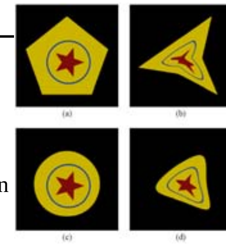
Mass-Spring Systems

Last Time?

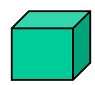
- Subdivision Surfaces
  - Catmull Clark
  - Semi-sharp creases
  - Texture Interpolation
- Interpolation vs. Approximation



Today

- Particle Systems
  - Equations of Motion (Physics)
  - Numerical Integration (Euler, Midpoint, etc.)
  - Forces: Gravity, Spatial, Damping
- Mass Spring System Examples
  - String, Hair, Cloth
- Stiffness
- Discretization

Types of Dynamics

- Point 
- Rigid body 

Witkin & Kass, "Spacetime Constraints", 1988.

- Deformable body
  (include clothes, fluids, smoke, etc.)

Carlson, Mucha, Van Horn, & Turk 2002

What is a Particle System?

- Collection of many small simple particles that maintain *state* (position, velocity, color, etc.)
- Particle motion influenced by external *force fields*
- *Integrate* the laws of mechanics (ODE Solvers)
- To model: sand, dust, smoke, sparks, flame, water, etc.

Particle Motion

- mass m, position x, velocity v
- equations of motion:

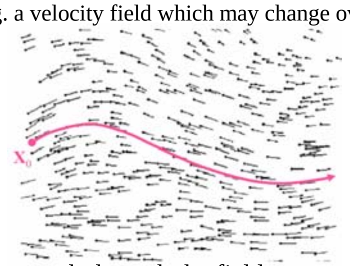
$$\frac{d}{dt}x(t) = v(t)$$

$$\frac{d}{dt}v(t) = \frac{1}{m}F(x,v,t)$$

- Analytic solutions can be found for some classes of differential equations, but most can't be solved analytically
- Instead, we will numerically approximate a solution to our *initial value problem*

Path Through a Field

- $f(\mathbf{X},t)$ is a vector field defined everywhere
  - E.g. a velocity field which may change over time



- $\boldsymbol{X}(t)$ is a path through the field

Higher Order ODEs

- Basic mechanics is a 2nd order ODE:

$$\frac{d^2}{dt^2}x = \frac{1}{m}F$$

- Express as 1st order ODE by defining $v(t)$:

$$\frac{d}{dt}x(t) = v(t)$$
$$\frac{d}{dt}v(t) = \frac{1}{m}F(x,v,t)$$

$$\mathbf{X} = \begin{pmatrix} x \\ v \end{pmatrix} \quad f(X,t) = \begin{pmatrix} v \\ \frac{1}{m}F(x,v,t) \end{pmatrix}$$

For a Collection of 3D particles…

$$\mathbf{X} = \begin{pmatrix} p_x^{(1)} \\ p_y^{(1)} \\ p_z^{(1)} \\ v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \\ p_x^{(2)} \\ p_y^{(2)} \\ p_z^{(2)} \\ v_x^{(2)} \\ v_y^{(2)} \\ v_z^{(2)} \\ \vdots \end{pmatrix} \quad f(\mathbf{X},t) = \begin{pmatrix} v_x^{(1)} \\ v_y^{(1)} \\ v_z^{(1)} \\ \frac{1}{m_1}F_x^{(1)}(\mathbf{X},t) \\ \frac{1}{m_1}F_y^{(1)}(\mathbf{X},t) \\ \frac{1}{m_1}F_z^{(1)}(\mathbf{X},t) \\ v_x^{(2)} \\ v_y^{(2)} \\ v_z^{(2)} \\ \frac{1}{m_2}F_x^{(2)}(\mathbf{X},t) \\ \frac{1}{m_2}F_y^{(2)}(\mathbf{X},t) \\ \frac{1}{m_2}F_z^{(2)}(\mathbf{X},t) \\ \vdots \end{pmatrix}$$

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Euler's Method

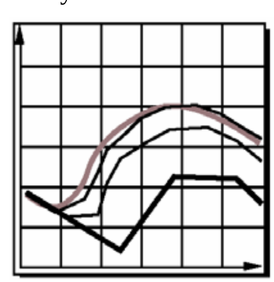
- Examine $f(\mathbf{X},t)$ at (or near) current state
- Take a step of size h to new value of $\mathbf{X}$:

$$t_1 = t_0 + h$$
$$\mathbf{X}_1 = \mathbf{X}_0 + h\, f(\mathbf{X}_0, t_0)$$

- Piecewise-linear approximation to the curve

Effect of Step Size

- Step size controls accuracy
- Smaller steps more closely follow curve
- For animation, we may want to take many small steps per frame



Euler's Method: Inaccurate

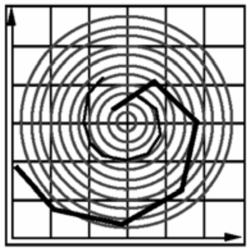
- Moves along tangent & can leave curve, e.g.:

$$f(\mathbf{X},t)=\begin{pmatrix}-y\\x\end{pmatrix}$$

- Exact solution is circle:

$$\mathbf{X}(t)=\begin{pmatrix}r\cos(t+k)\\r\sin(t+k)\end{pmatrix}$$

- Euler's spirals outward no matter how small *h* is



Euler's Method: Unstable

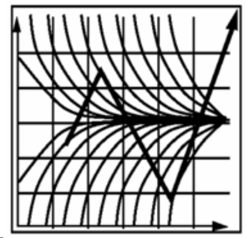
- Problem: $f(x,t)=-kx$
- Solution: $x(t)=x_0e^{-kt}$

- Limited step size:

$$x_1=x_0(1-hk)$$

$$\begin{cases}h\le 1/k & \text{ok}\\ h>1/k & \text{oscillates}\pm\\ h>2/k & \text{explodes}\end{cases}$$

- If *k* is big, *h* must be small



Analysis using Taylor Series

- Expand exact solution **X**(*t*)

$$\mathbf{X}(t_0+h)=\mathbf{X}(t_0)+h\left(\tfrac{d}{dt}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^2}{2!}\left(\tfrac{d^2}{dt^2}\mathbf{X}(t)\right)\Big|_{t_0}+\tfrac{h^3}{3!}(\cdots)+\cdots$$

- Euler's method:

$$\mathbf{X}(t_0+h)=\mathbf{X}_0+h\,f(\mathbf{X}_0,t_0)\qquad \ldots+O(h^2)\,\text{error}$$

$$h\to h/2\;\Rightarrow\; error\to error/4\,\text{per step}\times\text{twice as many steps}$$
$$\to error/2$$

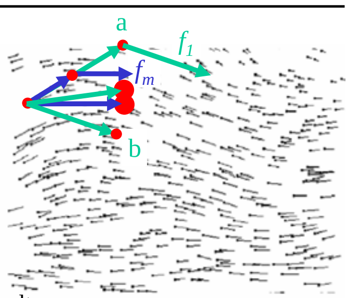
- First-order method: Accuracy varies with *h*
  - To get 100x better accuracy need 100x more steps

Can we do better than Euler's Method?

- Problem: *f* has varied along the step
- Idea: look at *f* at the arrival of the step and compensate for variation

2nd-Order Methods

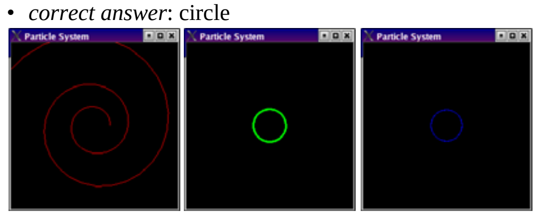
- Midpoint:
  - ½ Euler step
  - evaluate f_m
  - full step using f_m
- Trapezoid:
  - Euler step (a)
  - evaluate f_1
  - full step using f_1 (b)
  - average (a) and (b)
- Not exactly same result
- Same order of accuracy



Comparison: Euler, Midpoint, Runge-Kutta

A 4th order method!

- *initial position*: (1,0,0)
- *initial velocity*: (0,5,0)
- *force field*: pulls particles to origin with magnitude proportional to distance from origin
- *correct answer*: circle

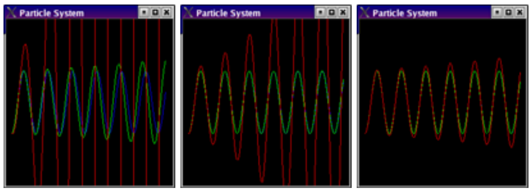


Euler will always diverge (even with small dt)

Comparison: Euler, Midpoint, Runge-Kutta

A 4th order method!

- *initial position*: (0,-2,0)
- *initial velocity*: (1,0,0)
- *force field*: pulls particles to line y=0 with magnitude proportional to distance from line
- *correct answer*: sine wave



Decreasing the timestep (dt) improves the accuracy

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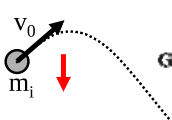
Forces: Gravity

For smoke, flame: make gravity point up!

- Simple gravity: depends only on particle mass



$$\text{Gravity: } f^{(i)} = \begin{pmatrix} 0 \\ 0 \\ -m_i G \end{pmatrix}$$

- N-body problem: depends on all other particles
  - Magnitude inversely proportional to square distance
  - $F_{ij} = G\, m_i m_j / r^2$

Forces: Spatial Fields

- Force on particle *i* depends only on position of *i*
  - wind
  - attractors
  - repulsers
  - vortices
- Can depend on time
- Note: these add energy, may need damping too

Forces: Damping

$$f^{(i)} = -dv^{(i)}$$

- Force on particle *i* depends only on velocity of *i*
- Force opposes motion
- Removes energy, so system can settle
- Small amount of damping can stabilize solver
- Too much damping makes motion too glue-like

Questions?

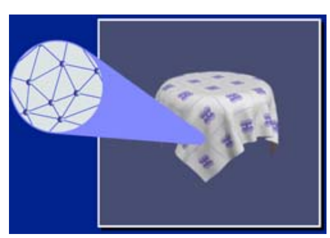


Image by
Baraff, Witkin, Kass

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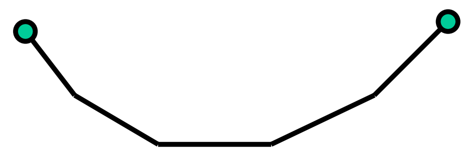
How would you simulate a string?

- Each particle is linked to two particles
- Forces try to keep the distance between particles constant
- What force?

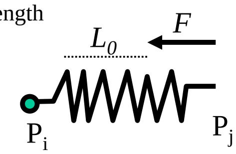


Spring Forces

- Force in the direction of the spring and proportional to difference with rest length L_0

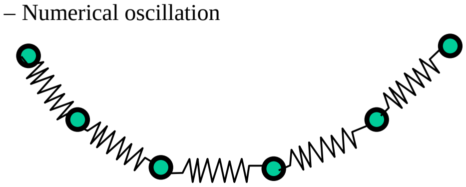
$$F(P_i, P_j) = K(L_0 - ||\vec{P_iP_j}||)\frac{\vec{P_iP_j}}{||\vec{P_iP_j}||}$$

- K is the stiffness of the spring
  - When K gets bigger, the spring really wants to keep its rest length



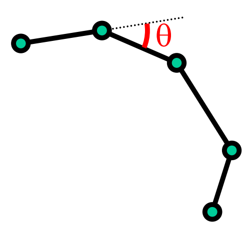
How would you simulate a string?

- Springs link the particles
- Springs try to keep their rest lengths and preserve the length of the string
- Problems?
  - Stretch, actual length will be greater than rest length
  - Numerical oscillation



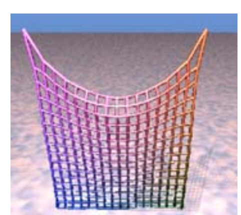
How would you simulate hair?

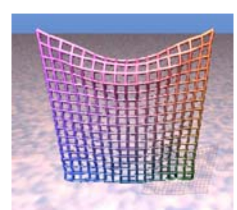
- Similar to string…
- Deformation forces proportional to the angle between segments



Reading for Today

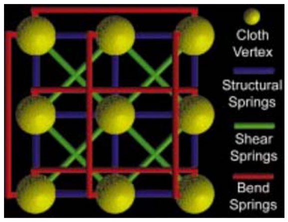
- "Deformation Constraints in a Mass-Spring Model to Describe Rigid Cloth Behavior", Provot, 1995.


Simple mass-spring system


Improved solution

Cloth Modeled with Mass-Spring

- Network of masses and springs
- Structural springs:
  - link (i, j) & (i+1, j) and (i, j) & (i, j+1)
- Shear springs
  - link (i, j) & (i+1, j+1) and (i+1, j) & (i, j+1)
- Flexion (Bend) springs
  - link (i, j) & (i+2, j) and (i, j) & (i, j+2)



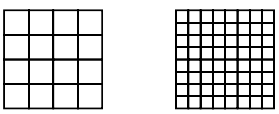
From Lander
http://www.darwin3d.com/gamedev/articles/col0599.pdf

The Stiffness Issue

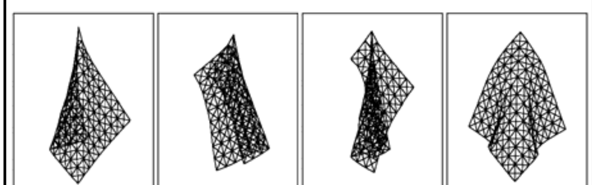
- What relative stiffness do we want for the different springs in the network?
- Cloth is barely elastic, shouldn't stretch so much!
- Inverse relationship between stiffness & Δt
- We really want a constraints (not springs)
- Many numerical solutions
  - reduce Δt
  - use constraints
  - implicit integration
  - …

The Discretization Problem

- What happens if we discretize our cloth more finely, or with a different mesh structure?
- Do we get the same behavior?
- Usually not! It takes a lot of effort to design a scheme that does not depend on the discretization.



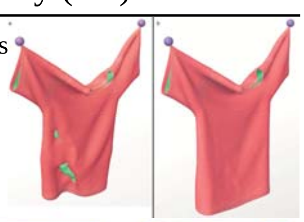
Questions?



Interactive Animation of Structured Deformable Objects
Desbrun, Schröder, & Barr 1999

Reading for Tuesday (2/5)

- Baraff, Witkin & Kass *Untangling Cloth,* SIGGRAPH 2003



- Post a comment or question on the LMS discussion by 10am on Tuesday 2/5