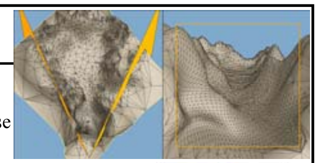
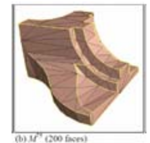
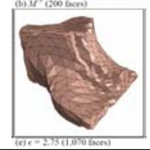
Curves & Surfaces

Last Time?



- Mesh Simplification
  - vertex split / edge collapse
  - preserving topology & discontinuities
- Geomorphs
- Progressive Transmission
- Compression
- Selective View-Dependent Refinement



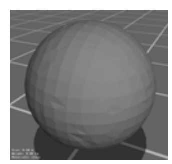


Today

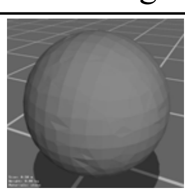
- Motivation
  - Limitations of Polygonal Models
  - Some Modeling Tools & Definitions
- Curves
- Surfaces / Patches
- Subdivision Surfaces

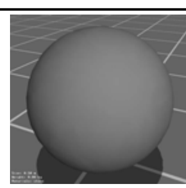
Limitations of Polygonal Meshes

- Planar facets (& silhouettes)
- Fixed resolution
- Deformation is difficult
- No natural parameterization (for texture mapping)



Can We Disguise the Facets?

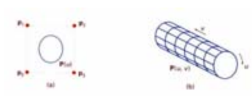




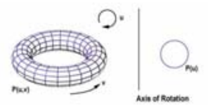
Better, but not always good enough

- Still low, fixed resolution (missing fine details)
- Still have polygonal silhouettes
- Intersection depth is planar (e.g. ray tracing visualization)
- Collisions problems for simulation
- Solid Texturing problems
- ...

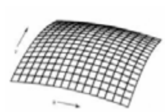
Some Non-Polygonal Modeling Tools



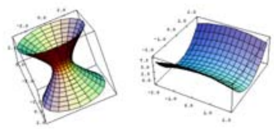
Extrusion



Surface of Revolution



Spline Surfaces/Patches



Quadrics and other implicit polynomials

Continuity definitions:

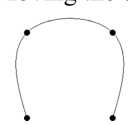
- C^0 continuous
  - curve/surface has no breaks/gaps/holes
- G^1 continuous
  - tangent at joint has same direction
- C^1 continuous
  - curve/surface derivative is continuous
  - tangent at joint has same direction *and* magnitude
- C^n continuous
  - curve/surface through n^{th} derivative is continuous
  - important for shading

Today

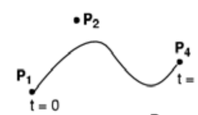
- Motivation
- Curves
  - What's a Spline?
  - Linear Interpolation
  - Interpolation Curves vs. Approximation Curves
  - Bézier
  - BSpline (NURBS)
- Surfaces / Patches
- Subdivision Surfaces

Definition: What's a Spline?

- Smooth curve defined by some control points
- Moving the control points changes the curve

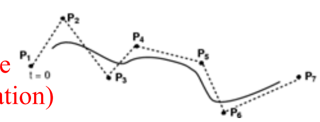


Interpolation

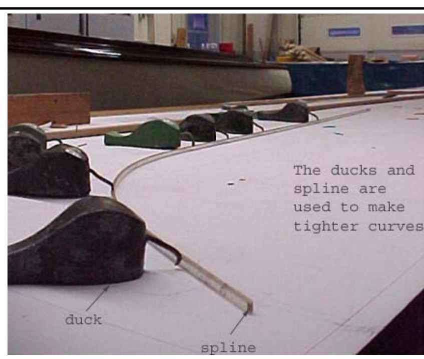


Bézier (approximation)

BSpline (approximation)

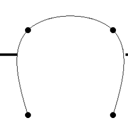


Interpolation Curves / Splines



www.abm.org

Interpolation Curves

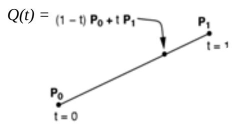


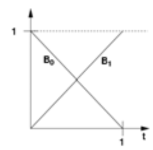
- Curve is constrained to pass through all control points
- Given points P_0, P_1, ... P_n, find lowest degree polynomial which passes through the points

$$x(t) = a_{n-1}t^{n-1} + ... + a_2t^2 + a_1t + a_0$$
$$y(t) = b_{n-1}t^{n-1} + ... + b_2t^2 + b_1t + b_0$$

Linear Interpolation

- Simplest "curve" between two points





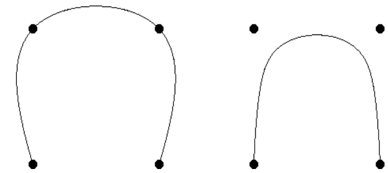
Spline Basis Functions

a.k.a. Blending Functions

$$Q(t) = \begin{pmatrix} Q_x(t) \\ Q_y(t) \\ Q_z(t) \end{pmatrix} = \begin{pmatrix} (P_0) & (P_1) \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

Interpolation vs. Approximation Curves

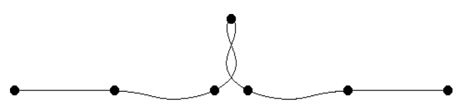


Interpolation: curve must pass through control points

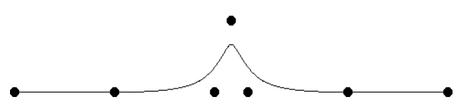
Approximation: curve is influenced by control points

Interpolation vs. Approximation Curves

- Interpolation Curve – over constrained → lots of (undesirable?) oscillations

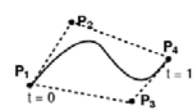


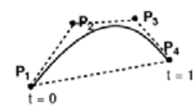
- Approximation Curve – more reasonable?

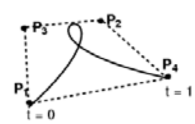


Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $\mathbf{P_0}$ to ($\mathbf{P_0}$-$\mathbf{P_1}$) and at $\mathbf{P_4}$ to ($\mathbf{P_4}$-$\mathbf{P_3}$)



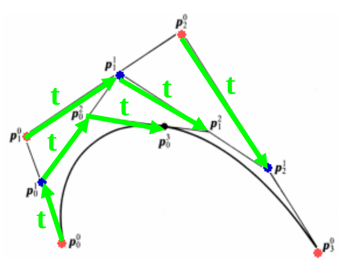




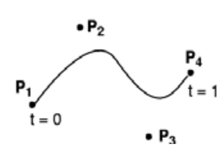
A Bézier curve is bounded by the convex hull of its control points.

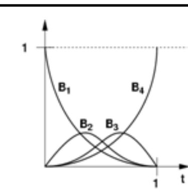
Cubic Bézier Curve

- de Casteljau's algorithm for constructing Bézier curves



Cubic Bézier Curve





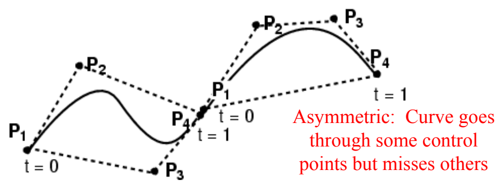
$$Q(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

$$Q(t) = \mathbf{GBT(t)} \qquad B_{Bezier} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Bernstein Polynomials

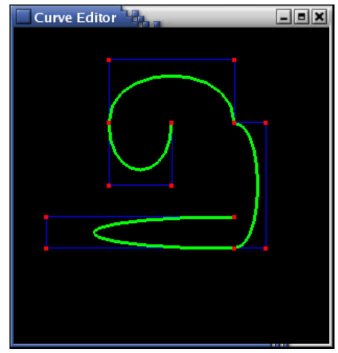
$$B_1(t) = (1-t)^3;\ B_2(t) = 3t(1-t)^2;\ B_3(t) = 3t^2(1-t);\ B_4(t) = t^3$$

Connecting Cubic Bézier Curves



- How can we guarantee C^0 continuity?
- How can we guarantee G^1 continuity?
- How can we guarantee C^1 continuity?
- Can't guarantee higher C^2 or higher continuity

Connecting Cubic Bézier Curves



- Where is this curve
  - C^0 continuous?
  - G^1 continuous?
  - C^1 continuous?
- What's the relationship between:
  - the # of control points, and
  - the # of cubic Bézier subcurves?

Higher-Order Bézier Curves

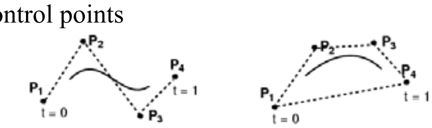
- > 4 control points
- Bernstein Polynomials as the basis functions

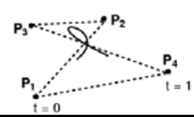
$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}, \qquad 0 \le i \le n$$

- Every control point affects the entire curve
  - Not simply a local effect
  - More difficult to control for modeling

Cubic BSplines

- ≥ 4 control points
- Locally cubic
- Curve is not constrained to pass through any control points

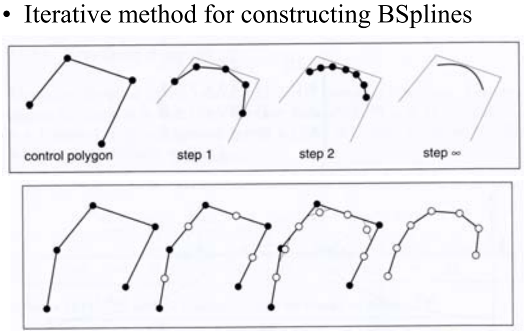




A BSpline curve is also bounded by the convex hull of its control points.

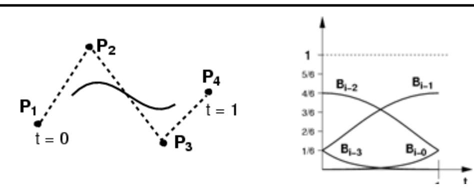
Cubic BSplines

- Iterative method for constructing BSplines



Shirley, Fundamentals of Computer Graphics

Cubic BSplines

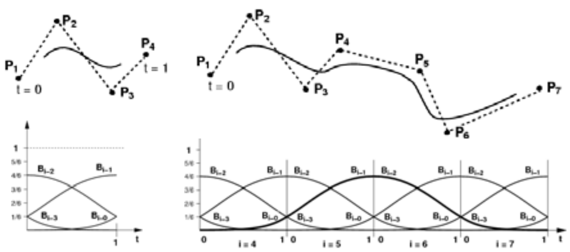


$$Q(t) = \frac{(1-t)^3}{6} P_{i-3} + \frac{3t^3 - 6t^2 + 4}{6} P_{i-2} + \frac{-3t^3 + 3t^2 + 3t + 1}{6} P_{i-1} + \frac{t^3}{6} P_i$$

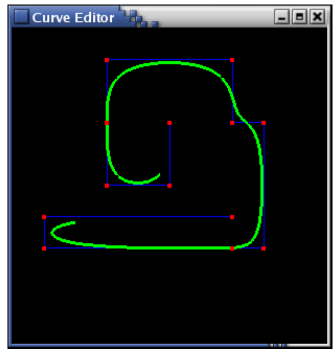
$$Q(t) = \mathbf{GBT(t)} \qquad B_{B-Spline} = \frac{1}{6} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Connecting Cubic BSpline Curves

- Can be chained together
- Better control locally (windowing)

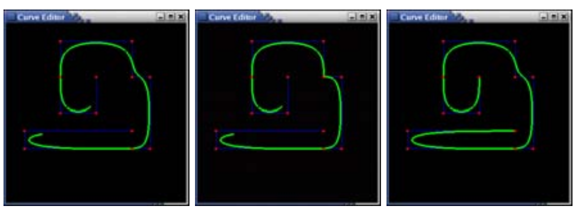


Connecting Cubic BSpline Curves



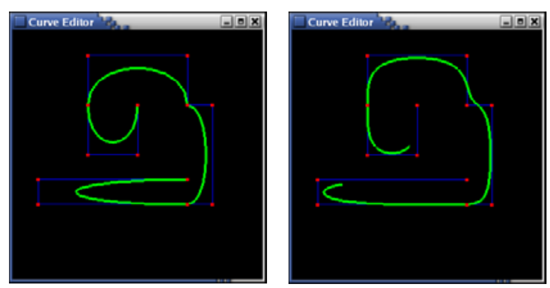
- What's the relationship between
  - the # of control points, and
  - the # of cubic BSpline subcurves?

BSpline Curve Control Points



Default BSpline	BSpline with Discontinuity	BSpline which passes through end points
	Repeat interior control point	Repeat end points

Bézier is not the same as BSpline

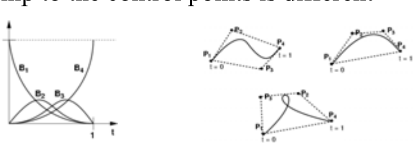


Bézier   BSpline

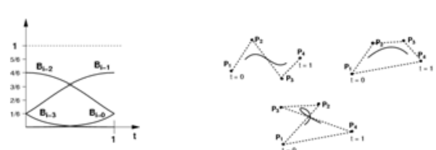
Bézier is not the same as BSpline

- Relationship to the control points is different

Bézier



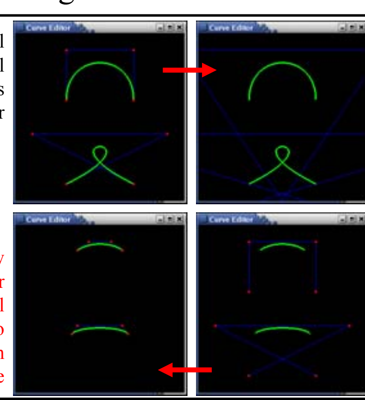
BSpline



Converting between Bézier & BSpline

original control points as Bézier

new BSpline control points to match Bézier



new Bézier control points to match BSpline

original control points as BSpline

Converting between Bézier & BSpline

- Using the basis functions:

$$B_{Bezier} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B_{B-Spline} = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 0 & 4 \\ -3 & 3 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

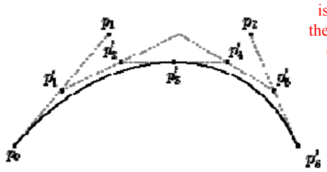
$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

NURBS (generalized BSplines)

- BSpline: uniform cubic BSpline
- NURBS: Non-Uniform Rational BSpline
  - non-uniform = different spacing between the blending functions, a.k.a. knots
  - rational = ratio of polynomials (instead of cubic)

Neat Bezier Spline Trick

- A Bezier curve with 4 control points:
  - P_0 P_1 P_2 P_3
- Can be split into 2 new Bezier curves:
  - P_0 P'_1 P'_2 P'_3
  - P'_3 P'_4 P'_5 P_3



A Bézier curve is bounded by the convex hull of its control points.

Questions?

Reading for Tuesday (1/30)

- DeRose, Kass, & Truong, "Subdivision Surfaces in Character Animation", SIGGRAPH 1998

Figure 5: Geri's hand as a piecewise smooth Catmull-Clark surface. Infinitely sharp creases are used between the skin and the finger nails.

- Additional Reference: SIGGRAPH course notes Subdivision for Modeling and Animation