumptions, which are quite restrictive. In ongoing work [32], the assumptions of perfect knowledge and control have been removed, and linear approximations of the effects of errors on the velocity of the object and the contact forces have been derived.

Assumptions:

1) All bodies are rigid polyhedra.
2) The surface geometry and the position of the center of gravity within each body are known.
3) Friction is negligible.
4) The object is acted upon by a gravitational force.
5) Dynamic effects are negligible.
6) The points of contact and their normal directions are known.
7) The kinematic parameters of the manipulators are known.
8) Controller errors are negligible.

Let the position and orientation of the grasped object, relative to an inertial frame, be denoted by the vector $\boldsymbol{q} \in E^{n_q}$, where E^m represents m-dimensional Euclidean space and n_q is the number of degrees of freedom of motion of the uncontacted object (six in the spatial case). Then $\boldsymbol{q}$ represents the position of the origin of a body-fixed frame (not necessarily coincident with the object's center of mass) and the orientation of its axes, expressed with respect to the inertial frame. Further, we define $\boldsymbol{\theta} \in E^{n_\theta}$ to be the vector of joint displacements of the hand, where n_θ is the number of joints. Together $\boldsymbol{q}$ and $\boldsymbol{\theta}$ define the configuration of the hand/object system. Assuming that the hand is composed of a set of independent manipulators in point contact with a single object, then the dynamic model of the hand can be written by combining the equations of motion for the individual fingers to yield

$$\tau = \boldsymbol{M}_{\text{hand}}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \boldsymbol{V}_{\text{hand}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \boldsymbol{G}(\boldsymbol{\theta}) + \boldsymbol{J}_n(\boldsymbol{\theta}, \boldsymbol{q})^T \boldsymbol{c}_n \quad (1)$$

where $\tau \in E^{n_\theta}$ is the vector of joint torques, $M_{\text{hand}} \in E^{n_\theta \times n_\theta}$ is the positive definite inertia matrix, $V_{\text{hand}} \in E^{n_\theta}$ is the vector of Coriolis and centripetal torques, $G \in E^{n_\theta}$ is the vector of gravity torques, $J_n \in E^{n_\theta \times n_c}$ is the global grasp Jacobian matrix [9] relating normal force components (or wrench intensities) at the contacts to joint torques, $c_n \in E^{n_c}$ is the vector of the normal components of the forces acting at the contacts, the superscript T represents the matrix transposition operation, and the overdot indicates differentiation with respect to time.

The dynamics of the grasped rigid object are given by the Newton–Euler equation

$$W(q,\theta)_n c_n + g_{\text{ext}} = M_{\text{obj}}\ddot{q} + V_{\text{obj}}(q,\dot{q}) \tag{2}$$

where $W_n \in E^{n_q \times n_c}$ is the global grasp wrench matrix [9], $M_{\text{obj}} \in E^{n_q \times n_q}$ is the inertia matrix of the object, $V_{\text{obj}} \in E^{n_q}$ is the vector of angular velocity products arising in Euler's equation, and $g_{\text{ext}} \in E^{n_q}$ is the external wrench acting on the object.

The dynamic differential equations (1) and (2) are coupled through contacts that give rise to kinematic velocity constraints [9] that prevent the interpenetration of bodies during motion

$$W_n^T \dot{q} - J_n \dot{\theta} \geq 0. \tag{3}$$

Neglecting the inertial terms in (1) and (2) yields the equilibrium equations, which must be satisfied by all quasi-static system motions:

$$J_n^T c_n = \tau - G \tag{4}$$

$$W_n c_n = -g_{\text{ext}} \tag{5}$$

$$c_n \geq 0 \tag{6}$$

where inequality (6) implies that contact loads must be compressive.

Assuming that the hand controller is always stable, (3), (5), and (6) can be used, in what we call the "classical" approach, to determine the quasi-static, instantaneous velocity of the object caused by the instantaneous joint velocities. This is accomplished by considering all 2^{n_c} possible combinations of contact interactions (i.e., sliding or breaking). For each combination, one must solve the for the object velocity $\dot{q}$, check that the contacts presumed to be breaking satisfy the kinematic constraints (3), and finally test the feasibility of the equilibrium equations, (5) and (6). As an alternative to the classical approach, we now derive a linear program whose solution is identical to that of the classical approach.

Let r be the homogeneous vector representing the position of the center of gravity of the object expressed with respect to its body-fixed frame, and let $T(q)$ represent the homogeneous transformation describing the position and orientation of the body-fixed frame with respect to the inertial frame. Further, let g be the homogeneous form of the gravitational acceleration vector, and let m be the mass of the object. Given these definitions and the assumptions stated above, stable equilibrium configuration(s), denoted by q^*, of an object within a hand of fixed known configuration θ^* can be determined as the solution(s) to the following optimization problem:

$$\min_q \quad y = -m r^T T(q)^T g \tag{7}$$

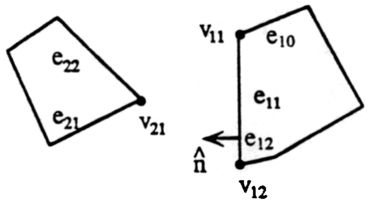

Fig. 1. Two convex polygons.

subject to

$$q \in C_{\text{valid}}(\theta^*) \tag{8}$$

where the objective function is equivalent to the usual mgh, but was written in the above form to expose the homogeneous transformation matrix. C_{valid} is the subset of configuration space corresponding to nonpenetrating configurations of the hand and object, i.e., the geometrically valid configurations. This subset is the union of the sets normally called the "free" space and "contact" space and can be defined through various logically linked combinations of C-function inequalities [4]. The C-functions, introduced by Lozano-Perez [13], represent the relative proximity of the vertices, edges, and faces of the object and hand. For example, consider the two convex polygons shown below (see Fig. 1). The C-function $f(q,\theta^*)$ is the distance between edge e_{11} and vertex v_{21} (measured from e_{11} to v_{21} along the outward-pointing normal, $\hat{n}$). A necessary condition for contact is that f equal zero. However, to restrict the contact to lie between the ends of e_{11}, the C-functions relating e_{10} to v_{21} and e_{12} to v_{21} must be nonpositive. Also, to ensure that the polygons do not interpenetrate, the C-functions relating e_{22} to v_{11} and e_{21} to v_{12} must be nonnegative. Naturally, v_{21} may separate from e_{11}. This condition is indicated by f becoming positive, at which point the additional four constraints discussed above become meaningless. Thus, in the planar case, five C-functions, all nonlinear functions of q and θ, are required to define all valid contacts between an edge and a vertex of convex polygons. C-function constraints for the spatial case are more numerous and more complicated in form, but their use in defining valid contacts is conceptually identical to their use in the planar case. Note that the number of C-functions required to define C_{valid} is the number of object and hand features that could ever possibly be in contact. However, the following development will show that, to determine the instantaneous motion or stability of the hand/object system, we need only consider the current set of contacts.

The nonlinear problem formulation given by relationships (7) and (8) cannot be used effectively for the prediction of the quasi-static motion of frictionless objects partially because the representation of the set C_{valid} is not conducive to efficient optimization techniques and partially because, when the configuration of the grasped frictionless object is "far" from an equilibrium configuration, its motion will be dominated by dynamic effects. These problems can be alleviated by employing the following line of reasoning. Since quasi-static dexterous manipulation will take place slowly, the object will never be "far" from a stable equilibrium configuration (unless it becomes unstable). To determine the effect on the nonlinear optimization problem, suppose that the system is in a stable state defined by q^* and θ^* and further suppose that the system is subjected to an infinitesimal joint angle perturbation $d\theta$. In the example discussed above, only the C-function describing the distance between the elements e_{11} and v_{21} constrains the perturbations of q. Since, at a contact, f equals zero, then valid differential changes in q and θ are constrained by the inequality $df \geq 0$. The other constraints, however, are not "tight," and thus will not be violated by any differential perturbation. Therefore, in linearizing the nonlinear optimization problem, we need only consider those zero-valued C-functions corresponding to the actual contacting elements. Thus, the perturbation in the object's configuration dq corresponding to the perturbation in the hand's configuration $d\theta$ can be determined by solving the following linearized problem:

$$\min_{dq} \quad -m\,dq^T T_v^T(q) g \tag{9}$$

subject to

$$f(q^*, \theta^*) + \left.\frac{\partial f}{\partial q}\right|_{\substack{q=q^* \\ \theta=\theta^*}} dq + \left.\frac{\partial f}{\partial \theta}\right|_{\substack{q=q^* \\ \theta=\theta^*}} d\theta \geq 0 \tag{10}$$

where $T_v(q)$ is the velocity transformation matrix relating the velocity of the center of mass to that of the origin of the body-fixed frame and f is the set of zero-valued C-functions corresponding to the contacts. The objective function now represents the differential change in the potential energy due to a differential change in its configuration. Note that associating T_v^T with g allows us to interpret dq as the (spatial transpose of the) differential twist of the body and the product $mT_v^T(q)g$ as the external wrench g_{ext} acting on the body [8]. Therefore, the objective function can be viewed as the virtual work expended on the object.

Realizing that, for contact points, the corresponding elements of $f(q^*, \theta^*)$ are zero, the above linear program reduces to the following one:

$$\min_{dq} \quad -dq^T g_{\text{ext}} \tag{11}$$

subject to

$$\left.\frac{\partial f}{\partial q}\right|_{\substack{q=q^* \\ \theta=\theta^*}} dq + \left.\frac{\partial f}{\partial \theta}\right|_{\substack{q=q^* \\ \theta=\theta^*}} d\theta \geq 0 \tag{12}$$

where f represents the current set of active nonintersection constraints (i.e., contacts). Here we note that the rows of $\partial f/\partial q$ are the unit wrenches corresponding to the normal components of the velocities of the contact points on the object [1]. Similarly, the rows of $\partial f/\partial \theta$ are the negatives of the rows of the Jacobian matrices corresponding to the normal components of the velocities of the contact points on the hand [20]. Next, if we view q and θ as functions of time, then we may rewrite linear program (11) and (12) in terms of time derivatives as follows

$$\min_{\dot{q}} \quad -\dot{q}^T g_{\text{ext}} \tag{13}$$

subject to

$$W_n^T(q^*, \theta^*)\dot{q} \geq J_n(q^*, \theta^*)\dot{\theta}. \tag{3'}$$

This formulation is called the *velocity formulation* of the frictionless object motion problem and can also be derived from Peshkin's minimum power principle [29]. Given the instantaneous joint velocities $\dot{\theta}$, the solution of this linear program provides not only the instantaneous velocity of the object $\dot{q}$ but also the wrench intensities and the nature of the contact interactions. These may be determined by noting that the elements of the vector of Lagrange multipliers (i.e., the optimal dual variables) associated with inequality (3) are the wrench intensities. While not considered in detail here, the effects of errors in $\dot{\theta}$ (control errors) can be readily determined through solution sensitivity analyses, which are well developed in the field of linear programming [14]. However, it is quite common during manipulation that W_n is nonsingular, in which case the exact error relationship is given as

$$\delta\dot{q} = W_n^{-T} J_n \delta\dot{\theta} \tag{14}$$

where $\delta\dot{q}$ is the error in the predicted object velocity due to the error $\delta\dot{\theta}$ in the joint velocity.

The dual linear program, called the *force formulation*, is written in terms of the unknown vector of wrench intensities c_n and is stated as follows:

$$\max_{c_n} \quad \dot{\theta}^T J_n^T c_n \tag{15}$$

subject to

$$W_n c_n = -g_{\text{ext}} \tag{5}$$

$$c_n \geq 0. \tag{6}$$

In this formulation, the input $\dot{\theta}$ has moved to the objective function, the primal variables are the elements of the wrench intensity vector, and the vector of dual variables is equivalent to $\dot{q}$. According to the theory of linear programming, both formulations are equivalent, so either one may be solved for the wrench intensities and the instantaneous velocity of the object. The theory of linear programming also allows us to deduce an important property of these formulations as follows. The primal's objective function represents the power expended in lifting the object's center of gravity, while the dual's represents the power input by the contact wrenches. The duality theorem of linear programming states that the primal and dual objective functions are equal at feasible bounded solutions. Thus, we see that the linear programs derived above predict motions that conserve energy [14]. Also, since primal and dual constraints are satisfied at a feasible optimal solution, solving either formulation results in a solution satisfying both the kinematic constraints and the equilibrium equations. Note that the error in the wrench intensities δc_n due to errors in the external wrench δg_{ext} can also be determined using sensitivity analysis. In the case that W_n is nonsingular, the expression analogous to (14) is

$$\delta c_n = -W_n^{-1} \delta g_{\text{ext}}. \tag{16}$$

We conclude this section with one final consideration. Smale [25] has shown that the "average" time complexity of the simplex algorithm in solving linear programs with a fixed number of constraints is proportional to the the number of unknowns. Since the force formulation of the object motion problem always has n_q constraints, we expect the average solution time to be proportional to n_c, the number of contacts, also the number of unknown elements in the wrench intensity vector $\boldsymbol{c}_n$. To determine the instantaneous velocity of a quasi-statically manipulated object via the classical analysis, it is necessary to consider 2^{n_c} systems of equations and inequalities. Even chosing to forgo the possibility of degenerate solutions one must still solve the $\binom{n_c}{n_q}$ systems corresponding to the possible ways to maintain n_q contacts. This number, $\binom{n_c}{n_q}$, is also the maximum number of systems solved by the simplex method (through basis change operations) in the worst case for both formulations. Therefore, for grasp configurations with a large numbers of contacts, one would expect the solution of the force formulation via the simplex algorithm to be much more efficient than solutions computed via an algorithm based on the "classical" solution approach.

These two approaches were implemented in "C" using code provided in [22] and were used to compare the computation time for 10 000 frictionless planar grasps with from 3 to 18 contacts per grasp. The computation time for the solution of the force formulation increased linearly with n_c while the computation time for the solution of the classical formulation appeared to increase exponentially. We found, however, that the solution of the linear programming algorithm imposed a large overhead cost, so that, on average, grasps with nine or fewer contacts, were solved more quickly by the classical method. Grasps with more than nine contacts were solved more quickly using the linear programming algorithm (10 times faster for grasps with 18 contacts).

III. Stability

In grasp and dexterous manipulation planning, the qualities with which we are most concerned are known as form closure and force closure. Form-closure grasps (or the objects so grasped) are stable in the face of all possible external wrenches acting on the grasped object, whereas force closure grasps are only stable for a subset of all possible external wrenches [11], [16], [23]. One reason that this distinction is important is that maintaining form closure requires some form of compliant control.

To help distinguish between form- and force-closure grasps, in Section III-A, we present a new quantitative test for form closure, the output of which indicates whether or not a grasp has form closure, and if it does have form closure, the objective value provides a crude measure of how far the grasp is from losing form closure. In Section III-B, we show that the existence of a unique, bounded, feasible solution to the velocity formulation of the frictionless object motion problem is sufficient for stability. In Section III-C, we define strong force-closure grasps and present a test for its identification and quantification. Finally, in Section III-D, we present sufficient conditions for the instability of force-closure grasps based on the linear programming formulation of the object motion problem.

A. Form Closure

According to the definition of Reuleaux and Salisbury, given in the Introduction, a grasp has form closure if and only if object equilibrium is possible regardless of the external wrench, that is, the following relationships hold:

$$\boldsymbol{W}_n\boldsymbol{c}_n = -\boldsymbol{g}_{\text{ext}}, \qquad \text{for all } \boldsymbol{g}_{\text{ext}} \in E^{n_q} \tag{5}$$

$$\boldsymbol{c}_n \geq \boldsymbol{0}. \tag{6}$$

The dual statement of form closure was previously derived [1], [11], [16], [19] using the concepts of contrary, reciprocal, and repelling screws. However, it can also be derived from the "theorem of the alternative," which can be stated as follows.

Theorem 1 (from Strang [27]): Given the matrix $\boldsymbol{A}$, $\boldsymbol{x}$, and $\boldsymbol{z}$, either $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution with all elements of $\boldsymbol{x}$ nonnegative or else there exists a vector $\boldsymbol{z}$ such that $\boldsymbol{z}^T\boldsymbol{A} \geq \boldsymbol{0}$ and $\boldsymbol{z}^T\boldsymbol{b} < 0$.

Replacing $\boldsymbol{A}$, $\boldsymbol{x}$, and $\boldsymbol{b}$ with $\boldsymbol{W}_n$, $\boldsymbol{c}_n$, and $-\boldsymbol{g}_{\text{ext}}$, respectively, we see that either a solution to the equilibrium equations exists such that all elements of $\boldsymbol{c}_n$ are nonnegative or a vector $\boldsymbol{z}$ exists satisfying the following relationships:

$$\boldsymbol{z}^T\boldsymbol{W}_n \geq \boldsymbol{0} \tag{17}$$

$$-\boldsymbol{z}^T\boldsymbol{g}_{\text{ext}} < 0. \tag{18}$$

Interpreting $\boldsymbol{z}$ as the instantaneous velocity $\dot{\boldsymbol{q}}$ of the grasped object implies the following. If there exists an external wrench $\boldsymbol{g}_{\text{ext}}$ such that equilibrium is infeasible, then there must exist a kinematically admissible motion that will reduce the object's potential energy. For form closure, however, the contact points must balance every possible external wrench. As a consequence, the theorem of the alternative implies that no $\dot{\boldsymbol{q}}$ may exist that satisfies the following system of linear inequalities:

$$\boldsymbol{W}_n^T\dot{\boldsymbol{q}} \geq \boldsymbol{0} \tag{19}$$

$$-\boldsymbol{g}_{\text{ext}}^T\dot{\boldsymbol{q}} < 0 \tag{20}$$

which, in turn, implies that a grasp has form closure if and only if inequality (19) admits only the trivial solution.[1] Since inequality (19) represents the special case of inequality (3), with $\dot{\boldsymbol{\theta}}$ set equal to zero, we may restate form closure as follows: a grasp is said to have form closure if, when locking the hand's joints, it is impossible to move the object, even infinitesimally, regardless of the external wrench applied to the object.

From Somoff's work, a necessary condition for form closure is that the wrench matrix $\boldsymbol{W}_n$ has more columns than rows and, therefore, a nontrivial null space. Thus, we may rewrite the form closure requirements as follows:

$$\boldsymbol{W}_n\boldsymbol{c}_{n,\text{row}} = -\boldsymbol{g}_{\text{ext}}, \qquad \text{for all } \boldsymbol{g}_{\text{ext}} \in E^{n_q} \tag{21}$$

$$\boldsymbol{W}_n\boldsymbol{c}_{n,\text{null}} = \boldsymbol{0} \tag{22}$$

$$\boldsymbol{c}_{n,\text{row}} + \boldsymbol{c}_{n,\text{null}} \geq \boldsymbol{0} \tag{23}$$

where $\boldsymbol{c}_{n,\text{null}} \in E^{n_c}$ and $\boldsymbol{c}_{n,\text{row}} \in E^{n_c}$ are elements of the null and row spaces of $\boldsymbol{W}_n$, respectively. With relationships

[1] To see that this is so, suppose that a nonzero $\dot{\boldsymbol{q}}$ exists satisfying inequality (19). Then inequality (20) defines all external wrenches that cannot be balanced by the grasp.

(21)–(23) in mind. Salisbury showed that a sufficient condition for form closure is the existence of a vector c_n, with all positive elements. This is equivalent to the result of Mishra *et al.* requiring that the origin of the wrench space lie strictly within the convex hull defined by the columns W_n [17]. Next we use the facts that W_n and its pseudoinverse provide one-to-one and onto mappings between the spaces of g_{ext} and $c_{n,row}$, and that g_{ext} is arbitrary. As such, any or all elements of $c_{n,row}$ can be made negative by the proper choice of g_{ext}. This places the onus of c_n's nonnegativity squarely on $c_{n,null}$. Therefore, form closure requires that (22) admit at least one strictly positive solution, i.e., the following relationships must be feasible:

$$W_n c_{n,\mathrm{null}} = \mathbf{0} \tag{22}$$

$$c_{n,\mathrm{null}} > \mathbf{0} \tag{24}$$

If no such solution exists, then one can easily find a g_{ext} that cannot be balanced. If a strictly positive $c_{n,null}$ does exist, then it may be arbitrarily scaled to make c_n nonnegative for any finite choice of g_{ext}. In fact, in this case, all wrench intensities may be increased without bound, which in turn, implies that the joint torques may be increased without bound, too. This observation turns out to be quite useful in trajectory planning for dexterous manipulation, as it implies that we can squeeze as hard as we like without disturbing the form-closure character of the grasp. This fact considerably reduces the accuracy required of the controller. However, it is important to note that Cutkosky has shown that compliant effects can cause grasp instability as the joint torques increase [3].

The form-closure measure we propose is the scalar value d of the minimum element of $c^*_{n,null}$, where $c^*_{n,null}$ is the vector satisfying inequalities (22) and (24) with maximum minimum element. If d is strictly positive, then the grasp has form closure. This measure is the objective value of the following linear program:

$$\max_{c_{n,\mathrm{null}},d} \; d \tag{25}$$

subject to

$$W_n c_{n,\mathrm{null}} = \mathbf{0} \tag{22}$$

$$c_{n,\mathrm{null}} - \boldsymbol{d} \geq \mathbf{0} \tag{26}$$

$$d \geq 0 \tag{27}$$

$$A c_{n,\mathrm{null}} \geq \boldsymbol{h} \tag{28}$$

where d is essentially a slack variable, and $\boldsymbol{d}$ is a vector with all elements equal to d. Inequality (28) may be any set of constraints that is feasible for $c_{n,null} = \mathbf{0}$. Its purpose is merely to prevent the linear program from becoming unbounded.

If inequality (28) approximates the unit sphere, then our measure is quite similar to the efficiency given in [10]. However, Kirkpatrick's measure is valid only for grasps with $2n_q$ (i.e., 12 for the spatial case) or more contact points, whereas our measure is valid for grasps with any number of contact points.

Example 1: Consider a rectangle subjected to a planar grasp with four contact points as shown in Fig. 2. This grasp has form closure if $1.052 < \alpha < \pi/2$. If $\alpha < 1.052$, then the object may rotate counterclockwise. If $\alpha = \pi/2$, then the object may translate vertically. Table I illustrates how our form-closure measure varies with α. For this example, A was chosen so that inequality (28) would represent a cube with edges of length 2. Using the coordinate directions shown and summing the moments about the upper right-hand corner of the rectangle, the wrench matrix is given by

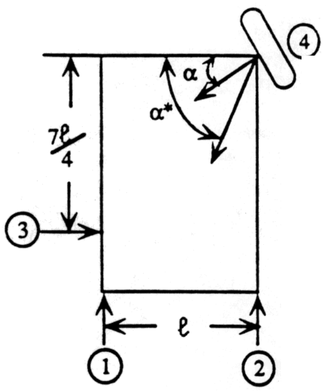

Fig. 2. Rectangle with four frictionless contact points.

TABLE I
QUANTIFICATION OF FORM CLOSURE

α	d	Closure Type
1.04	0.0	
1.06	0.020	
1.1	0.097	
1.2	**0.300**	
1.3	0.267	
1.4	0.170	
1.5	0.071	
1.56	0.010	
1.5707	0.0	
1.58	0.0	

$$W_n = \begin{bmatrix} 0 & 0 & 1 & -\cos(\alpha) \\ 1 & 1 & 0 & -\sin(\alpha) \\ -l & 0 & \frac{7l}{4} & 0 \end{bmatrix}.$$

The grasp "furthest" from losing form closure is the one which maximizes d, i.e., $\alpha \approx 1.2$ rad.

One might think that it should be possible to test for form closure during prediction of the velocity of the object using the frictionless object motion problem. This is *not* possible. Consider using the velocity formulation to predict the motion of a grasped object. Any planned finger motions would either tend to crush the object (kinematically infeasible motions) or release it (kinematically feasible motions). If the planned joint motions would tend to crush the object, then the finger motions would be prevented by the system's rigidity. This occurrence would not signal that the current grasp has form closure, because kinematic infeasibility can also occur with some force closure grasps. If the planned joint motions would release the object, then the fingers' motions would proceed and form closure would be lost. Since the object motion problem only predicts the imminent state of the grasp, the fact that the object was in form closure just prior to finger motion would not be revealed.

B. Force Closure

In the case of form closure, the grasp is stable regardless of the external wrench applied to the object. In contrast, a force-closure grasp can only satisfy the object's equilibrium relationships for a specific subset of possible external wrenches. The dual statement is that, when locking the hand's joints, it *is* possible to move the object. By definition, then, even grasps with one contact point have force closure. Thus, knowledge that particular grasp has force closure is not, in itself, particularly useful. What is important, both for static grasping and quasi-static manipulation, is whether a grasp with force closure will exhibit stable equilibrium under the application of a particular external wrench.

Given our assumptions, a grasped object will be stable if and only if its configuration corresponds to a local minimum of the constrained potential energy. However, we would prefer to not solve the nonlinear optimization problem given by (7) and (8) just to check stability. What we will conclude below is that if the object motion problem has a unique, bounded, feasible, solution $\dot{q}^*$, then the object is guaranteed to be stable; however, if the solution is not unique, then the object may or may not be stable. In the case of a unique $\dot{q}^*$, the stability is of first order in the sense that all infinitesimal perturbations of the object's configuration increase the potential energy infinitesimally (e.g., a ball at rest inside a cubical container tilted so that one corner is lower than all others). In the case of nonunique $\dot{q}^*$, if the object is stable, it is due to higher order effects in the sense that at least one infinitesimal perturbation of the object's configuration does not increase the potential energy even though a finite perturbation in the same direction does (e.g., a ball at rest in a hemispherical container). As one might expect, higher order stability may not be determined by solving the linear program but requires examination of the full nonlinear problem given by (1) and (2). The following definitions are relevant to the statement and proof of Theorem 2 below.

Definition: A set of contacts is said to be linearly independent if and only if the rank of the associated wrench matrix $\boldsymbol{W}_n$ is equal to the number of contacts.

Definition: A frictionless grasp is *stable to first order*, or is *first-order stable*, if the object's configuration corresponds to a stationary point of its constrained potential energy and if every feasible infinitesimal perturbation of the object's configuration away from that stationary configuration strictly increases the potential energy.

Definition: A frictionless grasp is *stable to higher order*, or is *higher-order stable*, if the object's configuration corresponds to a relative minimum of the object's constrained potential energy and the grasp is not first-order stable.

Theorem 2: A frictionless grasp is first-order stable if and only if the velocity formulation of the object motion problem has a unique, bounded, feasible solution.

Proof: If the frictionless object motion problem does not have a bounded, feasible solution, then the grasp configuration is either unstable or the requested finger motions are kinematically infeasible. If the object motion problem has a bounded, feasible solution, then the corresponding object configuration $\boldsymbol{q}_s$ represents a stationary point of the constrained potential energy function, because the equilibrium relationships (5) and (6) present in the force formulation must also be satisfied.

Given stationarity, the grasp will be first-order stable if every infinitesimal, kinematically feasible perturbation of the object's configuration away from $\boldsymbol{q}_s$ infinitesimally increases the object's potential energy. Let the augmented potential energy F be given by

$$F = y(\boldsymbol{q}) - \boldsymbol{\lambda}^T \boldsymbol{f}(\boldsymbol{q}) \tag{29}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0} \tag{30}$$

where $\boldsymbol{\lambda}$ is a vector of Lagrange multipliers and $\boldsymbol{f}(\boldsymbol{q})$ is the vector of C-functions for which each zero element corresponds to a point of contact between the object and the hand. Note that, for this proof, we need not make use of the dependence of the C-functions $\boldsymbol{f}$ on $\boldsymbol{\theta}$, so henceforth, the dependence will not be explicitly indicated.

Next, recall that a positive Lagrange multiplier always corresponds to a tight noninterference constraint (2) and, therefore, to a point of contact, i.e., if $\lambda_j > 0$, then $f_j(\boldsymbol{q}) = 0$. Then since f_j must be nonnegative, differential changes in the object's configuration $\boldsymbol{q}$ are constrained by the following system of linear inequalities:

$$df_j = \left.\frac{\partial f_j}{\partial \boldsymbol{q}}\right|_{\boldsymbol{q}=\boldsymbol{q}_s} d\boldsymbol{q} \geq 0, \qquad \text{for all } j \in \{j \mid \lambda_j > 0\}. \tag{31}$$

Casting inequality (31) into matrix form yields

$$\boldsymbol{W}_{ac}^T(\boldsymbol{q}_s) d\boldsymbol{q} \geq \boldsymbol{0} \tag{32}$$

where $\boldsymbol{W}_{ac}$ has dimension $(n_{ac} \times n_q), n_{ac}$ is the number of active contacts (those corresponding to $\lambda_j > 0$), and maintenance of the jth contact is indicated when the jth element of the product $\boldsymbol{W}_{ac}^T d\boldsymbol{q}$ is zero. The differential change of the the augmented objective function F due to differential changes in the objective function and the constraints yields the following equation:

$$dF = dy(\boldsymbol{q}) - \boldsymbol{\lambda}_{ac}^T d\boldsymbol{f}_{ac}(\boldsymbol{q}) \tag{33}$$

where $\boldsymbol{\lambda}_{ac}$ is the vector of Lagrange multipliers formed by removing the zero-valued elements from $\boldsymbol{\lambda}$. Equation (33) is required to be zero at the stationary point $\boldsymbol{q}_s$ [2]. To show how the potential energy depends on perturbations in $\boldsymbol{q}$, we rearrange (33) as

$$dy(\boldsymbol{q}_s) = \boldsymbol{\lambda}_{ac}^T \boldsymbol{W}_{ac}^T(\boldsymbol{q}_s) d\boldsymbol{q}. \tag{34}$$

Applying inequalities (32) and (30), we see that every feasible perturbation results in a nonnegative change in y. However, for first-order stability, dy must be strictly positive for every feasible nonzero perturbation. Since all elements of $\boldsymbol{\lambda}_{ac}$ are positive, then dy will be strictly positive if at least one element of the product $\boldsymbol{W}_{ac}^T d\boldsymbol{q}$ is positive. In other words, if there exists $d\boldsymbol{q} \neq \boldsymbol{0}$ that satisfies (35) below, then there exists a feasible perturbation of $\boldsymbol{q}$ such that the objective function does not increase. Thus, the grasp will have first-order stability if and only if the following equation has only the trivial solution:

$$W_{ac}^T(q_s)dq = 0. \tag{35}$$

Therefore, the rank of W_{ac} must be equal to n_q, which is equivalent to requiring that the grasp has n_q linearly independent active contacts. These contacts are active, since their corresponding Lagrange multipliers are positive.

To cast the above result into the light of the velocity formulation of the frictionless object motion problem, we reiterate that the requirement to have a bounded, feasible solution is equivalent to requiring stationarity of the grasp configuration. In the velocity formulation, the elements of the Lagrange multiplier vector (the wrench intensities) represent the rate of change of the objective function if the corresponding constraint is relaxed while all others are maintained. If, at a constrained stationary point, there are n_q positive Lagrange multipliers corresponding to n_q linearly independent contacts, and W_{ac} has a trivial null space, then the minimum, q^*, is guaranteed to be a local minimum [6]. Since the velocity formulation is a linear program, the local minimum is the unique global minimum [14]. In other words, first-order stability corresponds to nondegeneracy of the velocity formulation. Q.E.D.

Corollary 2.1: A grasp is stable if it has first-order stability.

Proof: The conditions stated above for first-order stability are clearly sufficient for stability. Q.E.D.

The most important practical implication of Theorem 2 is that the uniqueness of the instantaneous velocity of the object is guaranteed if there are n_q positive wrench intensities. Typically during manipulation, there will be n_q contacts. However, if there are more contacts, but n_q wrench intensities remain positive, then the instantaneous velocity of the object can be determined uniquely even though the wrench intensities are statically indeterminate. On the other hand, if there are fewer than n_q contacts, then there cannot be n_q positive wrench intensities, which implies that neither stability nor the instantaneous velocity of the object may be determined with first-order information alone.

C. Strong Force Closure

The wrench intensities of a form-closure grasp can be increased indefinitely while the grasp maintains equilibrium. For force-closure grasps, all wrench intensities have finite bounds. In terms of velocities, manipulation maintaining form closure requires compliant finger motion, whereas maintaining force closure usually does not. In this section, we define the *strong force closure* grasp. It is the subset of force-closure grasps for which some wrench intensities can be increased indefinitely. This class of grasps deserves recognition, because its maintenance during manipulation requires compliant motion, as does a form-closure grasp, but it can become unstable since it is, in fact, a force-closure grasp.

Definition: A grasp is said to have *strong force closure* if it has force closure and a subset of the contact wrench intensities can be increased without bound.

Given the definition of strong force closure, it is clear that no strictly positive solution of (22) may exist (i.e., relationships (22) and (24) are infeasible) but that at least one nonnegative solution must exist (i.e., relationship (24) is relaxed by allowing equality with zero). If a nonnegative solution does not exist, then no wrench intensity can be increased indefinitely.

Theorem 3: A grasp has strong force closure if and only if it does not have form closure and there exists a nontrivial vector $c_{n,\text{null}}$ in the null space of W_n with all nonnegative elements such that if the ith element of $c_{n,\text{row}}$ is negative, then the ith element of $c_{n,\text{null}}$ is positive.

Proof: The quasi-static assumption implies that the grasp under consideration is in equilibrium and therefore satisfies the relationships (5) and (6). Solving (5) and substituting into inequality (6) yields

$$c_n = c_{n,\text{row}} + c_{n,\text{null}} \geq 0$$

where $c_{n,\text{null}} = -W_n^\dagger g_{\text{ext}}, c_{n,\text{null}} = (W_n^\dagger W_n - I)\hat{k}_f\alpha, \alpha$ is a positive scalar, $\hat{k}_f$ is an arbitrary unit vector of compatible dimension, and $W_n^\dagger$ is the psuedoinverse of W_n.

If $c_{n,\text{null}} > 0$ exists, then by definition, the grasp has form closure: not force closure. Next, note that $c_{n,\text{null}}$ with all nonpositive elements (i.e., $c_{n,\text{null}} \leq 0$) prevents the unbounded increase of $c_{n,\text{null}}$. Therefore, for a grasp to have strong force closure, it is necessary that a nontrivial $c_{n,\text{null}}$ exist such that $c_{n,\text{null}} \geq 0$. Finally, let $c_{in,\text{row}}$ be the ith element of $c_{n,\text{row}}$. It is clear from (36) that if $c_{in,\text{row}}$ is negative, then $c_{in,\text{null}}$ must be positive. Q.E.D.

In light of the above proof, a test for frictionless strong force closure must allow some of the elements of $c_{n,\text{null}}$ to remain zero while encouraging others to be positive. This can be accomplished with the following linear program:

$$\max_{c_{n,\text{null}}} \quad \mathbf{1}^T c_{n,\text{null}} \tag{37}$$

subject to

$$W_n c_{n,\text{null}} = 0 \tag{22}$$

$$W_n c_{n,\text{row}} + W_n c_{n,\text{null}} = -g_{\text{ext}} \tag{38}$$

$$c_{n,\text{row}} + c_{n,\text{null}} \geq 0 \tag{23}$$

$$c_{n,\text{null}} \geq 0 \tag{39}$$

$$Ac_{n,\text{null}} \geq h \tag{28}$$

where $\mathbf{1}$ is a vector with all elements equal to 1.

As illustrated in example 2 below, this strong force-closure test is binary in nature. This comes from the fact that the null space components of some wrench intensities are zero-valued. However, a quantitative result similar to that given earlier for form closure could be formulated in two stages. First, apply the binary test to identify the nonzero components of $c_{n,\text{null}}$ and then apply the form closure test with the slack variable d added only to those components.

Example 2: Consider a rectangle subjected to a planar grasp with four contact points as shown in Fig. 3. It can be shown that this grasp has strong force closure if $-1.00 \leq a \leq 1.00$, but not form closure, since the object may translate vertically. Table II illustrates how the components of $c_{n,\text{null}}$ and $c_{n,\text{row}}$ vary with a. For this example, A was chosen so that inequality

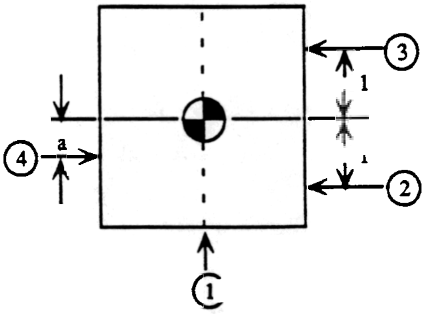

Fig. 3. Square in strong force closure grasp.

TABLE II
DETECTING STRONG FORCE CLOSURE

	Object Value	Closure Type	$c_{n,\text{null},1}$	$c_{n,\text{null},2}$	$c_{n,\text{null},3}$	$c_{n,\text{null},4}$
-1.001	0.0	force	0.0	0.0	0.0	0.0
-1.000	2.0	**strong force**	0.0	0.0	1.0	1.0
-0.5	2.0	**strong force**	0.0	0.25	0.75	1.0
0.0	**2.0**	**strong force**	**0.0**	**0.5**	**0.5**	**1.0**
0.4	2.0	**strong force**	0.0	0.7	0.3	1.0
0.7	2.0	**strong force**	0.0	0.85	0.15	1.0
1.00	2.0	**strong force**	0.0	1.0	0.0	1.0
1.001	0.0	force	0.0	0.0	0.0	0.0

(28) would represent a cube with edges of side 2. Using the coordinate directions shown and summing the moments about the object's center of mass yields the following wrench matrix:

$$W_n = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & a \end{bmatrix}.$$

Note that the nonzero components of $c_{n,\text{null}}$ can be increased indefinitely by squeezing.

During manipulation under compliant control, the matrix W_n is typically of full rank and often has more than n_q contacts. If the nullity of W_n is one, then the solution of the above linear program indicates which wrench intensities may be increased and which may not. When the nullity is greater than one, then the solution returned is just one of many possible and will not necessarily reflect which intensities may be increased. Brute force circumvention of this problem could be achieved by solving the linear program one time for each contact with the objective being to maximize the corresponding component of $c_{n,\text{null}}$ rather than the sum of the elements.

D. Instability

If the object is not in form closure during manipulation, then it must be stable in force closure or unstable. As we have seen, the frictionless object motion problem may be used as a sufficiency test for stability. If the velocity formulation has a unique, bounded, feasible solution, then the grasp is stable. If the velocity formulation has a nonunique, bounded, feasible solution (indicated by fewer than n_q positive wrench intensities), then the nonlinear optimization problem (7) and (8) must be used to determine stability. If the object's configuration corresponds to a relative minimum, then, by definition, the grasp has higher order stability. However, if the grasp configuration does not represent a local minimum, then the equilibrium of the object is unstable and dynamic effects would be required to determine the object's subsequent motion. Finally, if the velocity formulation has an unbounded solution (or equivalently, the force formulation is infeasible), then the grasp is unstable and again dynamic information is required for motion prediction.

Theorem 4: A grasp is unstable if the force formulation of the frictionless object motion problem is infeasible or, equivalently, if the velocity formulation is unbounded.

Proof: The Kuhn–Tucker necessary condition for stationarity of the object's configuration is composed of the equilibrium relationships [2]. Satisfaction of these equations is necessary for the object's stability, and therefore, their infeasibility is a sufficient condition for grasp instability. Since the infeasibility of the force formulation is equivalent to an unboundedness of the velocity formulation, this condition is also a sufficient condition for grasp instability. Q.E.D.

The physical interpretation of infeasibility of the force formulation is rather straightforward. However, the dual interpretation is less obvious. The physical interpretation of unboundedness in the velocity formulation is that there exists a feasible velocity of the object that causes reduction of the object's potential energy. Object motion in that direction increases the rate of potential energy reduction in direct proportion to the magnitude of $\dot{q}$, and thus the velocity formulation is unbounded. An example of such an unstable grasp is an object on a frictionless plane that is not perpendicular to the vector of gravitational acceleration.

IV. INDETERMINACIES

In the case of first-order stability, the object is stable, and its instantaneous velocity can be determined uniquely by solving the frictionless object motion problem. Typically, the matrix W_n is square and nonsingular, providing a unique solution to the wrench intensities also. In this section, we consider the cases in which things do not work out quite so nicely. We comment briefly on the well-studied case of statically indeterminate wrench intensities but concentrate primarily on the less well known case of velocity indeterminacy.

A. Force Indeterminacies

Rigid-body models lead to grasp configurations that are statically indeterminate with respect to the wrench intensities (see, for example, [16] and [7]). This happens when the wrench matrix W_n possesses a nontrivial null space. Practically speaking, for the frictionless case, this situation occurs when there are more contacts n_c than the number of degrees of freedom of the object n_q (i.e., when the wrench matrix W_n has more columns than rows). However, as a frictionless grasp evolves quasi-statically, this indeterminacy is often resolved by the breaking of redundant contacts. In some situations, though, it may be desirable to maintain "extra" contacts by compliant control, in which case it may be possible to determine the

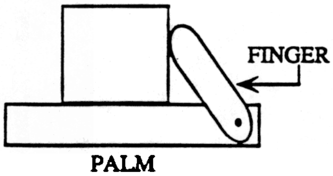

Fig. 4. Grasp with first-order statically indeterminate velocity.

wrench intensities of an otherwise statically indeterminate grasp by specifying some of the elements of τ and solving the following system of equations:

$$\begin{bmatrix} W_n & 0 \\ J_n^T & -I \end{bmatrix} \begin{bmatrix} c_n \\ \tau \end{bmatrix} = - \begin{bmatrix} g_{\text{ext}} \\ G \end{bmatrix} \tag{40}$$

where $I \in E^{n_\theta \times n_\theta}$ is the identity matrix and $0 \in E^{n_q \times n_\theta}$ is the zero matrix. Even still, if n_c is greater than $n_q + n_\theta$, the contact wrenches will be statically indeterminate. For a complete discussion of force indeterminacies, see [9].

B. Velocity Indeterminacies

Under certain geometric conditions, the nonlinear optimization problem described by (7) and (8) cannot provide a unique solution for the instantaneous velocity of the object. In this case, we say that the velocity of the object is *statically indeterminate.* If the velocity formulation of the frictionless object motion problem has a nonunique, bounded, feasible solution (i.e., a degenerate solution), then we say that the velocity is *statically indeterminate to first order.*

Definition: The instantaneous velocity of a manipulated frictionless object is said to be *statically indeterminate to first order* if the velocity formulation of the frictionless object motion problem has a nonunique, bounded, feasible solution.

In this case, if the object remains stable, then the velocity of the object may usually be computed uniquely by determining the local minimum of the original nonlinear problem in the neighborhood of the current stationary configuration. However, we would prefer to identify and avoid these grasp configurations, so, we characterize them mathematically.

Fig. 4 shows a frictionless planar grasp with first-order statically indeterminate velocity but with statically determinate wrench intensities. Note that if the finger is rotated counterclockwise, the object will translate to the left. However, the velocity and force formulations cannot predict the speed of the leftward motion, because leftward translation at any speed does not affect their objective values. In this case, even the nonlinear optimization problem cannot be used to determine the object's velocity without an auxiliary *ad hoc* condition such as minimizing the object's kinetic energy.

In the above example, the indeterminacy appears to arise from the fact that the palm's surface is perpendicular to the direction of gravity. Theorem 5 provides a more general statement of this condition.

Theorem 5: The instantaneous velocity of a frictionless object is statically indeterminate to first order if and only if the velocity formulation of the object motion problem is degenerate.

Proof: Suppose we solve the velocity formulation for the grasp in question. The result will be one of the following: no solution, an unbounded solution, or a bounded feasible solution. The result of concern here is that of a bounded feasible solution, at which there will always be a subset of the kinematic constraints that are active (i.e., corresponding to maintained contacts) and thus satisfy

$$W_{ak}^T \dot{q}^* - J_{ak}\dot{\theta} = 0 \tag{41}$$

where $W_{ak}^T \in E^{n_{ak} \times n_q}$ and $J_{ak} \in E^{n_{ak} \times n_\theta}$ are constructed from W_n^T and J_n by removing the rows corresponding to the inactive kinematic constraints (i.e., those corresponding to kinematic inequalities satisfied by strict inequality and therefore corresponding to breaking contacts), n_{ak} is the number of active kinematic constraints, and θ is fixed and known. From the theory of linear programming, we know that any kinematic constraints corresponding to zero-valued elements of the Lagrange multiplier vector λ may be relaxed without affecting the objective value. Removing those rows from W_{ak} and J_{ak} yields the following system of equations that the set of minimizing instantaneous velocities of the object must satisfy:

$$W_{ac}^T \dot{q} - J_{ac}\dot{\theta} = 0$$

where n_{ac} is the number of active contacts (those corresponding to positive elements of λ), and $W_{ac}^T \in E^{n_{ac} \times n_q}$ and $J_{ac} \in E^{n_{ac} \times n_\theta}$ are the wrench and Jacobian matrices of the active contacts. If (42) has a unique solution, then the velocity formulation is said to be nondegenerate, implying that the velocity formulation has a unique solution. Otherwise, the velocity formulation is degenerate, implying that the object's velocity is statically indeterminate to first order with

$$\dot{q}^* = (W_{ac}^T)^\dagger J_{ac}\dot{\theta} + [(W_{ac}^T)^\dagger W_{ac} - I]k_\nu \tag{43}$$

$$W_{\text{inac}}^T \dot{q}^* - J_{\text{inac}}\dot{\theta} \geq 0$$

where W_{inac} and J_{inac} are the partitions of W_n and J_n corresponding to the inactive contacts (those corresponding to the zero-valued elements of λ) and k_ν is an arbitrary vector of compatible length. Q.E.D.

Corollary 5.1: A sufficient condition for the instantaneous velocity to be statically indeterminate to first order is that there be fewer than n_q contacts.

Proof: W_{ac}^T has dimension $(n_{ac} \times n_q)$. Since $n_{ac} < n_q$, the rank of W_{ac} must be less than n_q. Therefore, W_{ac} will have a nontrivial left null space and $\dot{q}^*$ will statically indeterminate to first order, as shown by relationships (43) and (44). Q.E.D.

Note that usually n_{ac} is equal to the number of contacts, so that, practically speaking, the instantaneous velocity of a grasped object will be statically indeterminate to first order if it has fewer than n_q contacts; it is rare that the velocity formulation will be degenerate when there are n_q or more contacts. If this case does arise, reorientation of the palm will typically remove the problem.

Corollary 5.2: A necessary condition for stable quasi-static manipulation of a frictionless object with first-order statically indeterminate velocity is that the external wrench applied to the object be orthogonal to every nontrivial vector lying in the the left null space of the global grasp wrench matrix.

Proof: Continuing from the proof of Theorem 5, we may write the objective function by premultiplying both sides of (43) by g_{ext}^T yielding

$$g_{\text{ext}}^T \dot{q}^* = g_{\text{ext}}^T (W_{ac}^T)^\dagger J_{ac}\dot{\theta} + g_{\text{ext}}^T[(W_{ac}^T)^\dagger W_{ac}^T - I]k_\nu. \quad (45)$$

Since static indeterminacy is manifested in objective insensitivity to changes in $\dot{q}^*$, and since varying k_ν is equivalent to varying $\dot{q}^*$ (see (43)), we see that the condition of first-order static indeterminacy of velocity is equivalent to the condition that the second term on the right-hand side of (45) be zero for all values of k_ν. Equivalently, the external wrench applied to the object g_{ext} must be orthogonal to the null space of W_{ac}^T. Also, since the null space of W_{ac}^T is equivalent to the left null space of W_{ac}, and since the column and the left null spaces of a matrix are orthogonal complements, it is clear that Corollary 5.2 is a statement of equilibrium—a necessary condition for stability. Q.E.D.

Corollary 5.3: For a stable grasp with first-order statically indeterminate velocity, if the active contacts are linearly independent, then the wrench intensities may be computed uniquely.

Proof: If the object's velocity is indeterminate to first order, then all solutions to the velocity formulation imply the maintenance of the contacts corresponding to the matrix W_{ac}. Other contacts could be maintained, but their wrench intensities must be zero since the velocity formulation is degenerate. Also, because the velocity formulation has a bounded feasible solution, the dual relationship of the velocity and force formulations implies that the equilibrium relationships can be satisfied. If, in addition, the active contacts are linearly independent, then the solution of the equilibrium equations is unique and provided by the dual variables. Q.E.D.

A typical planar configuration particularly relevant to Theorem 5 is shown in Fig. 5. Note that the center of gravity of the object is directly below the intersection of the two contact normals. An infinitesimal rotation of the object about the instantaneous center of rotation (the point marked "COR" in Fig. 5) does not raise the object's center of gravity: the source of the static indeterminacy. However, a small finite rotation in either direction does. Corollary 5.2 suggests that to avoid static indeterminacy of the object's instantaneous velocity, W_{ac} be changed with respect to g_{ext}. In the situation depicted in Fig. 4, rotating the palm clockwise infinitesimally has the desired effect. In the case shown in Fig. 5, the object will respond to a rotation of the palm by moving to maintain stable equilibrium. However, if the palm is rotated far enough, the object will become stable with a third contact, thus making its velocity statically determinate.

Grasps with statically indeterminate velocity have arisen approximately 5% of the time during our simulations of the manipulation of random convex polygons [28], but their appearance clearly depends on the joint trajectories. This suggests that manipulation planners could be developed to recognize such grasps and to avoid using them when solving planning problems.

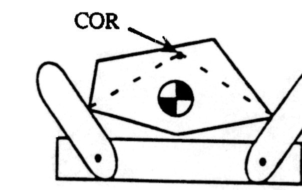

Fig. 5. Stable grasp with first-order statically indeterminate velocity.

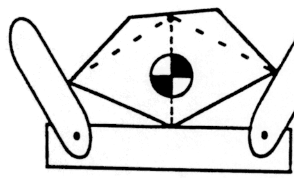

Fig. 6. Planar grasp with completely, statically, indeterminate velocity.

C. Completely Statically Indeterminate Velocity

Completely statically indeterminate velocity configurations are those for which the nonlinear optimization problem (7) and (8) has a nonunique minimum, i.e., the object may move in a number of directions all consistent with our assumptions. In this situation, higher order effects cannot indicate the correct solution. Instead, the actual value of $\dot{q}^*$ depends on unmodeled effects and uncertain parameters. A planar grasp with completely statically indeterminate velocity is shown in Fig. 6. If the fingers are spread slowly, then the object will rotate either clockwise or counterclockwise maintaining three contacts, but the direction of rotation is indeterminate as predicted by the "voting rule" proposed by Mason and Salisbury [16]. If the object's surface were curved at the contact, then higher order information available in the C-function constraints (8) could possibly be used to resolve the ambiguity. Unfortunately, we have found no good mathematical characterization of this type of grasp and are unable to estimate how often they arise. However, since we have not observed them in simulations, we conjecture that their rate of occurrence will depend on the manipulation planning algorithm employed.

V. Conclusion

A desire to solve frictionless manipulation planning problems has highlighted the need for efficient computational procedures for predicting the contact wrench intensities and the instantaneous velocity of a grasped rigid object. Toward this end, we have derived dual linear programs called the velocity formulation and the force formulation whose solutions provide the instantaneous velocity of a frictionless object moving quasi-statically in contact with frictionless points. In the case of force-closure grasps, these linear programs are particularly useful, because usually both linear programs are nondegenerate, implying that the instantaneous velocity and the wrench intensities have unique solutions. Also, we showed that when the velocity formulation is nondegenerate, the object's stability is guaranteed and its velocity may be uniquely determined even if the wrench intensities are statically indeterminate (i.e., the force formulation is degenerate). Under the antithetical condition that the velocity formulation is degenerate, the first-order information provided by the linear programs is insufficient to determine either stability or the velocity. In such cases, the full nonlinear optimization problem must be solved, but even then there are pathological situations that defy prediction. Interestingly, if the velocity formulation is statically indeterminate but the active contacts are linearly independent (which is the case when manipulating with fewer than n_q contacts), then the wrench intensities can be uniquely determined.

The velocity and force formulations also provide a sufficient condition for instability. If the velocity formulation is unbounded (equivalently, the force formulation is infeasible), then the grasp is unstable. Somewhat nonintuitively though, infeasibility of the velocity formulation does not imply that the grasp has form closure, since a subset of force-closure grasps called *strong force-closure* grasps can also be responsible for kinematic infeasibility. In manipulation planning, the distinction between form closure and strong force closure is important, because while form-closure grasps are always stable, strong force-closure grasps are not. Thus, during manipulation planning, if the grasp has form closure, one need not be concerned with the external wrench applied to the object, whereas otherwise, knowledge of the external wrench is crucial. Therefore, we have developed two tests: one capable of distinguishing strong force-closure grasps from other force-closure grasps and a second capable of distinguishing between force-closure and form-closure grasps. To our knowledge, the form-closure test is the first quantitative test valid for any number of contact points.

A. Future Work

The work presented here depends on a quite restrictive set of assumptions. In particular, the "frictionless" assumption leads to the formulation of the quasi-static motion problem as a linear program, which usually yields a unique solution for the object's velocity. When friction is significant, the linear program is invalid, and the "classical" approach must be used [30]. The assumption of perfect knowledge of part geometry is also too restrictive. Therefore, the sensitivities of the contact forces and the object's velocity to control, sensing, and modeling errors are currently being explored for both the frictional and frictionless cases.

In future work, we plan to address the problem of determining bounds in the coefficients of friction, which if satisfied, would allow the successful execution of dexterous manipulation plans generated under the frictionless assumption in the presence of significant friction. For preliminary results, refer to [5].

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REFERENCES

[1] H. Asada and A. By, "Kinematic analysis and design for automatic workpart fixturing in flexible assembly," in *2nd International Symposium on Robotics Research* (Kyoto, Japan, Aug. 20–23, 1984). Cambridge, MA: MIT Press, 1984, pp. 237–244.

[2] G. S. G. Beveridge and R. S. Schechter, in *Optimization: Theory and Practice.* New York: McGraw-Hill, 1970.

[3] M. R. Cutkosky, in *Robotic Grasping and Fine Manipulation.* Boston: Kluwer, 1985.

[4] B. R. Donald, "A search algorithm for motion planning with six degrees of freedom," in *Artificial Intell.*, vol. 31, pp. 295–353, 1987.

[5] A. O. Farahat and J. C. Trinkle, "An investigation of dexterous manipulation planning using C-functions and stability functions," in *Proc. Int. Conf. Intell. Robots Syst.* (Raleigh, NC), July 1992.

[6] P. E. Gill, W. Murray, and M. H. Wright, in *Practical Optimization.* New York: Academic, 1981.

[7] W. Holzmann and J. M. McCarthy, "Computing the friction forces associated with a three-fingered grip," in *Proc. Int. Conf. Robotics Automat.* (St. Louis, MO, Mar. 18–21, 1985), pp. 594–600.

[8] K. H. Hunt, in *Kinematic Geometry of Mechanisms.* Oxford, England: Clarendon, 1978.

[9] J. R. Kerr, "An analysis of multi-fingered hands," Ph.D. dissertation, Dept. of Mechanic. Eng., Stanford Univ., Stanford, CA, Dec. 1984.

[10] D. Kirkpatrick, S. R. Kosaraju, B. Mishra, and C-K. Yap, "Quantitative Steinitz's theorem with applications to multifingered grasping," TR 460, Robotics Rep. 210, Courant Inst. of Math. Sci., New York Univ., NY, Sept. 1989.

[11] K. Lakshminarayana, "Mechanics of form closure," ASME Rep. 78-DET-32, 1978.

[12] Z. Li, "Kinematics, planning and control of dexterous robot hands," UCB/ERL M89/127, Ph.D. dissertation, College of Eng., Univ. of California, Berkeley, CA, Nov. 29, 1989.

[13] T. Lozano-Perez, "A simple motion-planning algorithm for general robot manipulators," *IEEE J. Robotics Automat.*, vol. RA-3, no. 3, pp. 224–238, 1987.

[14] D. G. Luenberger, *Linear and Nonlinear Programming.* Reading, MA: Addison-Wesley, 1984.

[15] X. Markenscoff, L. Ni, and C. H. Papadimitriou, "The geometry of grasping," *Int. J. Robotics Res.*, vol. 9, no. 1, pp. 61–74, Feb. 1990.

[16] M. T. Mason and J. K. Salisbury, Jr., in *Robot Hands and the Mechanics of Manipulation.* Cambridge, MA: MIT Press, 1985.

[17] B. Mishra, J. C. Schwartz, and M. Sharir, "On the existence and synthesis of multifinger positive grips," TR 259, Robotics Rep. 89, Courant Inst. of Math. Sci., New York Univ., NY, Nov. 1986.

[18] V. D. Nguyen, "The synthesis of stable force-closure grasps," M.S. thesis, TR-905, MIT AI Lab., Cambridge, MA, May 16, 1986.

[19] M. S. Ohwovoriole, "An extension of screw theory and its application to the automation of industrial assemblies," Ph.D. dissertation, Stanford Univ., Stanford, CA, Apr. 1980.

[20] R. P. Paul, in *Robot Manipulators: Mathematics, Programming, and Control.* Cambridge, MA: MIT Press, 1981.

[21] J. Pertin-Troccaz, "Grasping: A state of the art," in *The Robotics Review: 1*, T. Lozano-Perez, Ed. Cambridge, MA: MIT Press, 1989.

[22] W. H. Press *et al.*, *Numerical Recipes: The Art of Scientific Computing.* Cambridge, U.K.: Cambridge Univ. Press, 1989.

[23] F. Reuleaux, *The Kinematics of Machinery.* New York: Macmillan, 1876; reprint, New York: Dover, 1963.

[24] J. K. Salisbury, "Kinematic and force analysis of articulated hands," Ph.D. dissertation, Dept. of Mechanic. Eng., Stanford Univ., Stanford, CA, Rep. STAN-CS-82-921, July 1982.

[25] S. Smale, "On the average number of steps of the simplex method of linear programming," *Math. Program.*, vol. 27, no. 3, pp. 241–262, 1983.

[26] P. Somoff, "Uber Gebiete von Schraubengeschwindigkeiten eines starren Korpers bie verschiedener Zahl von Stutzflachen," *Z. Math. Phys.*, vol. 45, pp. 245–306, 1900.

[27] G. Strang, in *Linear Algebra and its Applications.* New York: Academic, 1980.

[28] J. C. Trinkle, "The mechanics and planning of enveloping grasps," Ph.D. dissertation, Dept. of Syst. Eng., Univ. of Pennsylvania, Philadelphia, PA, MS-CIS-87-46, GRASP Lab. 108, June 1987.

[29] J. C. Trinkle and R. P. Paul, "Planning for dexterous manipulation with sliding contacts," *Int. J. Robotics Res.*, vol. 7, no. 3, pp. 24–48, June 1990.
[30] J. C. Trinkle and D. C. Zeng, "Planar quasistatic motion of a lamina with uncertain contact friction," in *Proc. Int. Conf. Intell. Robots Syst.* (Raleigh, NC), July 1992.
[31] J. C. Trinkle, "A quantitative test for form closure grasps," in *Proc. Int. Conf. Intell. Robots Syst.* (Raleigh, NC), July 1992.
[32] D. C. Zeng, "On the prediction of quasistatic motion in the presence of uncertain contact friction," M.S. thesis, Dept. of Comput. Sci., Texas A&M Univ., College Station, TX, Dec. 1991.

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