Simulation and Experiments in Vibratory Manipulation: Rigid Bodies on a Vibrating Surface

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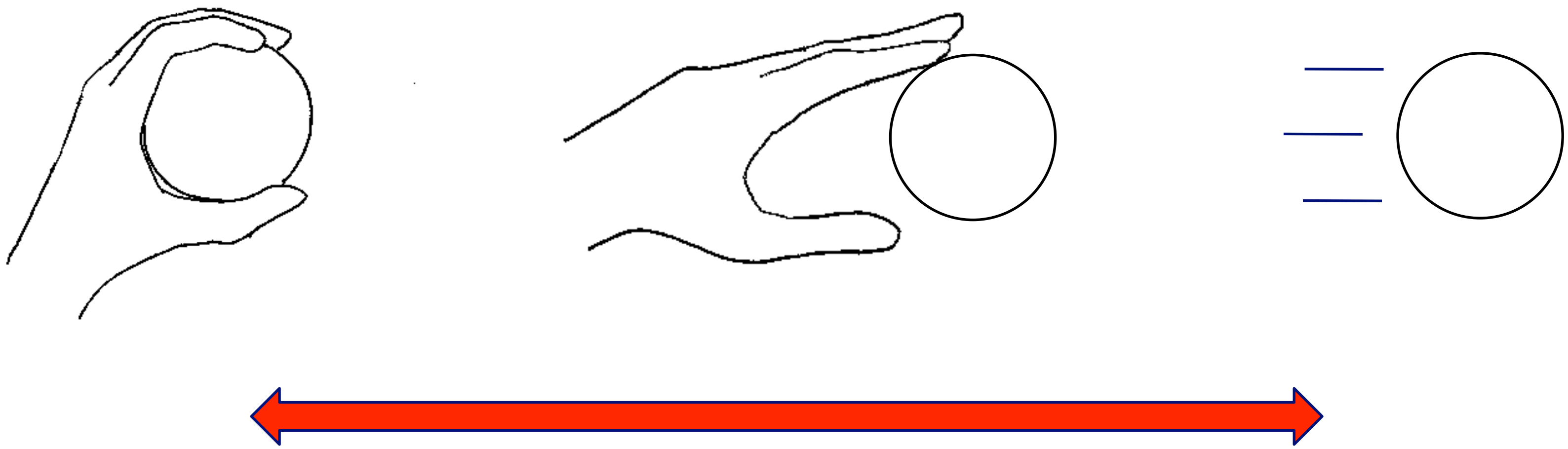


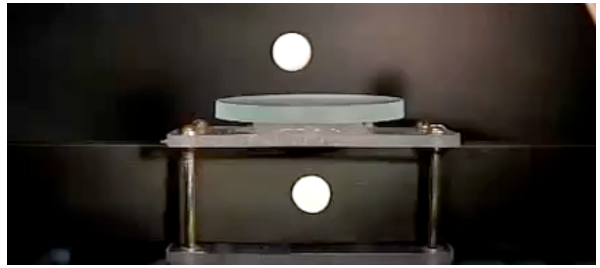
hand controls ball:
grasping

shared control:
nonprehensile manipulation

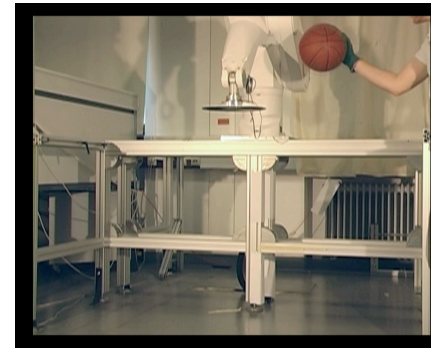
environment
controls ball

Examples

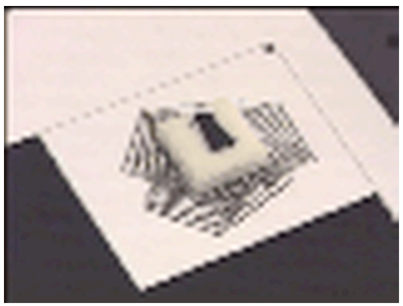
throwing and batting
(U Tokyo)



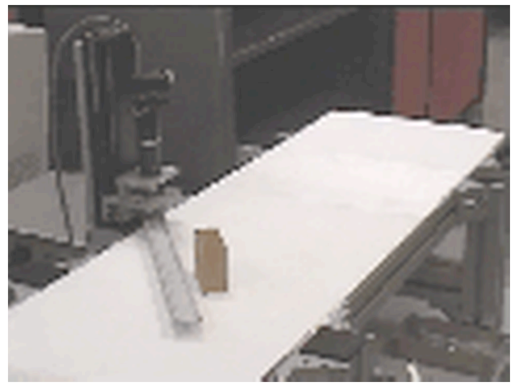
bat juggling



dribbling
(TU Munich)



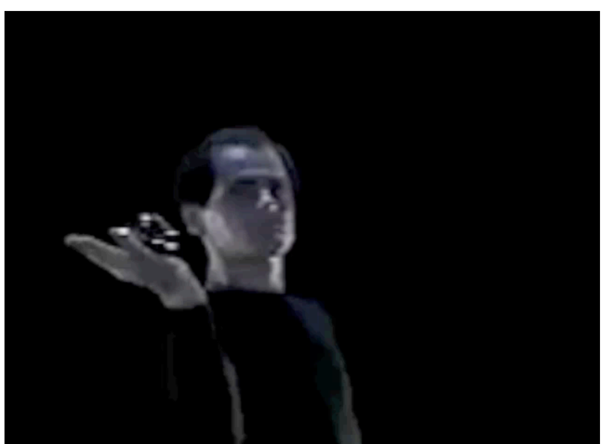
pushing



pushing and toppling

vibratory feeding

rolling
(Michael Moschen)



rolling on a
constraint
surface
(dung beetle,
Natl Geo)

Why Nonprehensile Manipulation?

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- Given a robot, increase the set of solvable tasks
- Given a task, use cheaper, simpler robots (automation)
- Most manipulation is nonprehensile! (pushing, throwing, tapping, sliding, rolling, batting, kicking, …)

Why Nonprehensile Manipulation?

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Research Topics

- sensing/observability/uncertainty
- mechanics and modeling
- motion planning
- feedback control
- understanding what tasks are solvable (e.g., accessibility, controllability)

Outline

a nonprehensile primitive: vibratory sliding

- asymptotic velocity fields
- velocity fields for rigid bodies

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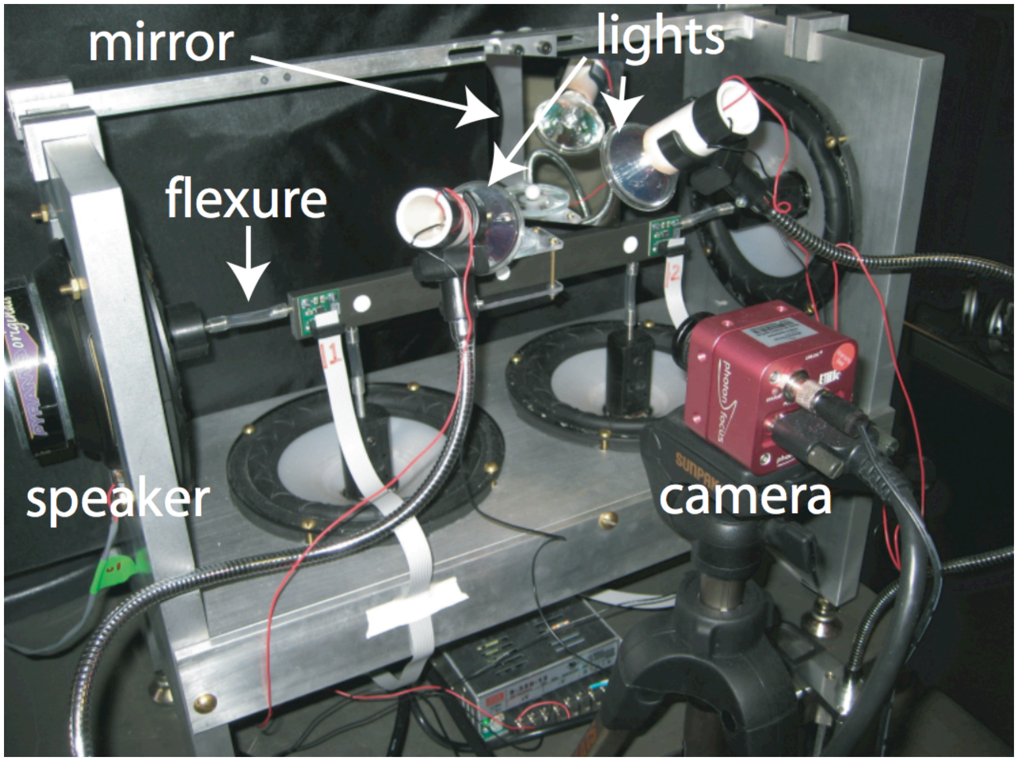
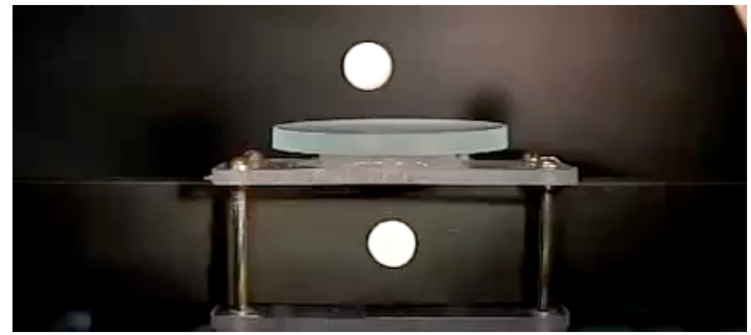
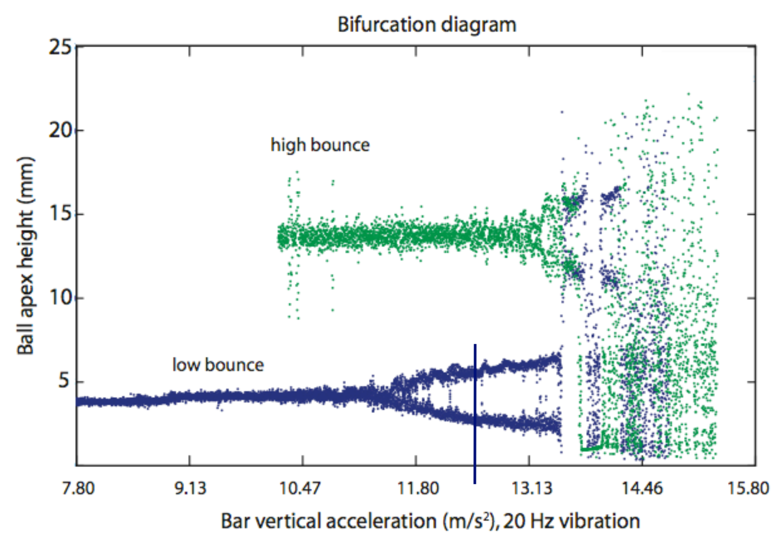
Batting and Sliding



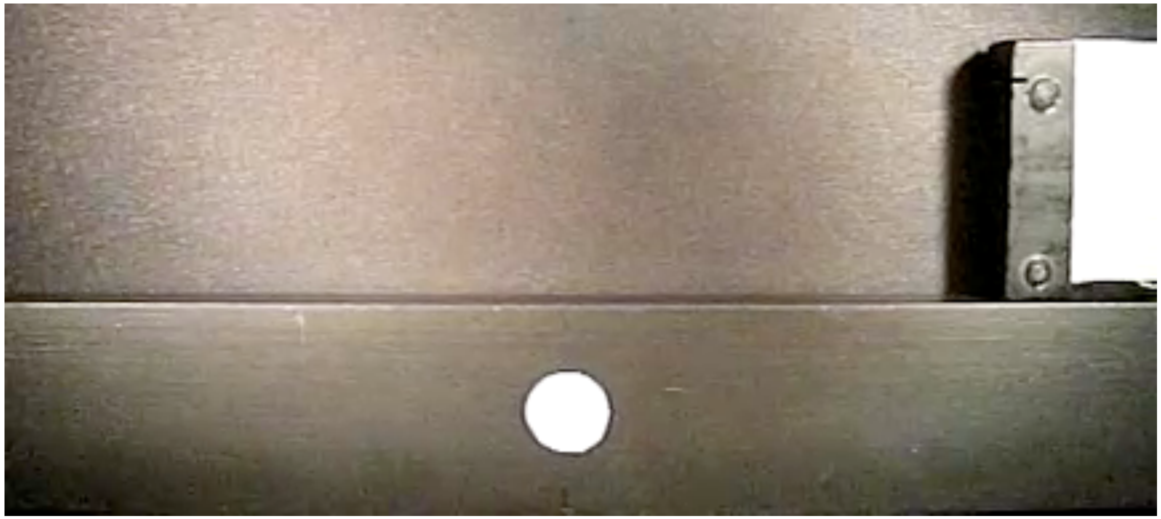
3-DOF "VPOD" vibratory vertical plane manipulator with 3D high-speed vision





Sliding Manipulation

size scale



15 Hz vibration, 20x slow motion

$$\mathbf{f}_{\mathrm{fric}} = \mu N \frac{\mathbf{v}_{\mathrm{rel}}}{\|\mathbf{v}_{\mathrm{rel}}\|}$$

friction force

friction coefficient

normal force magnitude

slipping direction

Sliding Manipulation

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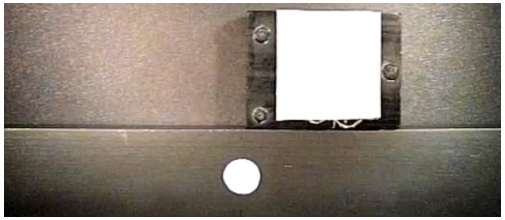
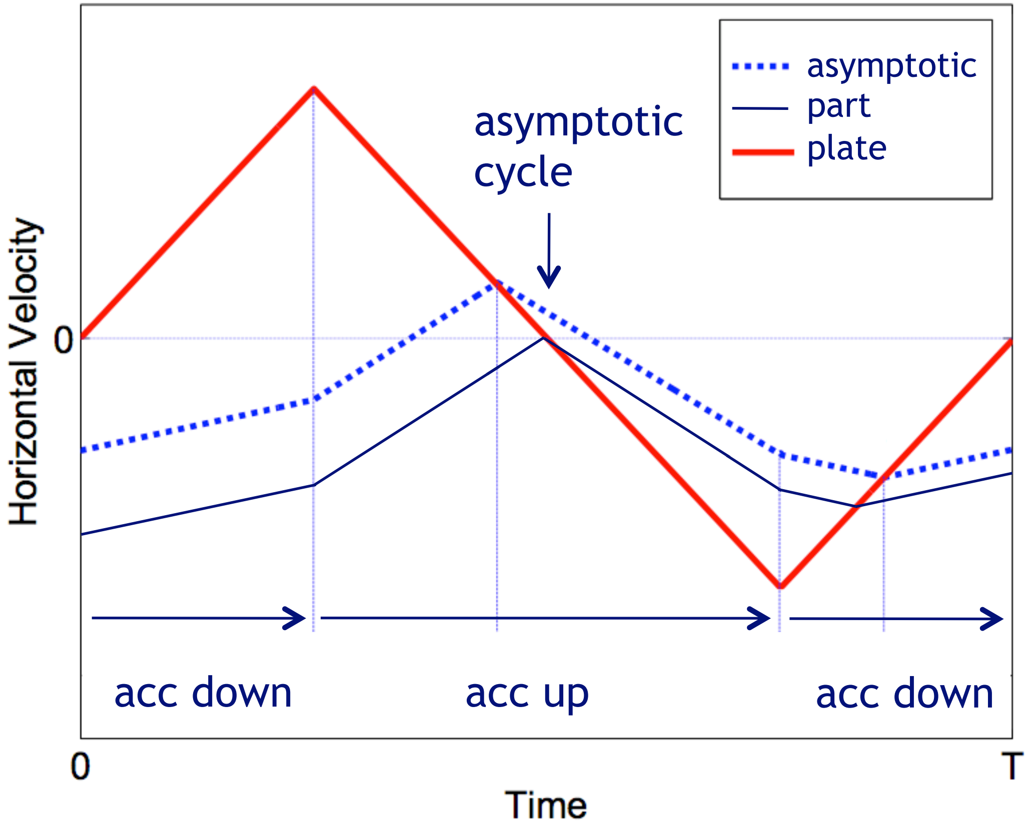


square wave vertical and horizontal acceleration



Sliding Manipulation

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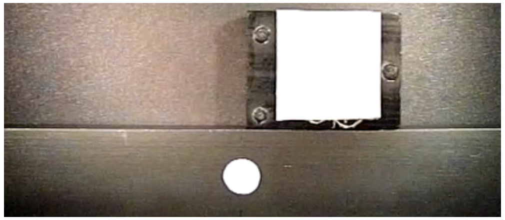

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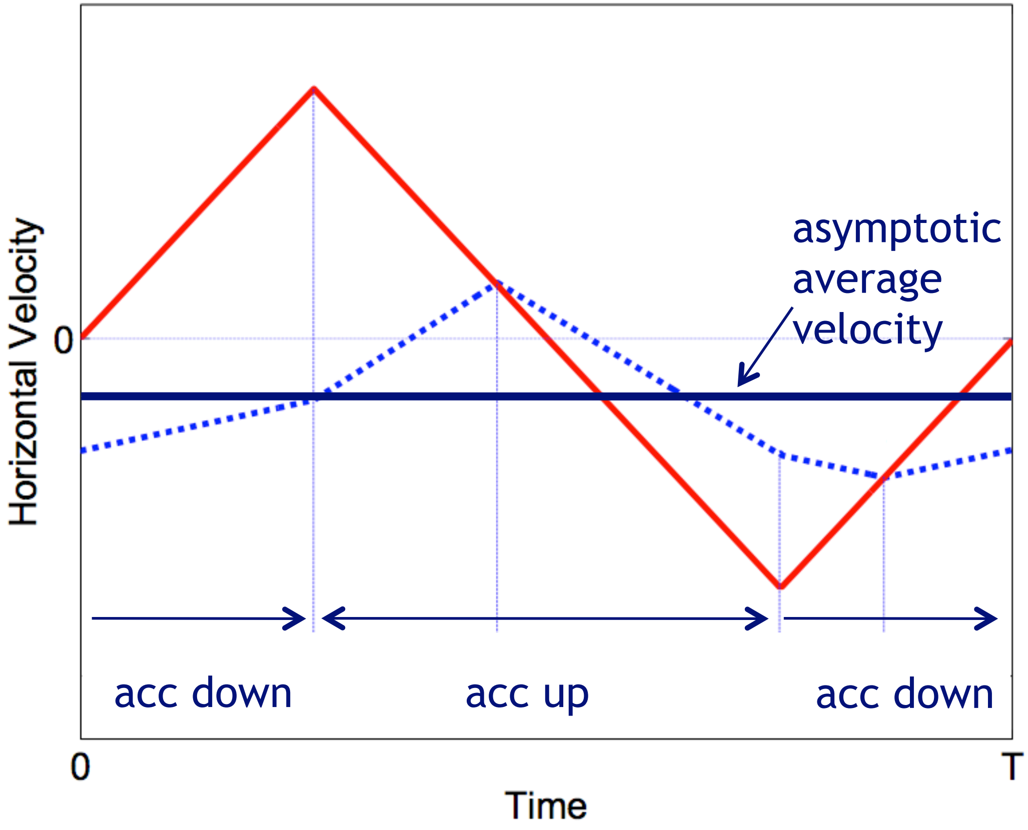
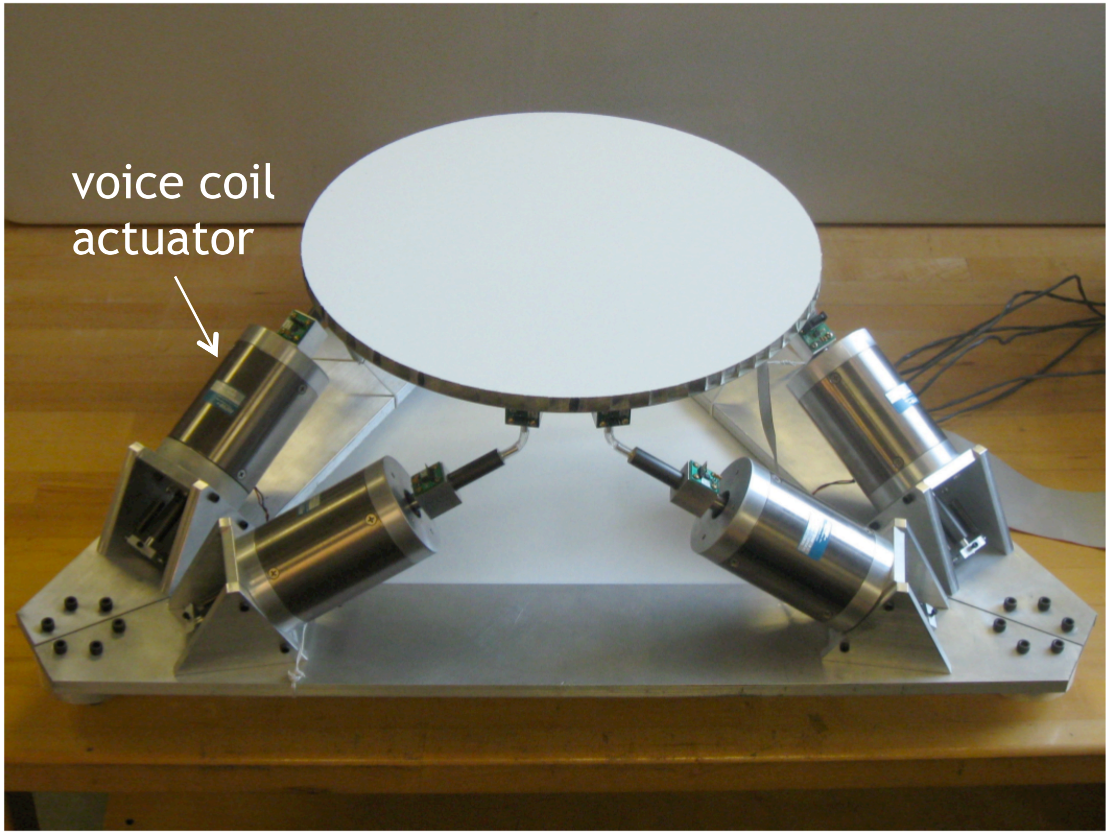
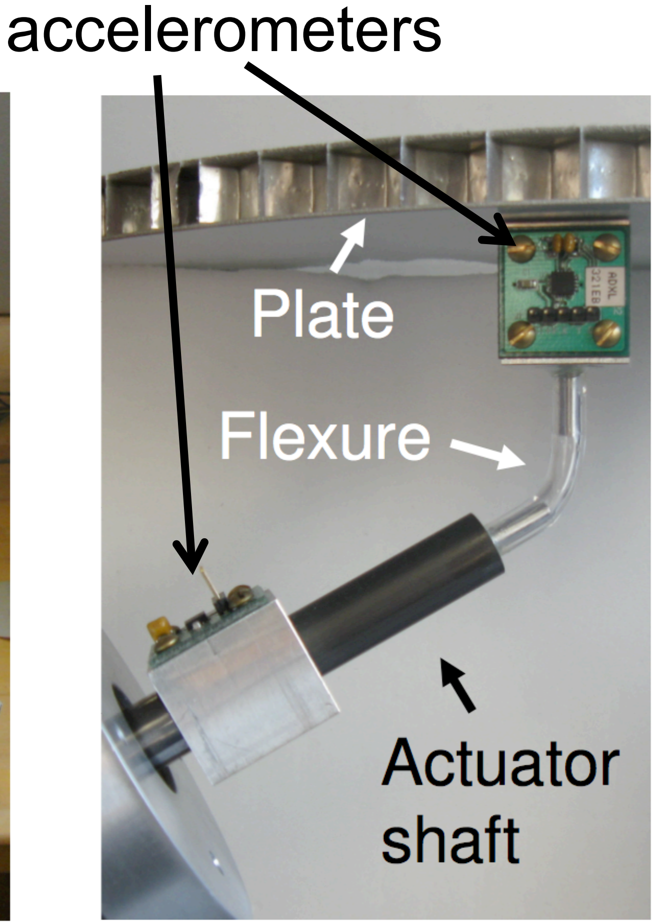


The 6-DOF PPOD
(Programmable Parts-feeding Oscillatory Device)





PPOD2: flexure-based Stewart platform

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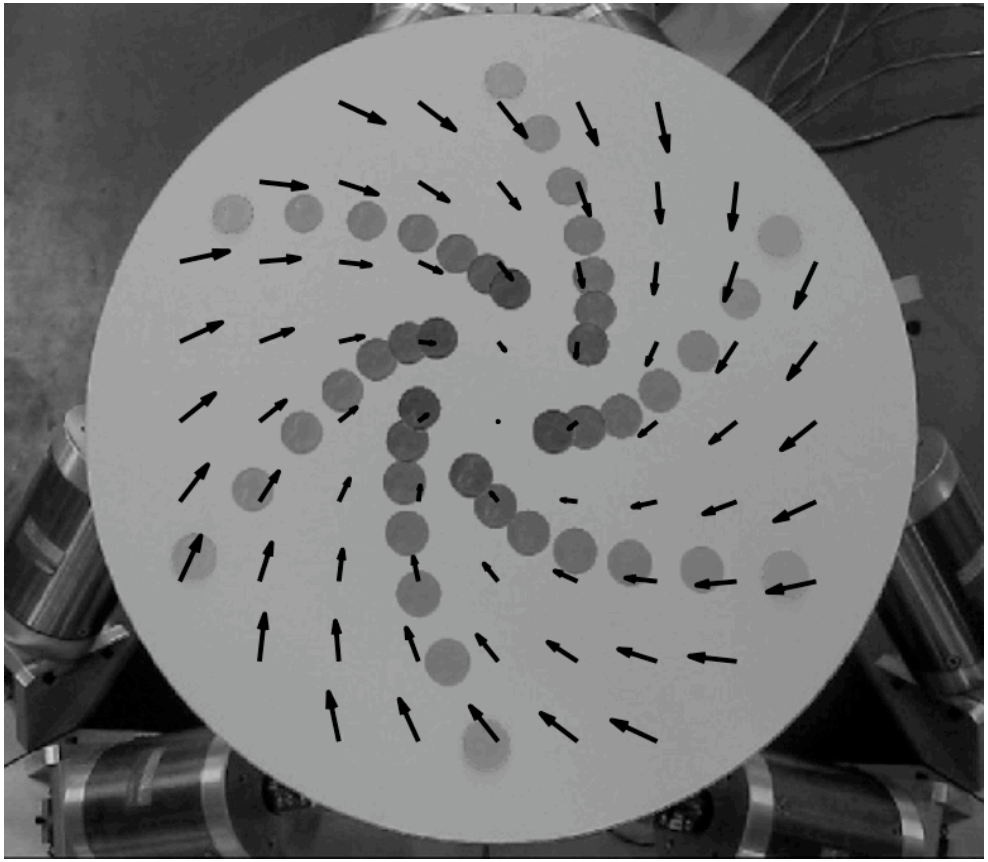

asymptotic average velocity field

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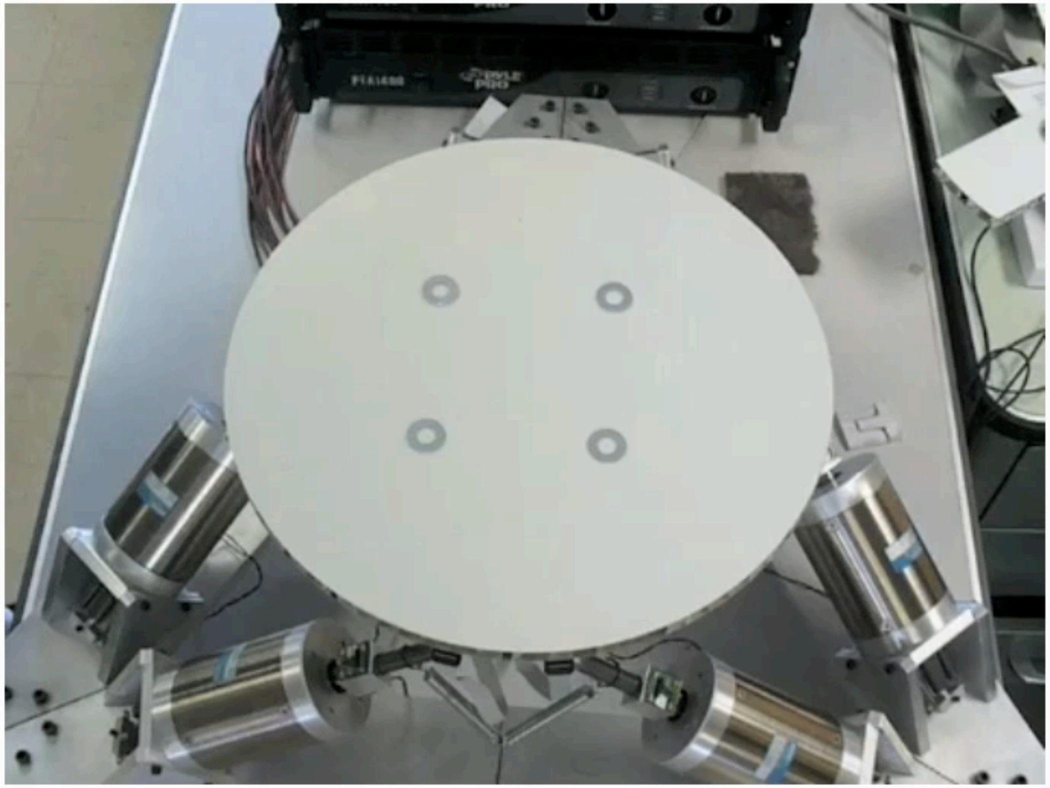
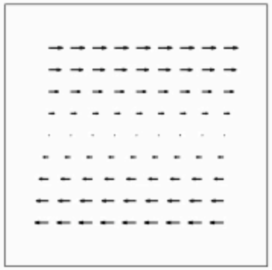
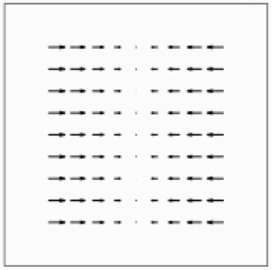
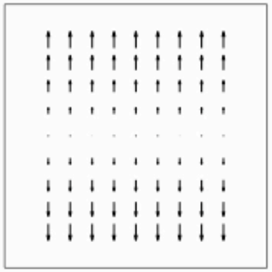
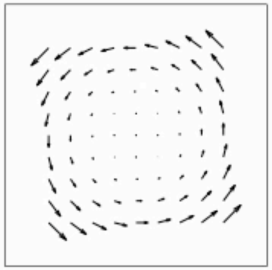
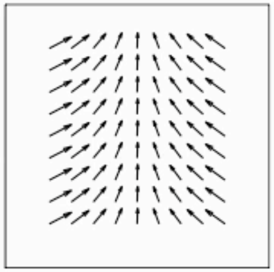






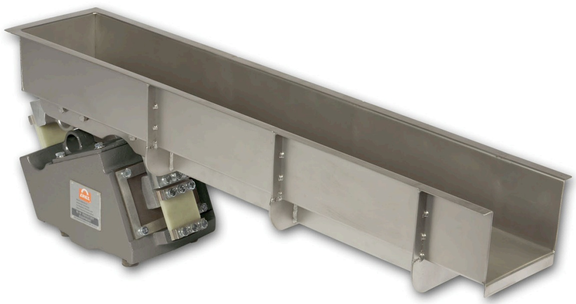




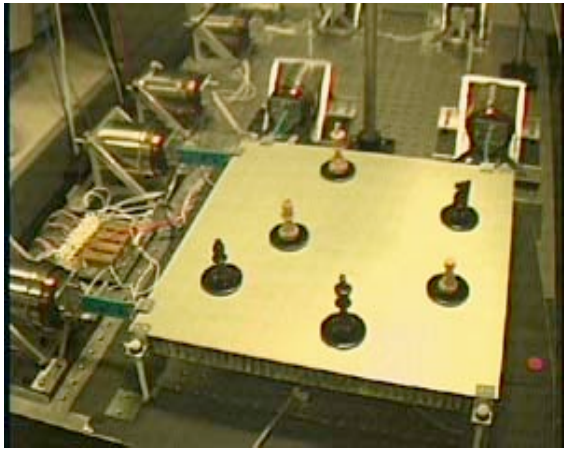


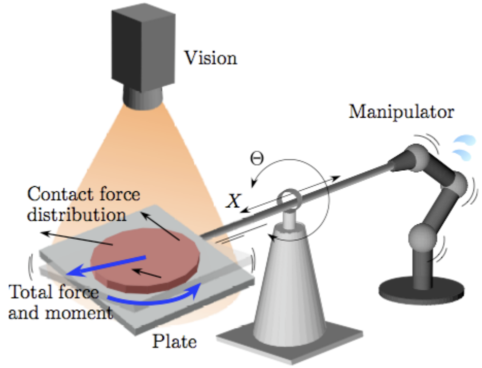
Related Work

vibratory linear conveyors

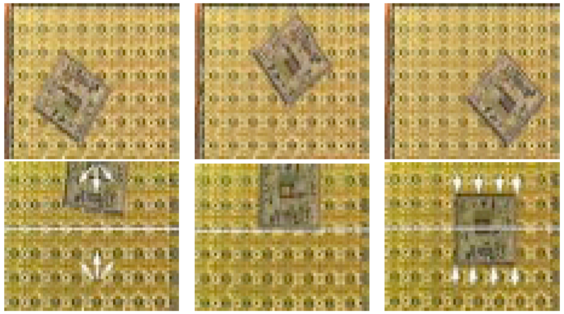


horizontally-vibrating plate
[Reznik, Canny
Bohringer, Goldberg, et al.]





pizza manipulation
[Higashimori, Utsumi, Kaneko]



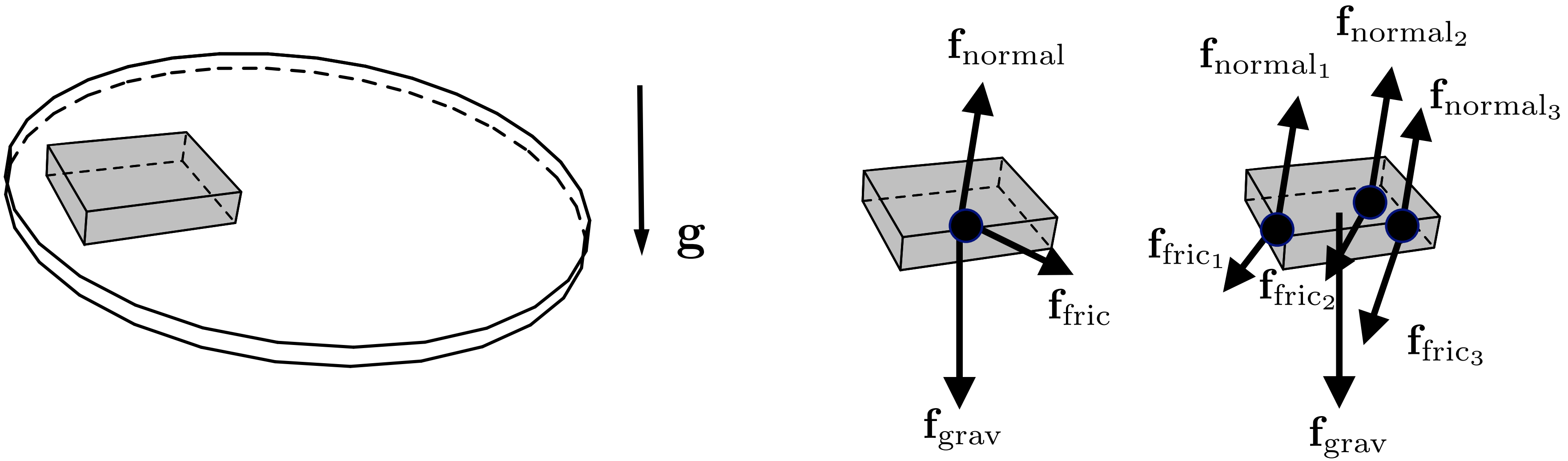
arrays of vibrating plates, MEMS, airjets, wheels
[Frei et al., Bohringer and Donald, Luntz et al.,
Murphey and Burdick, Kavraki, Goldberg et al.]

Outline

a nonprehensile primitive: vibratory sliding

- asymptotic velocity fields
- velocity fields for rigid bodies

Part Dynamics



$$\mathbf{f}_{\text{fric}_i} = \mu_i N_i \frac{\mathbf{v}_{\text{rel}_i}}{\|\mathbf{v}_{\text{rel}_i}\|}$$

- direction of **relative velocity** between part and plate determines **direction** of friction force
- vertical **acceleration** of plate determines normal force and therefore **magnitude** of friction force
- by exploiting full 6-DOF motion, the direction and magnitude of the friction forces on a part can be made **configuration-dependent**

Part Dynamics

Given:

1. Periodic control signal (plate acceleration)
2. Part parameters (inertia, contact locations)
3. Friction parameters (friction coefficients)

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$$\dot{x} = f(x, u)$$

$$x = (q_{\text{plate}}, q_{\text{part}}, v_{\text{plate}}, v_{\text{part}})$$

$$u = \dot{v}_{\text{plate}}$$

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Simplified dynamics:

1. Sliding at all contacts
2. No Coriolis or centripetal effects
3. Fixed plate and part configurations

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$$\boxed{\dot{v}_{\text{part}} = \mathbf{A}^{-1}\mathbf{b}}$$

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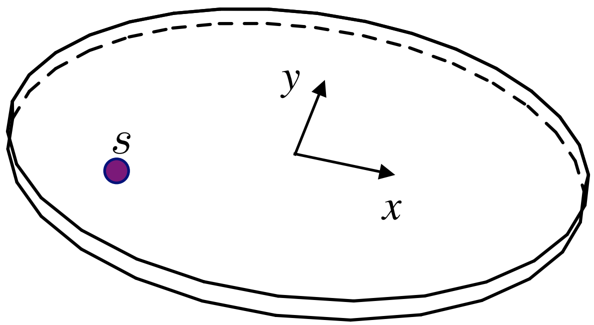
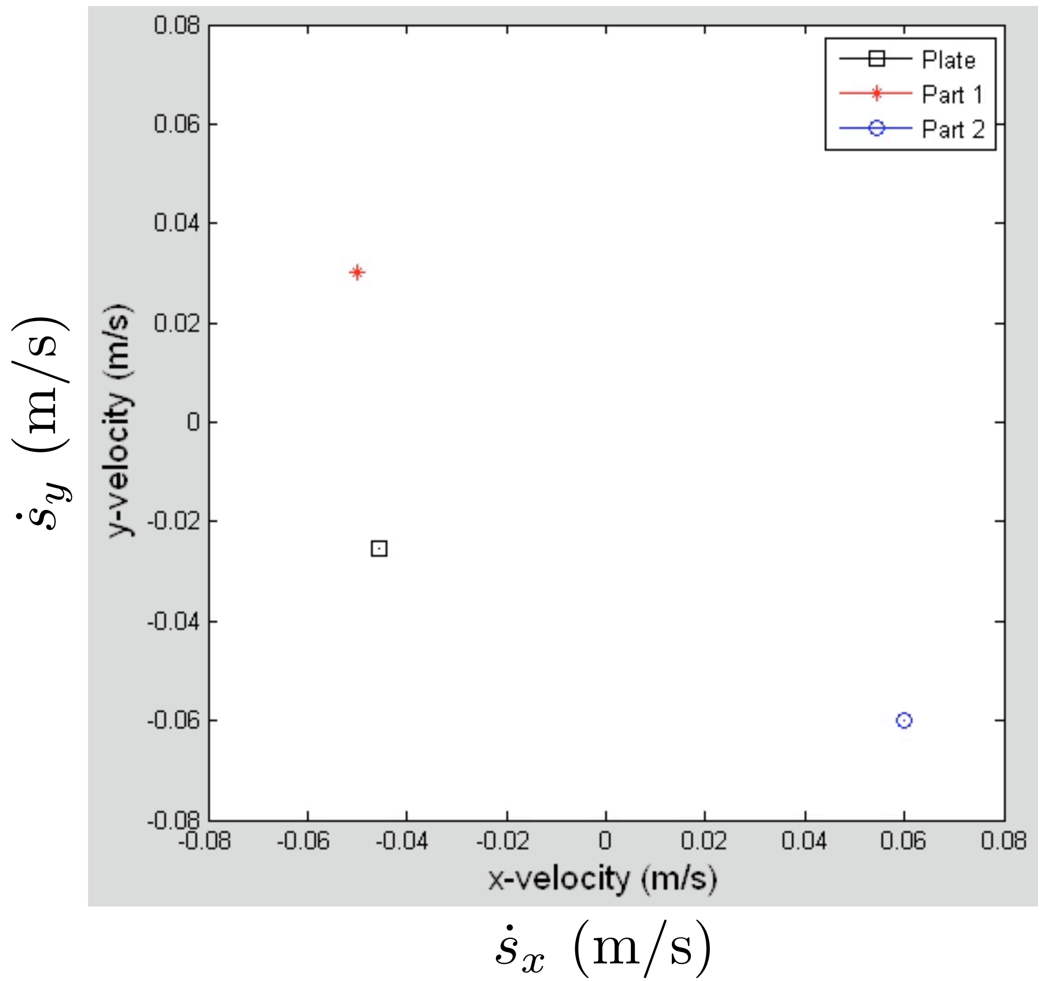
Natural representation of simplified part dynamics:

velocity field on part's **configuration space**
(not force field on plate surface)

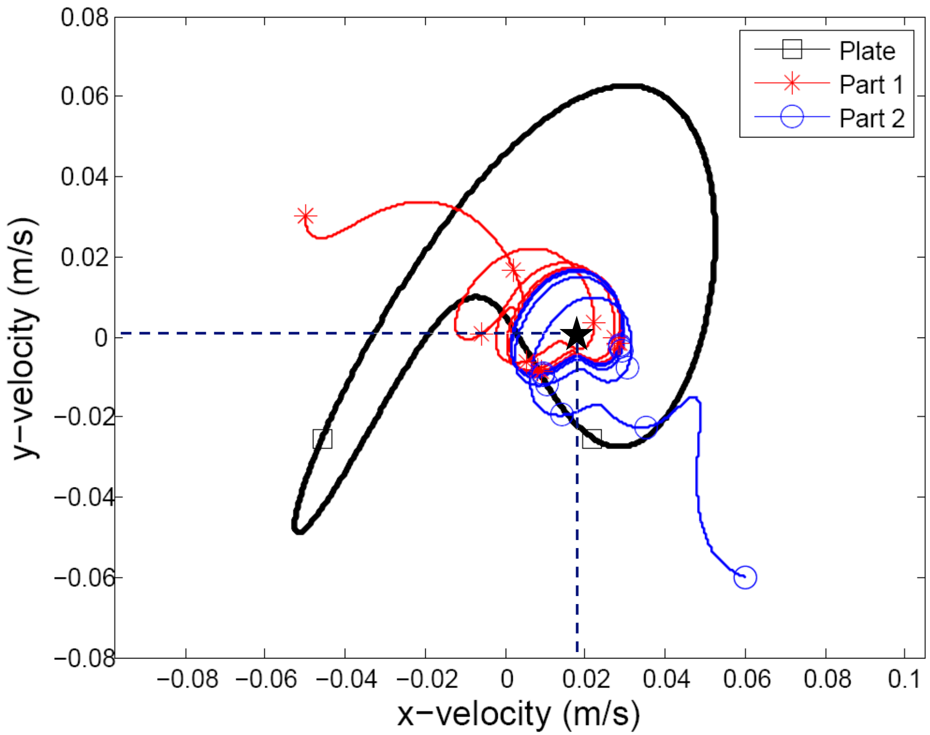
Asymptotic Behavior (Point Part)



Two velocity trajectories (red and blue) for the purple part shown at left, assuming its configuration does not change

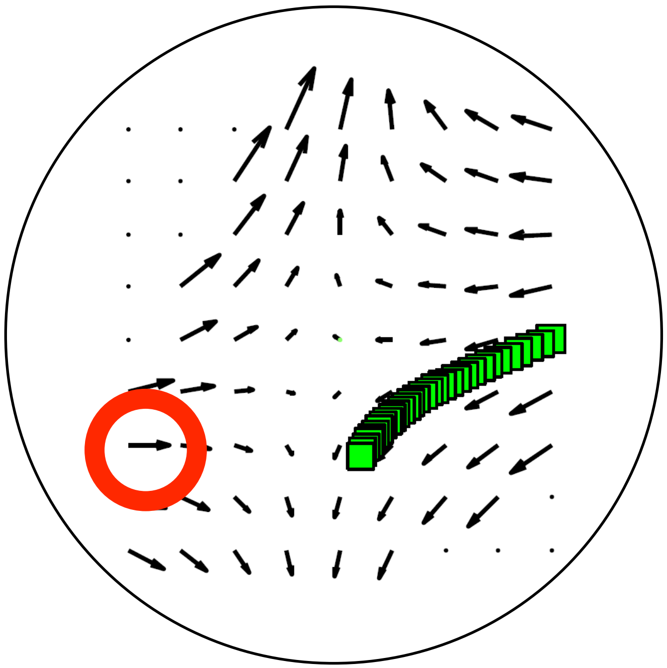


Asymptotic Velocity (Point Part)



Asymptotic velocity field for a point part

$$\mathbf{v}_a : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

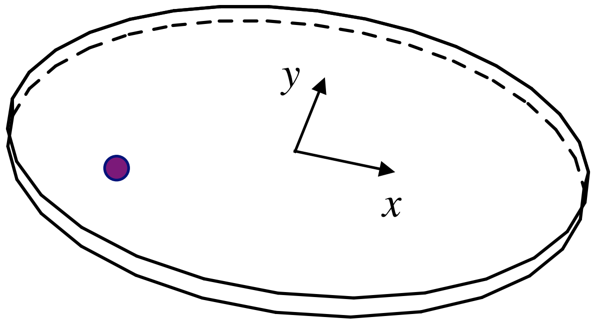


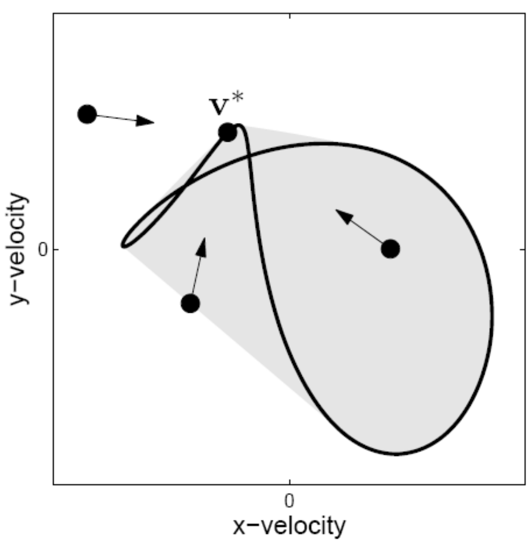
Asymptotic velocity at configuration (x,y):

$$\mathbf{v}_a(x,y) = \frac{1}{T}\int_0^T \mathbf{v}^{LC}(t)dt$$

Where $\mathbf{v}^{LC}(t)$ is the limit cycle.

Asymptotic Velocity (Point Part)





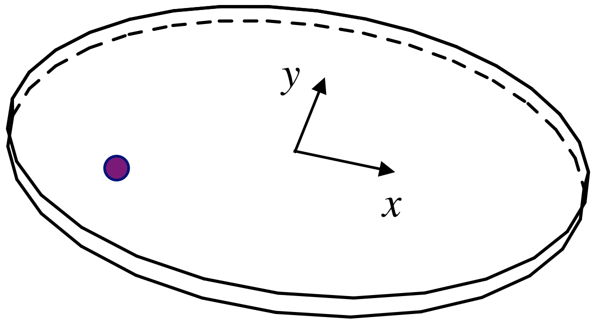
Theorem

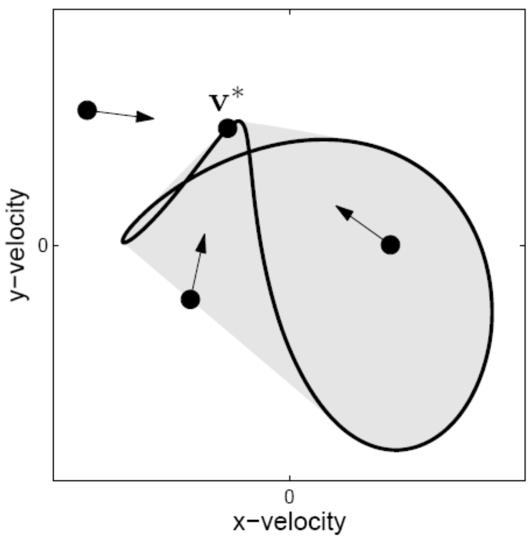
Given:

- simplified dynamics
- plate oscillation of period T

For every (valid) part configuration, the part's velocity trajectory asymptotically converges to a unique limit cycle of period T on or inside the convex hull of the plate's velocity trajectory.

Asymptotic Velocity (Point Part)

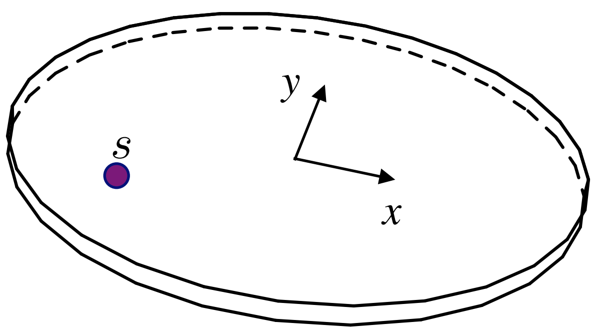




$$\mathbf{f}_{\text{fric}} = \mu N \frac{\mathbf{v}_{\text{rel}}}{\|\mathbf{v}_{\text{rel}}\|}$$

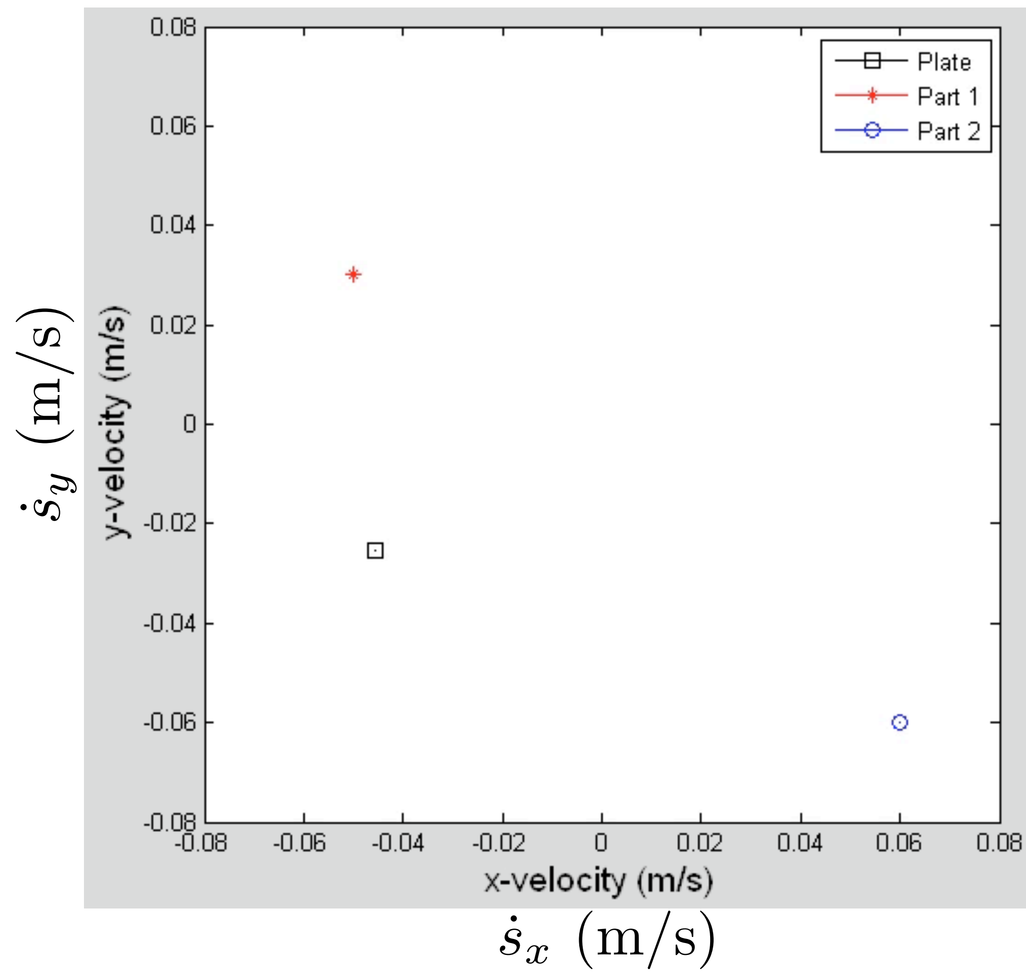
Normal force depends on part configuration and plate acceleration, but NOT part velocity

Asymptotic Behavior (Point Part)

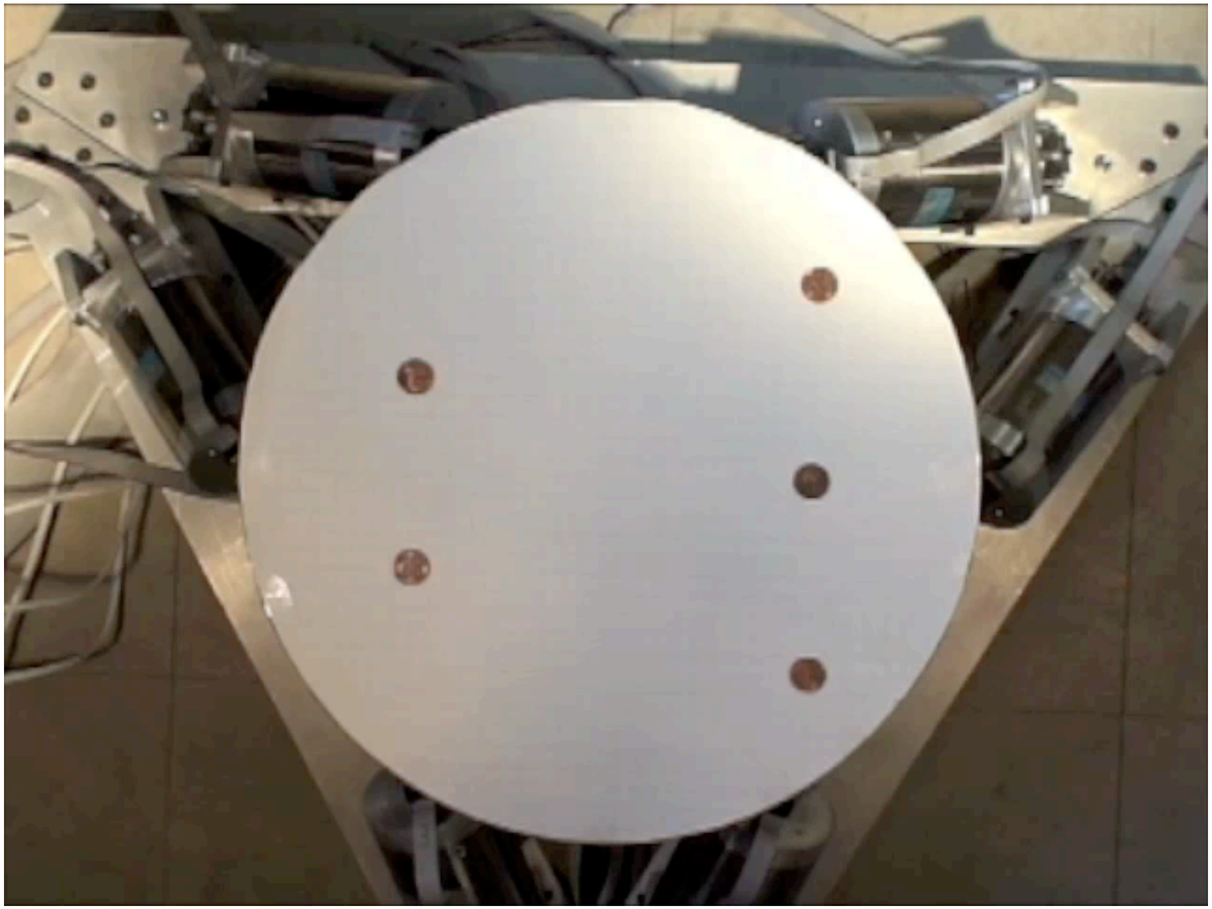


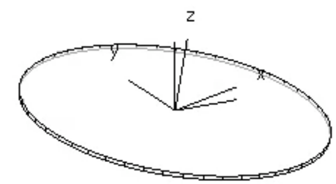
$$\mathbf{f}_{\mathrm{fric}} = \mu N \frac{\mathbf{v}_{\mathrm{rel}}}{\|\mathbf{v}_{\mathrm{rel}}\|}$$

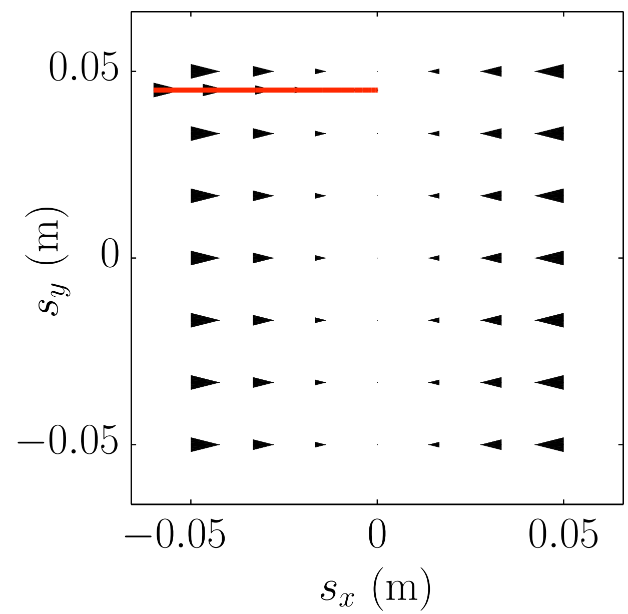
Pursuer-Evader: in velocity space, all parts "chase" the plate by moving directly toward it at the same speed



Example: LineSink

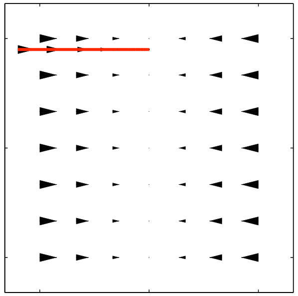


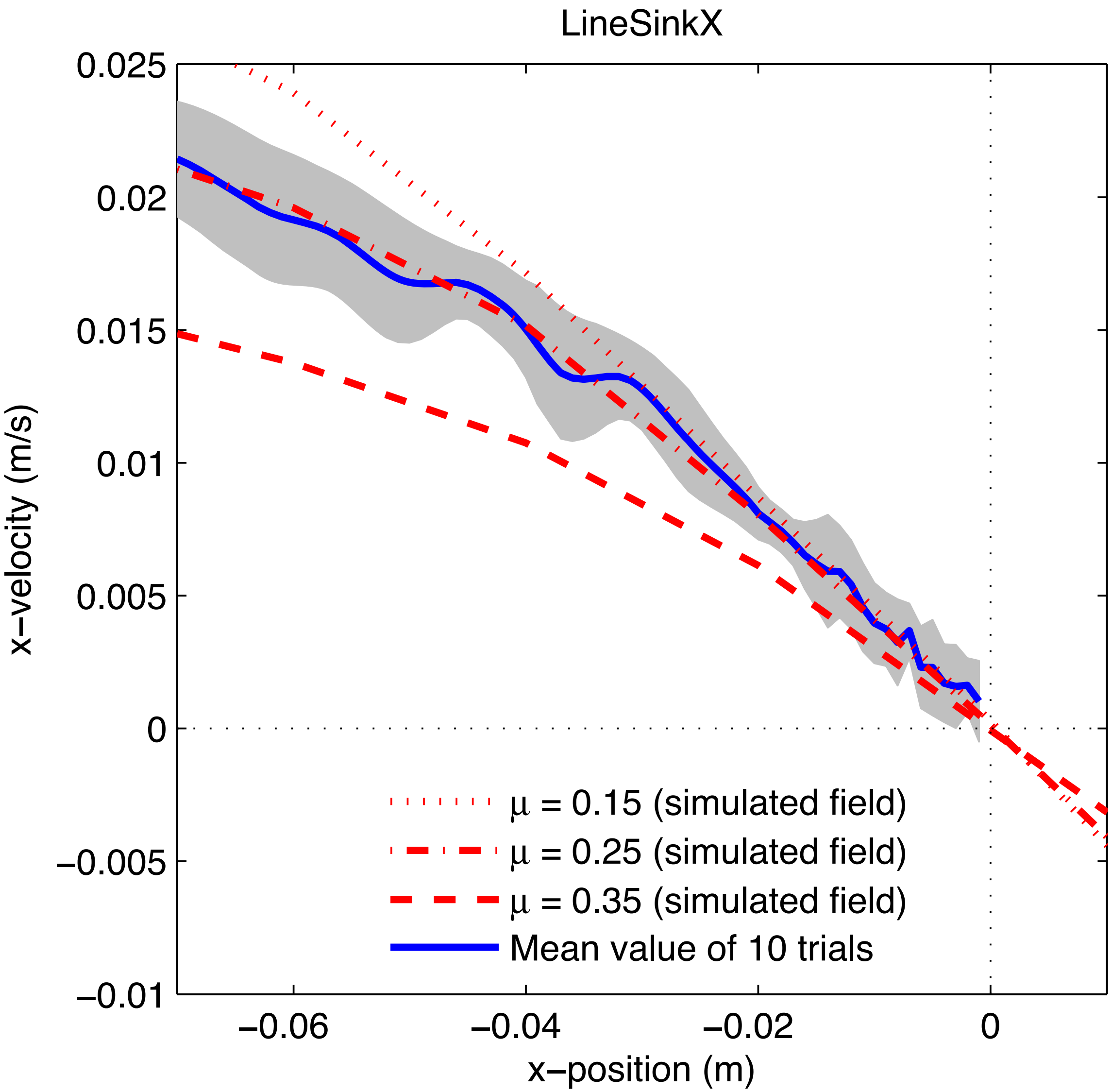




Asymptotics vs. Experiment

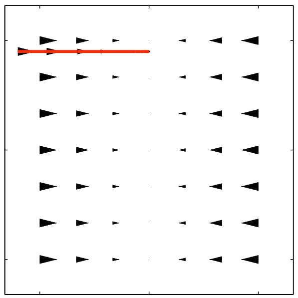
LineSink

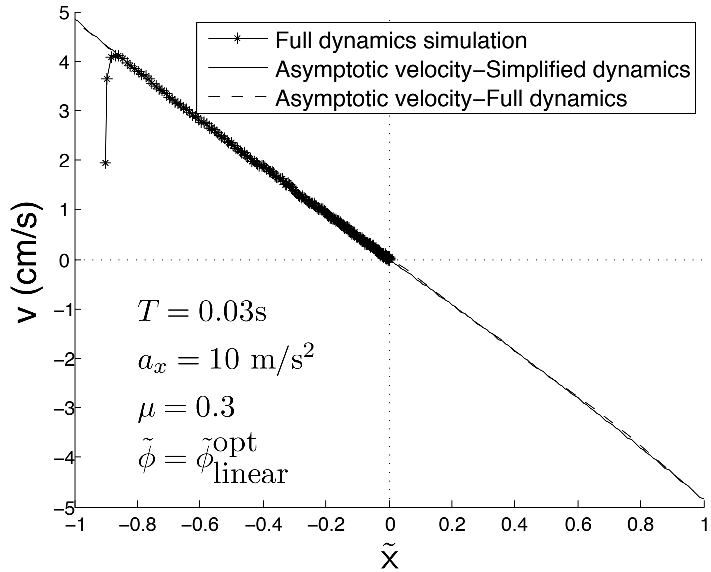


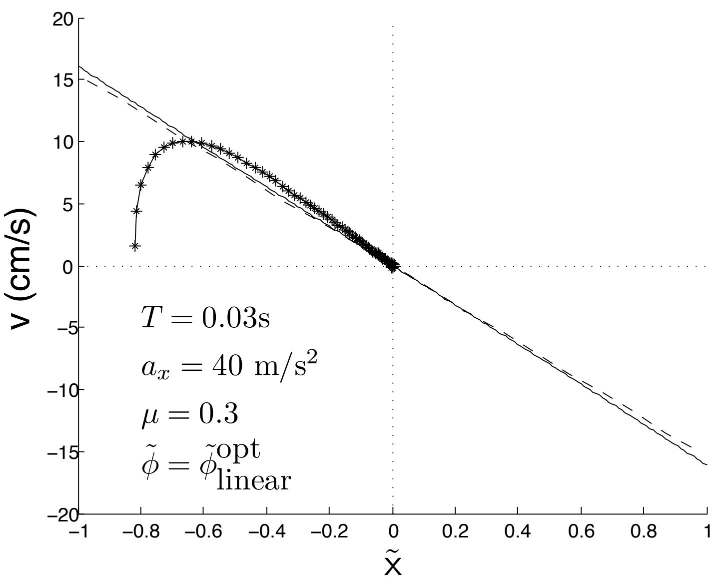


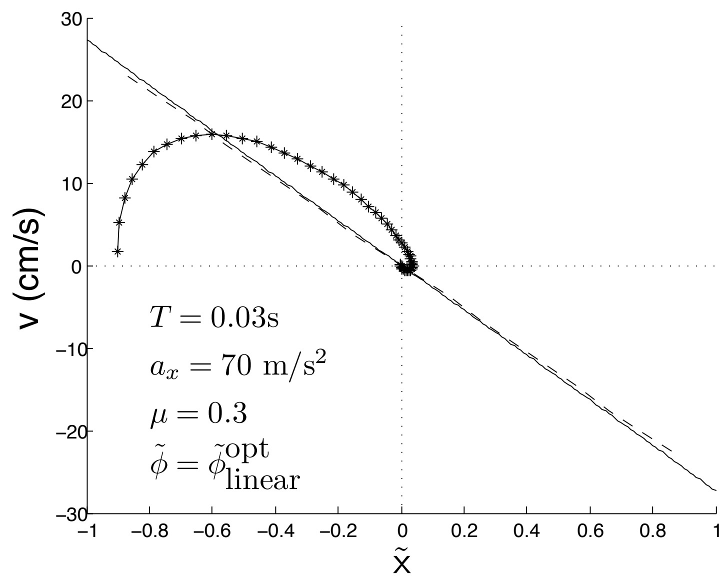
Asymptotics vs. Full Simulation

LineSink









Outline

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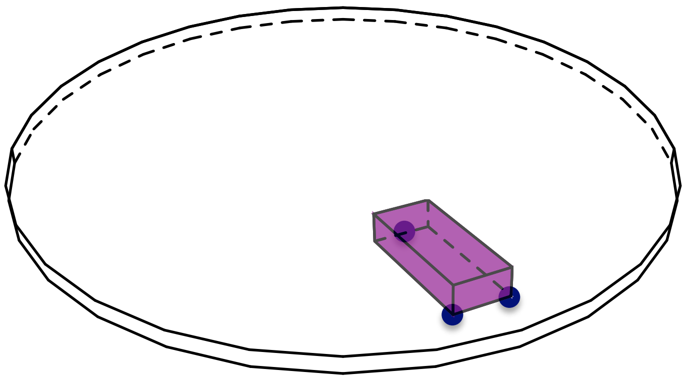
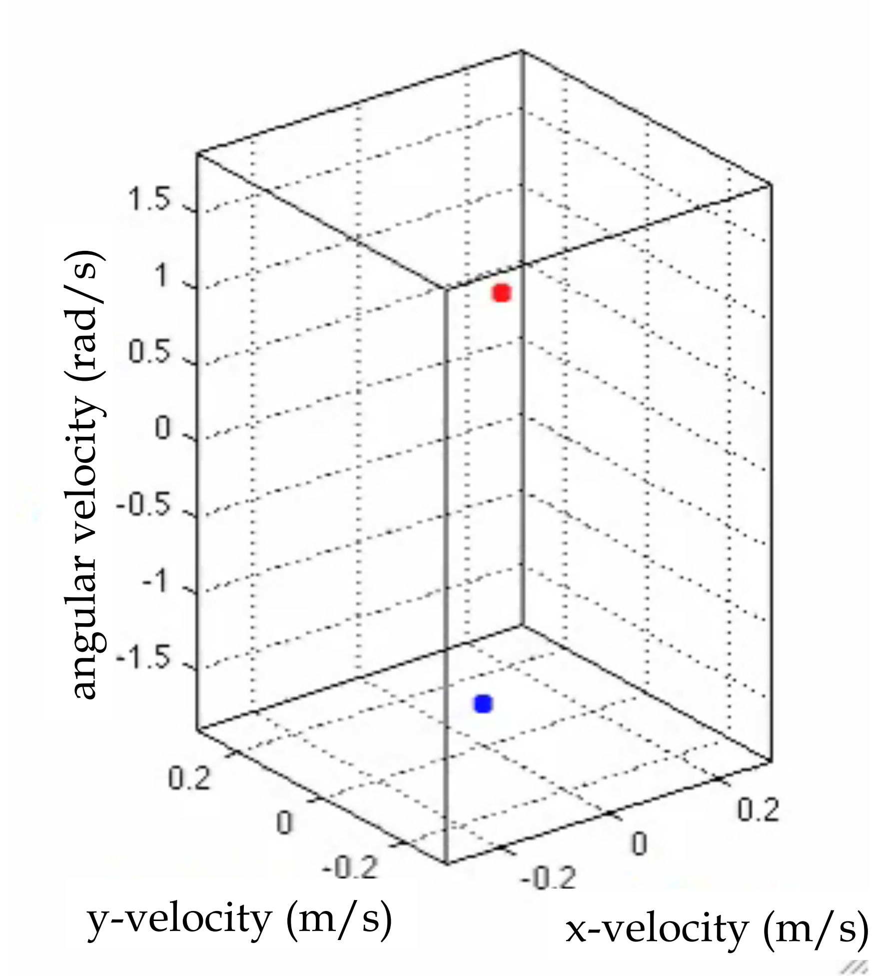
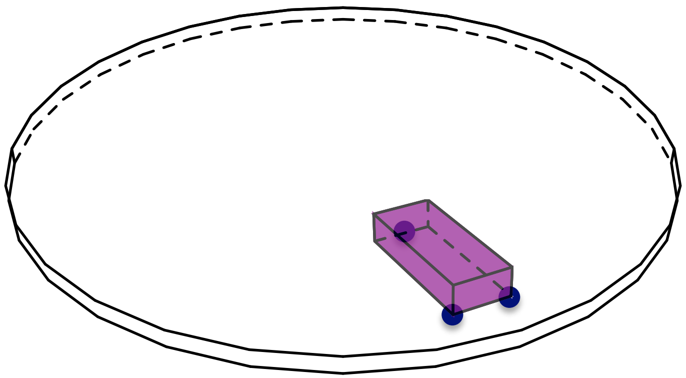
Asymptotic Behavior (Rigid Parts)



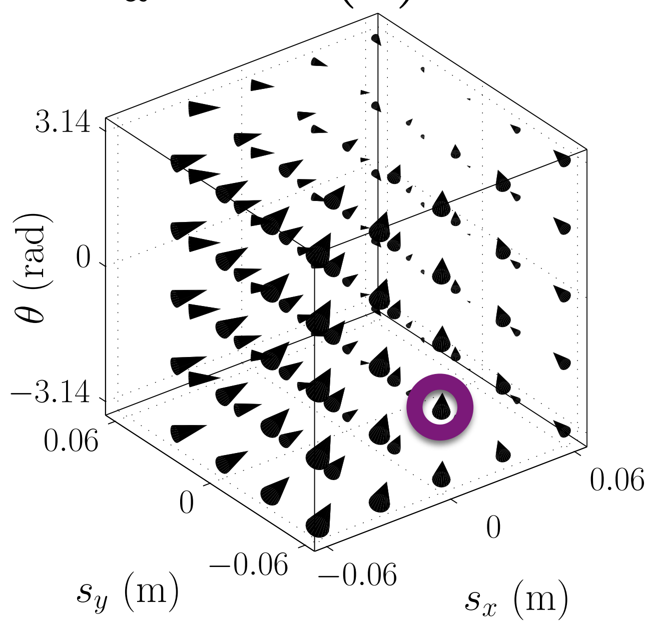
Two velocity trajectories for the purple part shown at left, assuming its configuration does not change



Asymptotic Velocity (Rigid Parts)



$$\mathbf{v}_a : SE(2) \to \mathbb{R}^3$$



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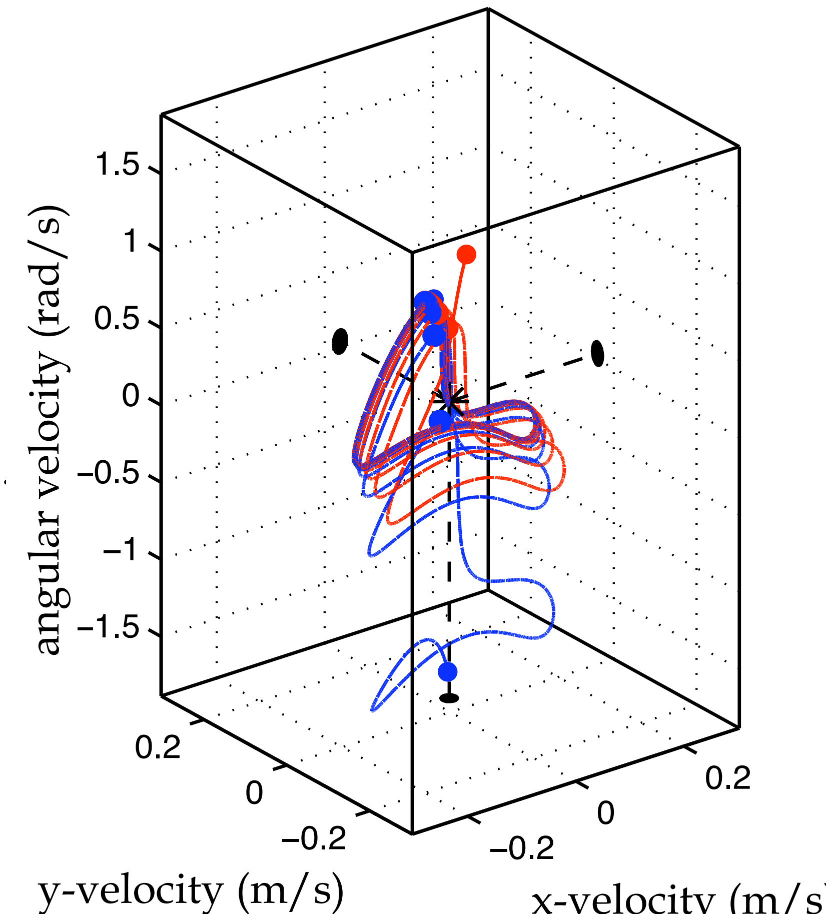
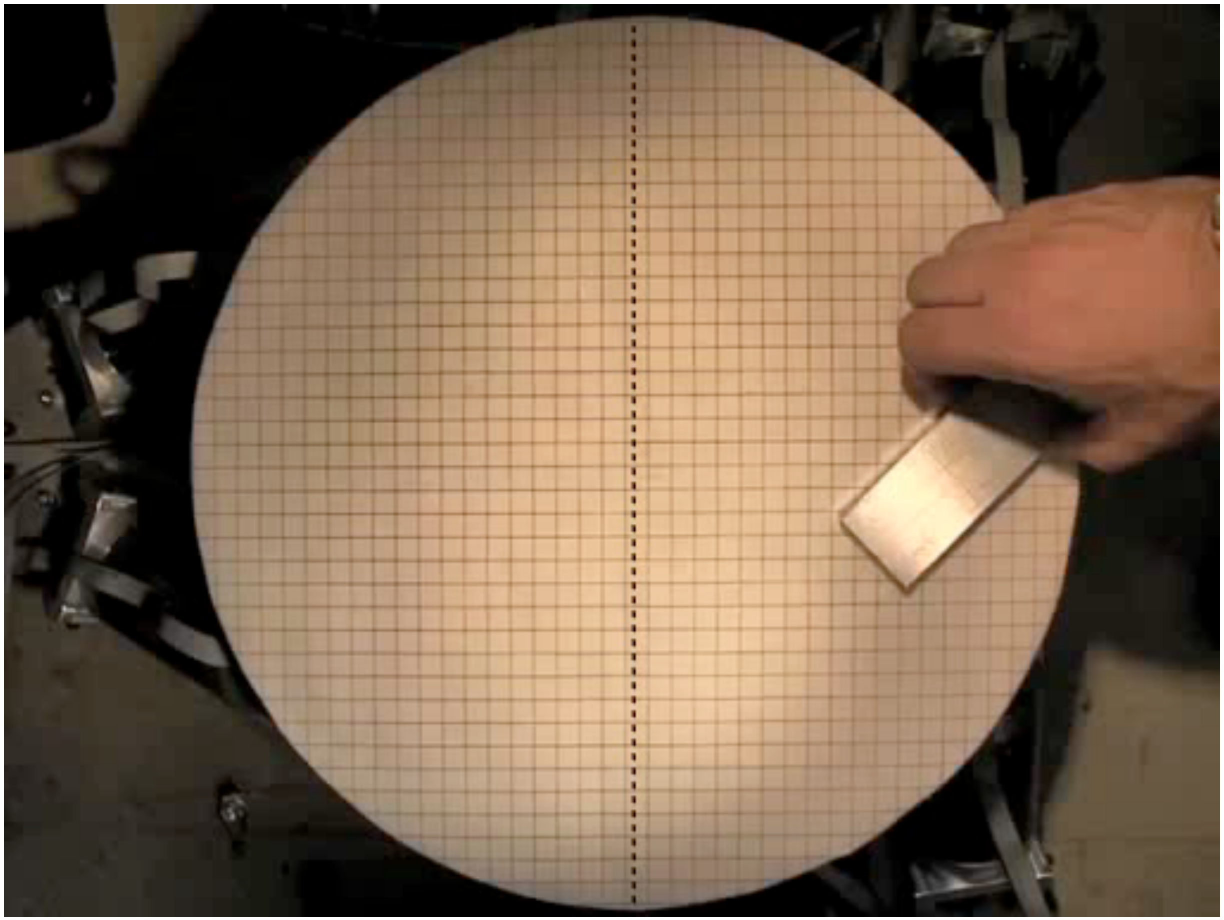
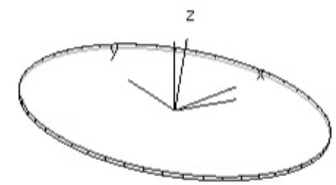
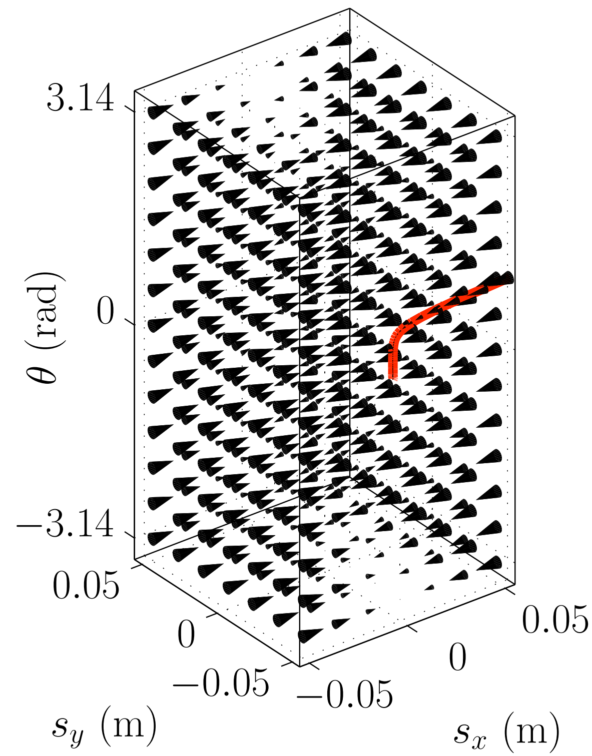


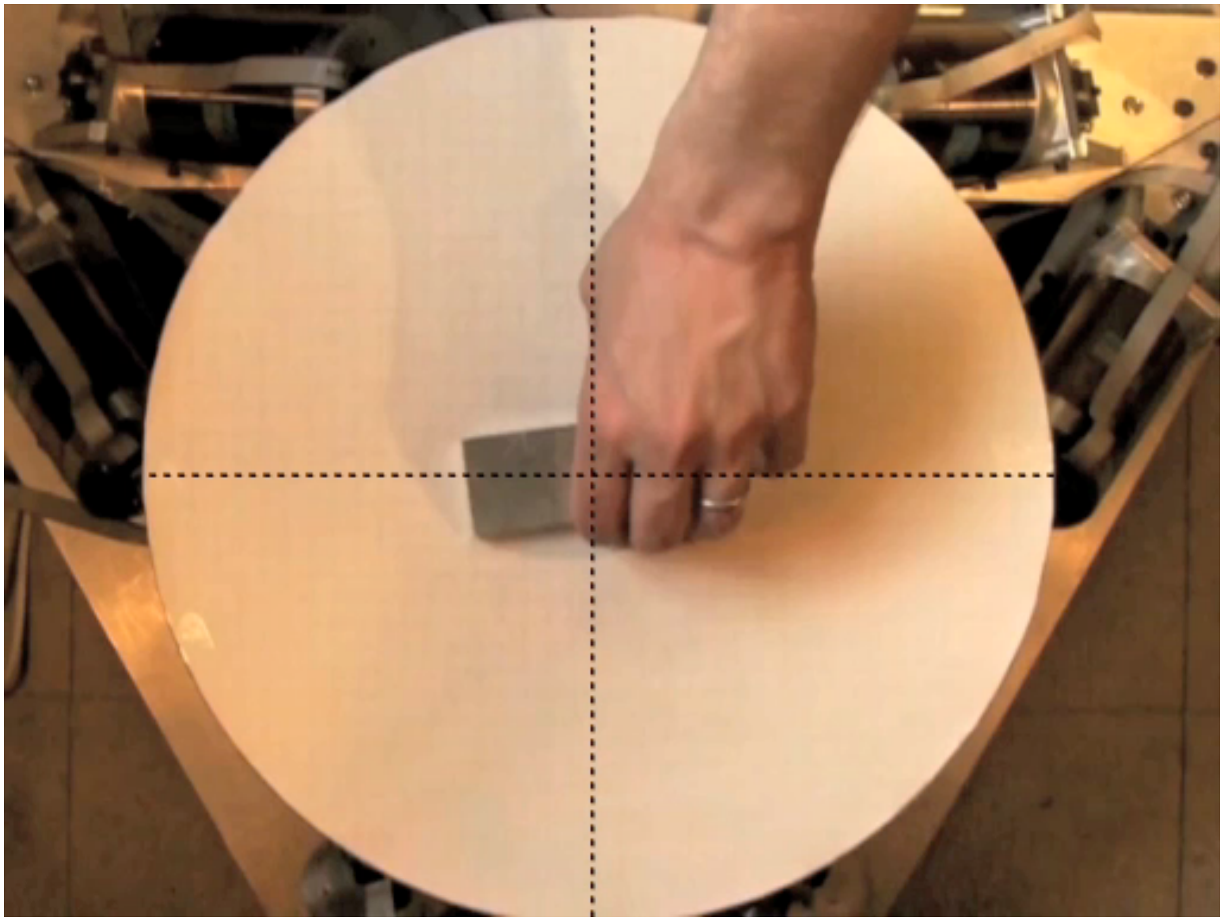
Sensorless Orienting and Positioning to a Line

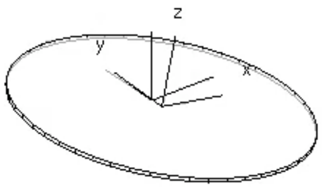


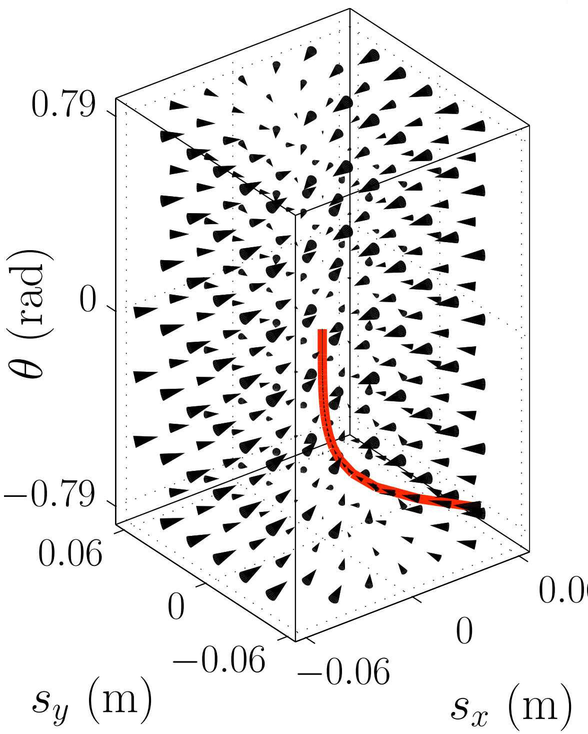




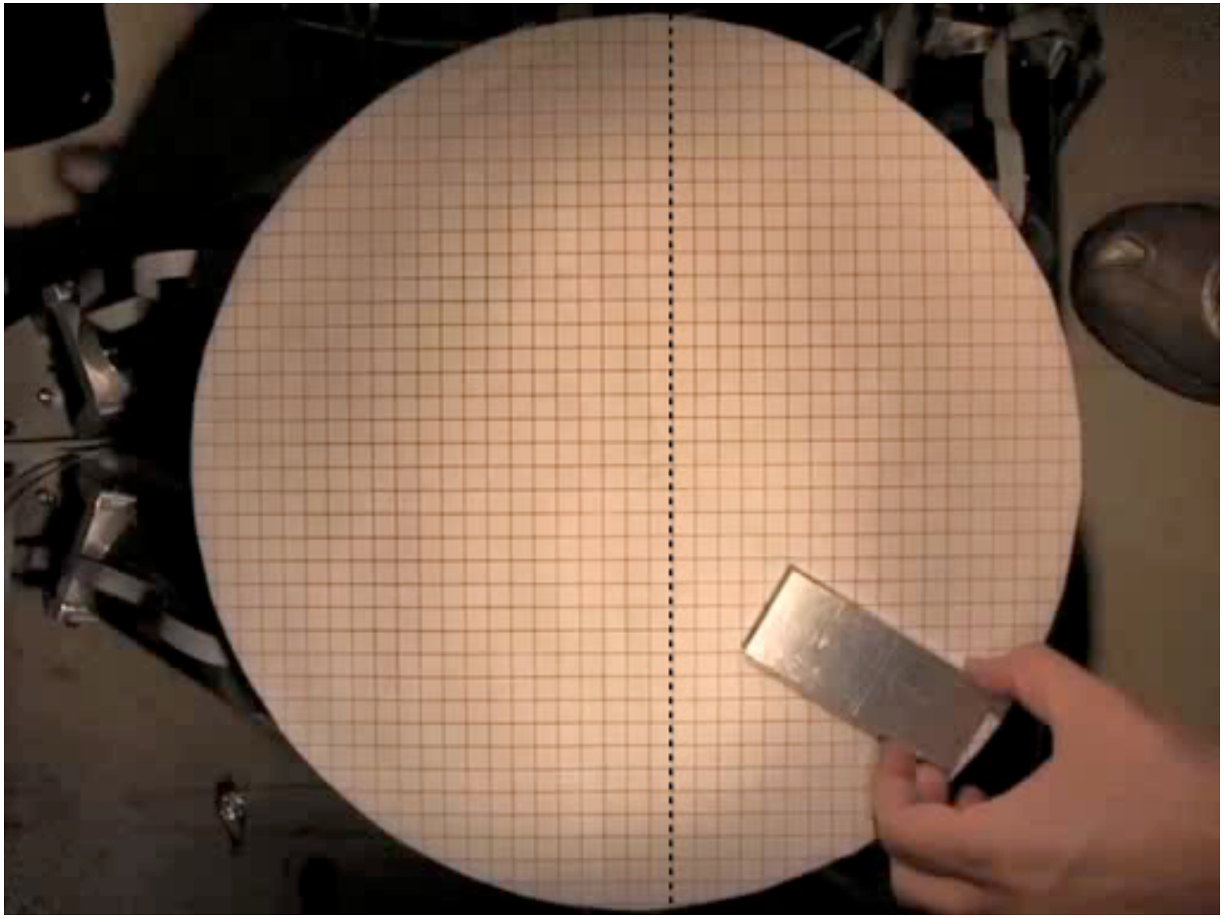
Sensorless Positioning and Orienting

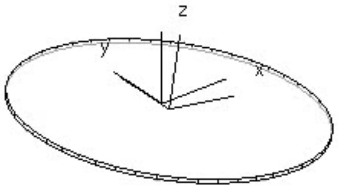


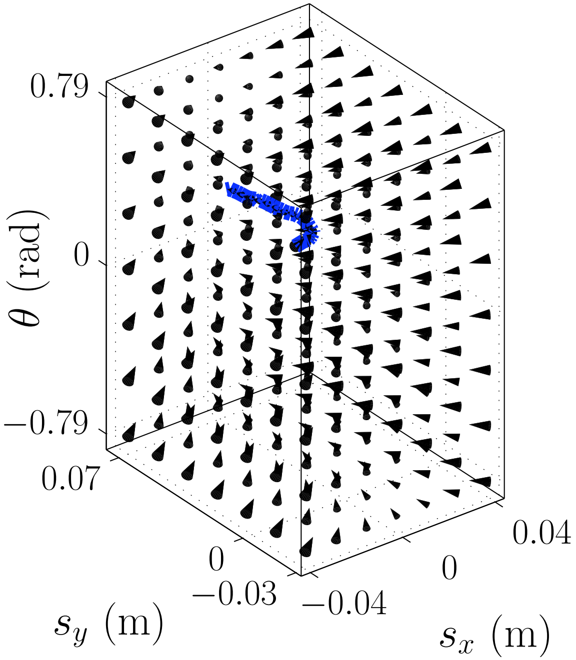




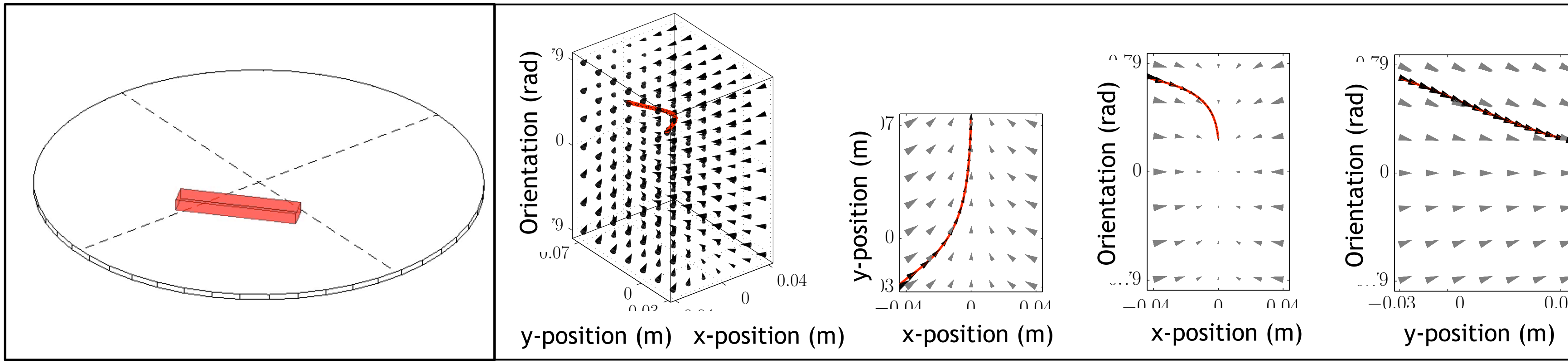
Sensorless Orientation and Transport





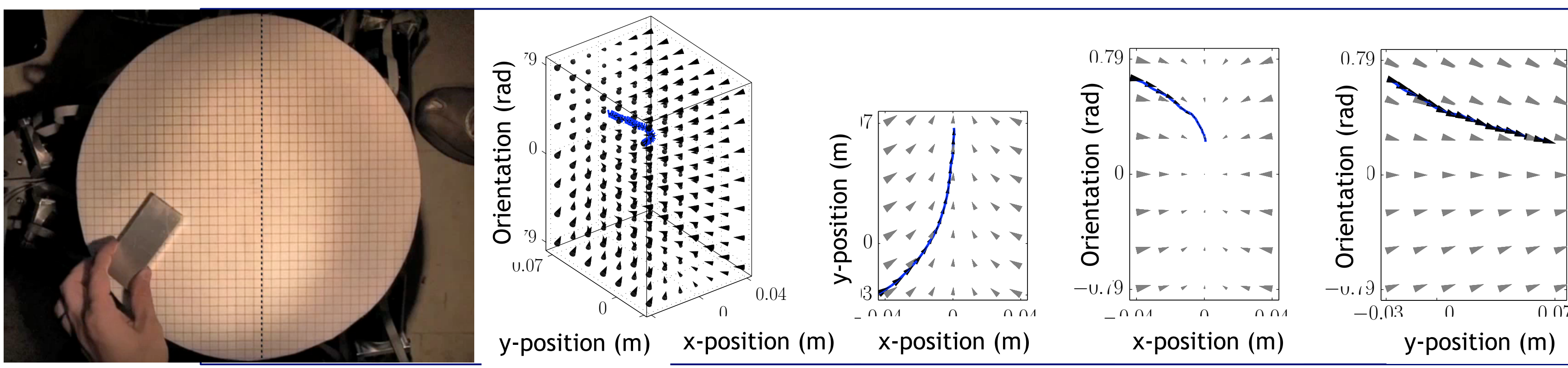


Full dynamic simulation vs. asymptotic velocity



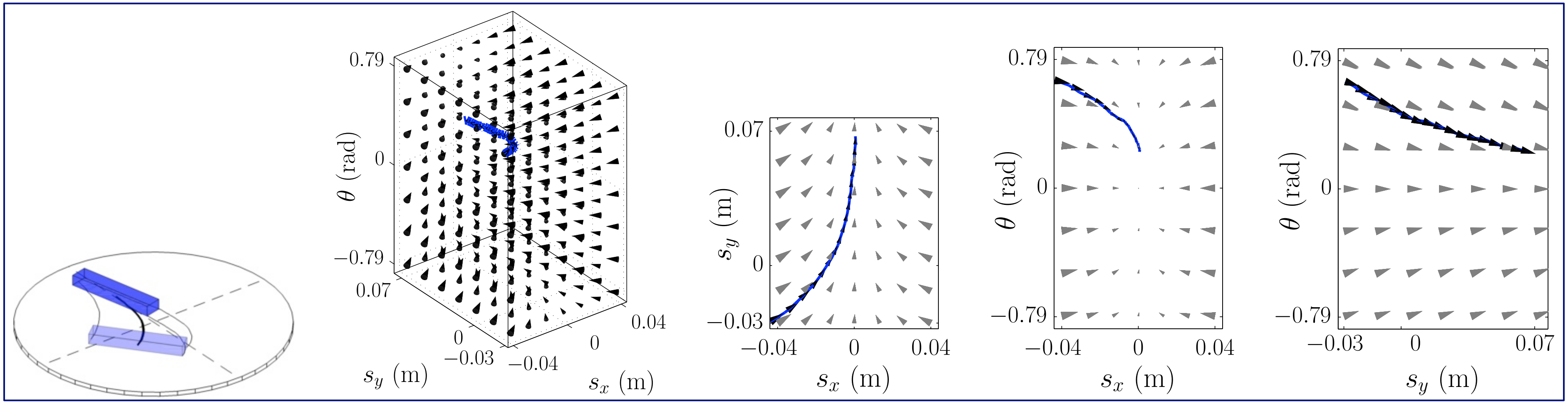
Red path = part trajectory simulated with full dynamics
Black arrows = asymptotic velocity vectors
Gray arrows = projections of asymptotic velocity vectors

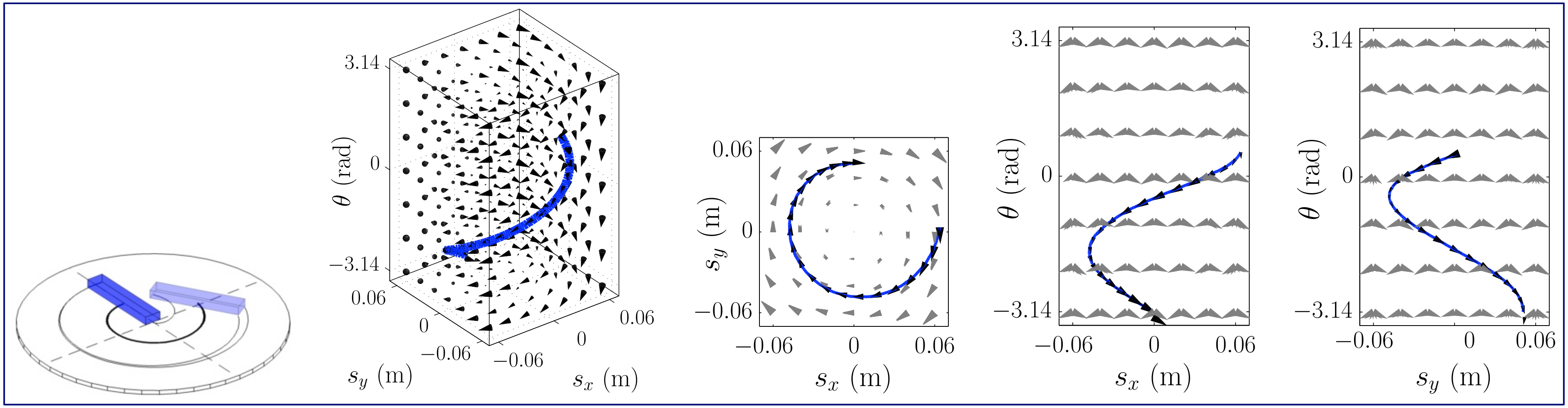
Experimental data vs. asymptotic velocity



Blue path = part trajectory obtained with an overhead camera
Black arrows = asymptotic velocity vectors
Gray arrows = projections of asymptotic velocity vectors

Asymptotics vs. Experimental Results





Outline

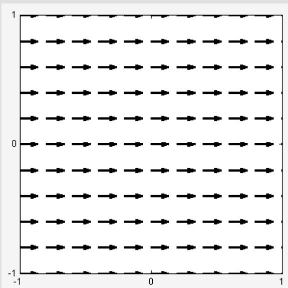
a nonprehensile primitive: vibratory sliding

- asymptotic velocity fields
- velocity fields for rigid bodies
- feasible velocity fields for point parts

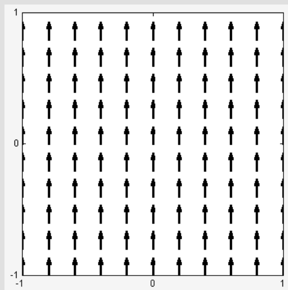
Basic Plate Motions/Basis Fields

(1 in-plane acceleration + 1 out-of-plane acceleration at the same frequency)

Translational

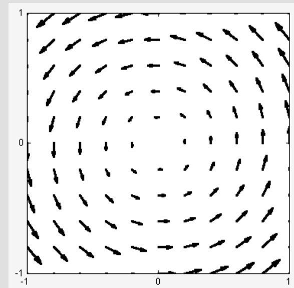


x-translation
+
z-translation



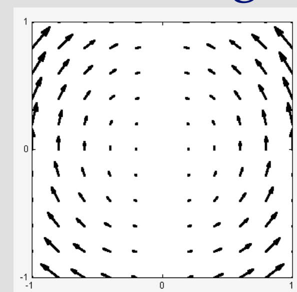
y-translation
+
z-translation

Circular

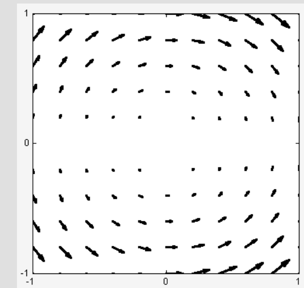


z-rotation
+
z-translation

Divergent Circular

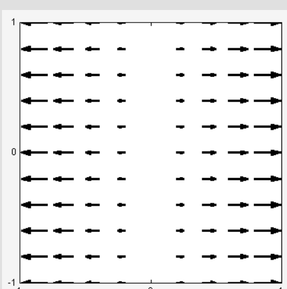


z-rotation
+
y-rotation

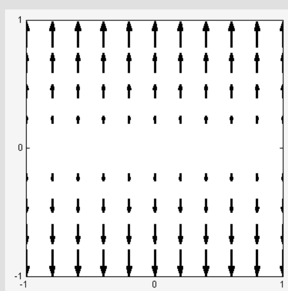


z-rotation
+
x-rotation

Line Sink/Line Source

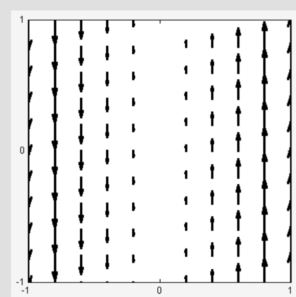


x-translation
+
y-rotation

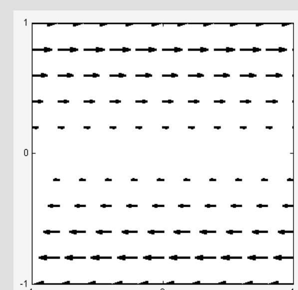


y-translation
+
x-rotation

Shear



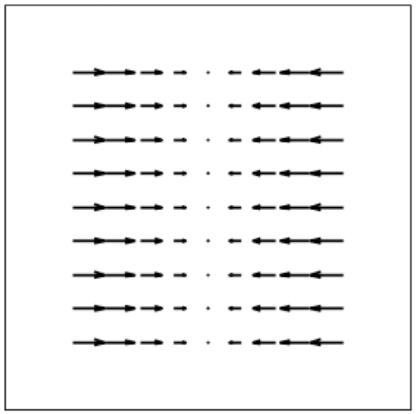
y-translation
+
y-rotation



x-translation
+
x-rotation

Dynamics Are Nonlinear

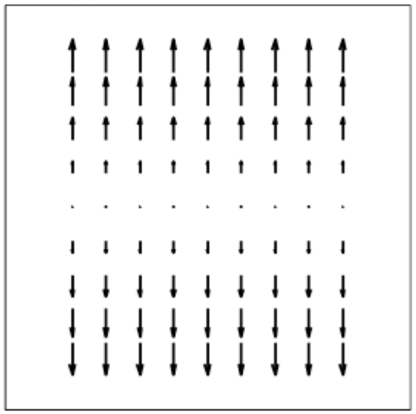
(a) LineSinkX



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 0$$
$$\alpha_x = 0$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$

$$\mathbf{v}_\mathrm{a} \approx \begin{bmatrix} -0.27x \\ 0 \end{bmatrix}$$

(b) LineSourceY

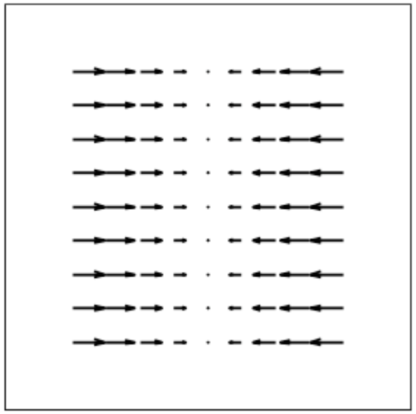


$$\ddot{p}_x = 0$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\alpha_y = 0$$

$$\mathbf{v}_\mathrm{a} \approx \begin{bmatrix} 0 \\ 0.27y \end{bmatrix}$$

Dynamics Are Nonlinear

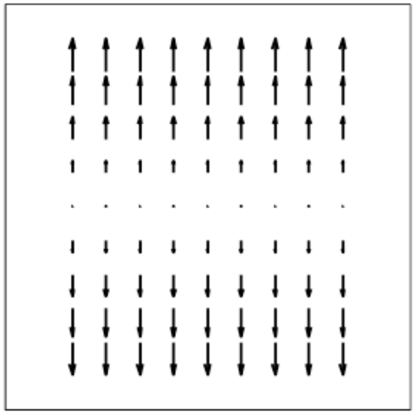
(a) LineSinkX



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 0$$
$$\alpha_x = 0$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$

$$\mathbf{v}_{\mathrm{a}} \approx \begin{bmatrix} -0.27x \\ 0 \end{bmatrix}$$

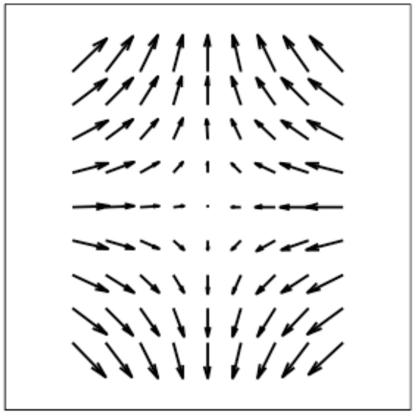
(b) LineSourceY



$$\ddot{p}_x = 0$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\alpha_y = 0$$

$$\mathbf{v}_{\mathrm{a}} \approx \begin{bmatrix} 0 \\ 0.27y \end{bmatrix}$$

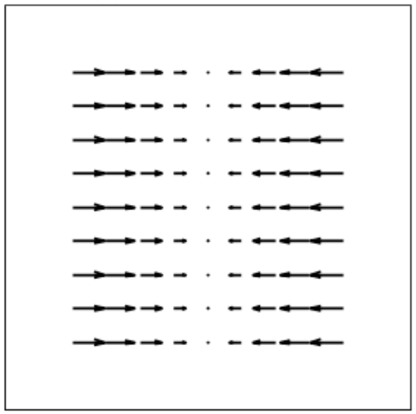
(d) Saddle



$$\mathbf{v}_{\mathrm{a}} \approx \begin{bmatrix} -0.27x \\ 0.27y \end{bmatrix}$$

Dynamics Are Nonlinear

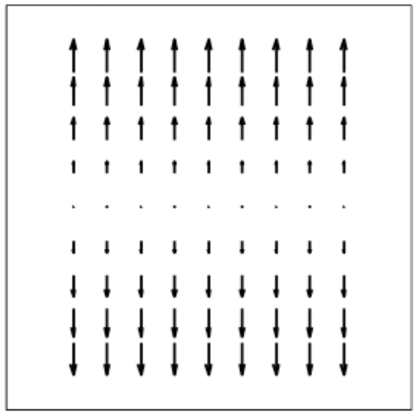
(a) LineSinkX



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 0$$
$$\alpha_x = 0$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$

$$\mathbf{v}_\mathrm{a} \approx \begin{bmatrix} -0.27x \\ 0 \end{bmatrix}$$

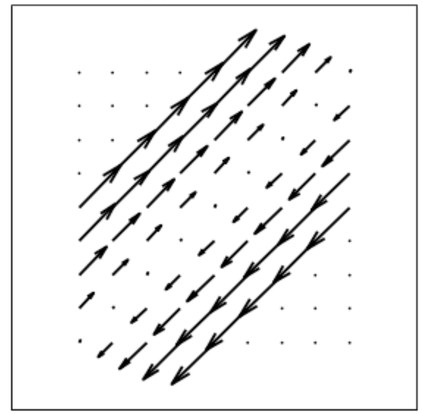
(b) LineSourceY



$$\ddot{p}_x = 0$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\alpha_y = 0$$

$$\mathbf{v}_\mathrm{a} \approx \begin{bmatrix} 0 \\ 0.27y \end{bmatrix}$$

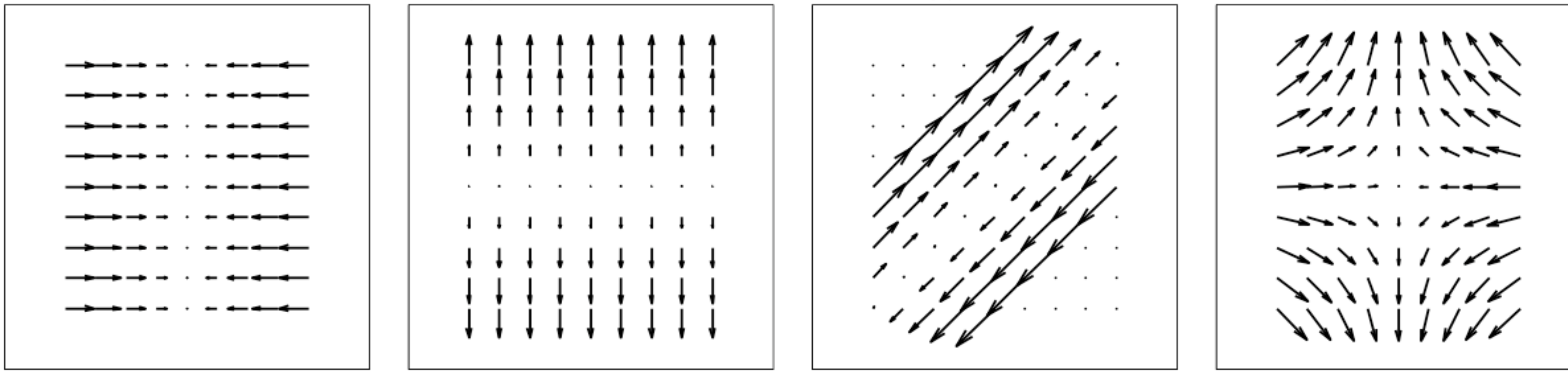
(c) Shear



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$

$$\mathbf{v}_\mathrm{a} \approx \begin{bmatrix} -0.35x + 0.35y \\ -0.34x + 0.35y \end{bmatrix}$$

Dynamics Are Nonlinear



(a) LineSinkX

$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 0$$
$$\alpha_x = 0$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$

$$\mathbf{v}_\text{a} \approx \begin{bmatrix} -0.27x \\ 0 \end{bmatrix}$$

(b) LineSourceY

$$\ddot{p}_x = 0$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\alpha_y = 0$$

$$\mathbf{v}_\text{a} \approx \begin{bmatrix} 0 \\ 0.27y \end{bmatrix}$$

(c) Shear

$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$

$$\mathbf{v}_\text{a} \approx \begin{bmatrix} -0.35x + 0.35y \\ -0.34x + 0.35y \end{bmatrix}$$

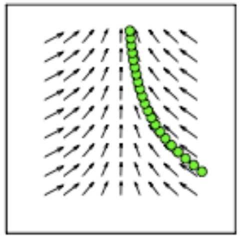
(d) Saddle

$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_x = 57\sin(60\pi t + \tfrac{5}{32}\pi)$$
$$\alpha_y = 57\sin(60\pi t + \tfrac{53}{32}\pi)$$

$$\mathbf{v}_\text{a} \approx \begin{bmatrix} -0.27x \\ 0.27y \end{bmatrix}$$

design by nonlinear optimization,
initial guess from linear superposition of “basis” fields

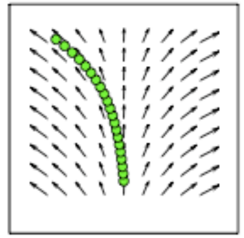
(a) SqueezeTrans



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\ddot{p}_z = 5\sin(60\pi t + \tfrac{3}{20}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{5}{3}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} -0.41x \\ 0.02 \end{bmatrix}$$

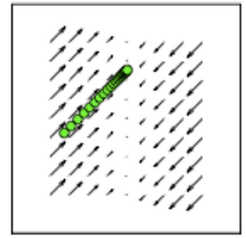
(b) DivTrans



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\ddot{p}_z = 5\sin(60\pi t + \tfrac{3}{20}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{2}{3}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} 0.41x \\ 0.02 \end{bmatrix}$$

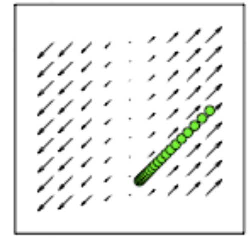
(c) SkewLineSink



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} -0.34x \\ -0.34x \end{bmatrix}$$

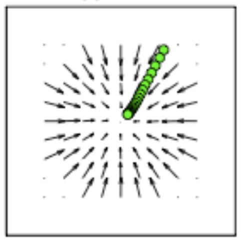
(d) SkewLineSource



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{1}{2}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} 0.34x \\ 0.34x \end{bmatrix}$$

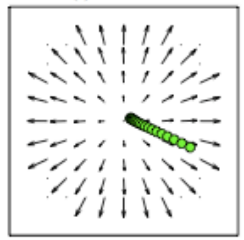
(e) Sink



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{75}{64}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{107}{64}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} -0.42x \\ -0.42y \end{bmatrix}$$

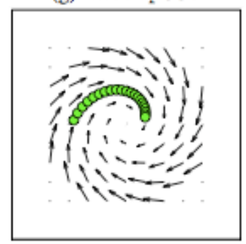
(f) Source



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{11}{64}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{43}{64}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} 0.42x \\ 0.42y \end{bmatrix}$$

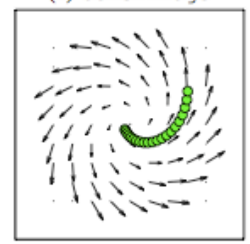
(g) Whirlpool



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\alpha_y = 100\sin(60\pi t)$$

$$\mathbf{v}_a \approx \begin{bmatrix} -0.22x + 0.36y \\ -0.36x - 0.22y \end{bmatrix}$$

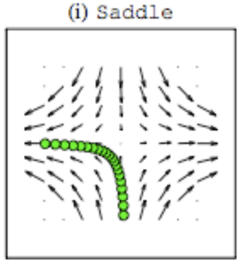
(h) Centrifuge



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} 0.22x - 0.36y \\ 0.36x + 0.22y \end{bmatrix}$$

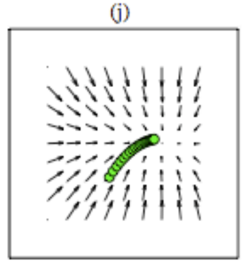
(i) Saddle



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{75}{64}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{43}{64}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} 0.42x \\ -0.42y \end{bmatrix}$$

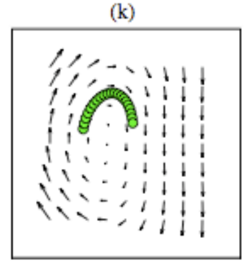
(j)



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\ddot{p}_z = 2\sin(60\pi t + \tfrac{107}{64}\pi)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{75}{64}\pi)$$
$$\alpha_y = 50\sin(60\pi t + \tfrac{107}{64}\pi)$$

$$\mathbf{v}_a \approx \begin{bmatrix} -0.22x + 0.01 \\ -0.42y \end{bmatrix}$$

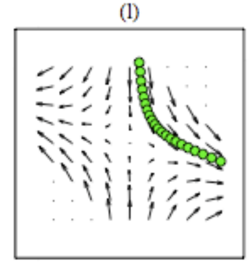
(k)



$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{3}{2}\pi)$$
$$\ddot{p}_z = 5\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \pi)$$
$$\alpha_z = 100\sin(60\pi t)$$

$$\mathbf{v}_a \approx \begin{bmatrix} -2.3xy + 0.17y \\ 2.3x^2 - 0.47x - 0.01 \end{bmatrix}$$

(l)



$$\ddot{p}_x = 10\sin(60\pi t)$$
$$\ddot{p}_y = 10\sin(60\pi t + \tfrac{1}{2}\pi)$$
$$\alpha_x = 100\sin(60\pi t + \tfrac{75}{64}\pi)$$
$$\alpha_y = 100\sin(60\pi t + \tfrac{30}{64}\pi)$$

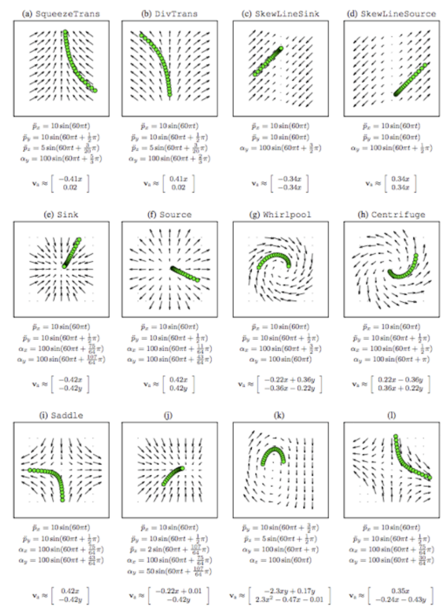
$$\mathbf{v}_a \approx \begin{bmatrix} 0.35x \\ -0.24x - 0.43y \end{bmatrix}$$

One-Frequency Plate Motions

11-dimensional space of plate motions
- 6 amplitudes
- 5 phases

$$\mathbf{u} = \begin{bmatrix} \ddot{p}_x \\ \ddot{p}_y \\ \ddot{p}_z \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} A_x \sin(2\pi f t) \\ A_y \sin(2\pi f t + \phi_y) \\ A_z \sin(2\pi f t + \phi_z) \\ A_\theta \sin(2\pi f t + \phi_\theta) \\ A_\varphi \sin(2\pi f t + \phi_\varphi) \\ A_\psi \sin(2\pi f t + \phi_\psi) \end{bmatrix}$$

8^{+}-dimensional space of fields
- All constant fields
- All linear fields
- Some quadratic fields
- others



$$v_x(x,y) = a_1 y^2 + a_2 xy + b_1 x + b_2 y + c_1$$

$$v_y(x,y) = a_2 y^2 + a_1 xy + b_3 x + b_4 y + c_2$$

Two-Frequency Plate Motions

23-dimensional space of plate motions

- 12 amplitudes
- 11 phases

$$\mathbf{u} = \begin{bmatrix} \ddot{p}_x \\ \ddot{p}_y \\ \ddot{p}_z \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} A_{x,1}\sin(2\pi f t) + A_{x,2}\sin(4\pi f t + \phi_{x,2}) \\ A_{y,1}\sin(2\pi f t + \phi_{y,1}) + A_{y,2}\sin(4\pi f t + \phi_{y,2}) \\ A_{z,1}\sin(2\pi f t + \phi_{z,1}) + A_{z,2}\sin(4\pi f t + \phi_{z,2}) \\ A_{\theta,1}\sin(2\pi f t + \phi_{\theta,1}) + A_{\theta,2}\sin(4\pi f t + \phi_{\theta,2}) \\ A_{\varphi,1}\sin(2\pi f t + \phi_{\varphi,1}) + A_{\varphi,2}\sin(4\pi f t + \phi_{\varphi,2}) \\ A_{\psi,1}\sin(2\pi f t + \phi_{\psi,1}) + A_{\psi,2}\sin(4\pi f t + \phi_{\psi,2}) \end{bmatrix}$$

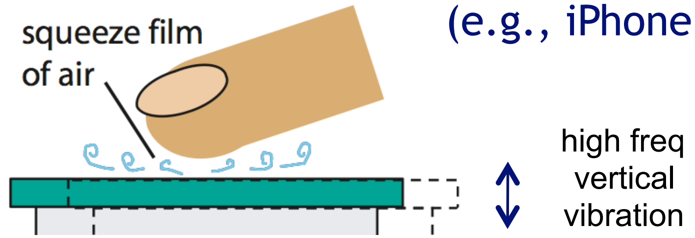
12^{+}-dimensional space of fields

- All constant fields
- All linear fields
- All quadratic fields?
- others

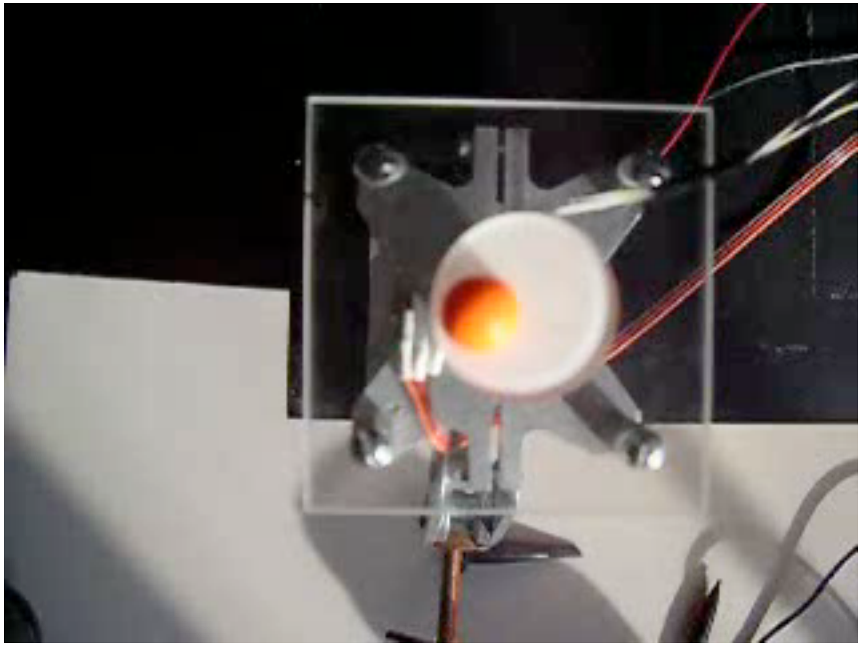
Extensions

- controlling friction

- part interaction, assembly



$$\mathbf{f}_{\mathrm{fric}} = \boxed{\mu} N \frac{\mathbf{v}_{\mathrm{rel}}}{\|\mathbf{v}_{\mathrm{rel}}\|}$$

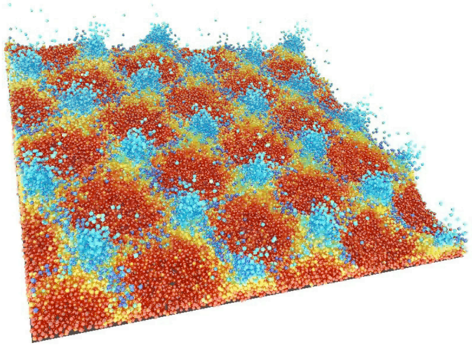


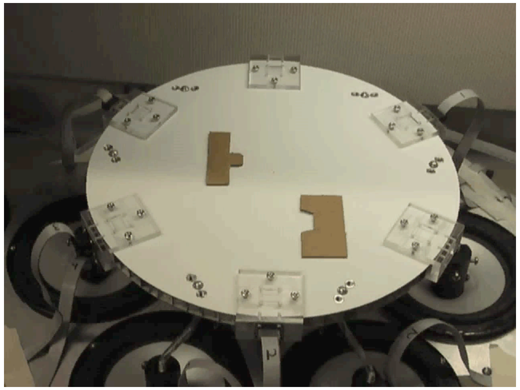
Colgate, Peshkin, et al.

Extensions

- controlling friction

- part interaction, assembly





(world's worst peg-in-hole)