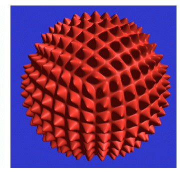
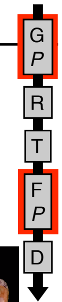
Monte Carlo Rendering

Last Time?

- Modern Graphics Hardware
- Cg Programming Language
- Gouraud Shading vs.
  Phong Normal Interpolation
- Bump, Displacement, & Environment Mapping

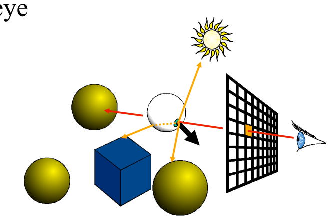



Today

- Does Ray Tracing Simulate Physics?
- Monte-Carlo Integration
- Sampling
- Advanced Monte-Carlo Rendering

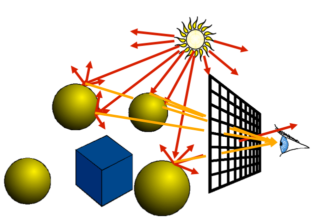
Does Ray Tracing Simulate Physics?

- No…. traditional ray tracing is also called "*backward*" *ray tracing*
- In reality, photons actually travel from the light to the eye



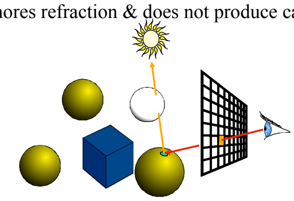
Forward Ray Tracing

- Start from the light source
  - But very, very low probability to reach the eye
- What can we do about it?
  - Always send a ray to the eye…. still not efficient



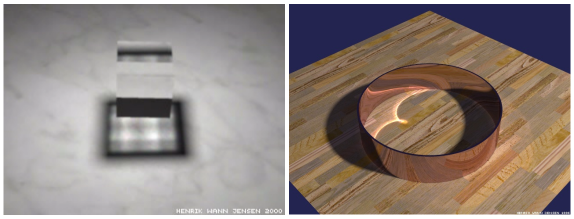

Transparent Shadows?

- What to do if the shadow ray sent to the light source intersects a transparent object?
  - Pretend it's opaque?
  - Multiply by transparency color?
    (ignores refraction & does not produce caustics)



- Unfortunately, ray tracing is full of dirty tricks

Is this Traditional Ray Tracing?

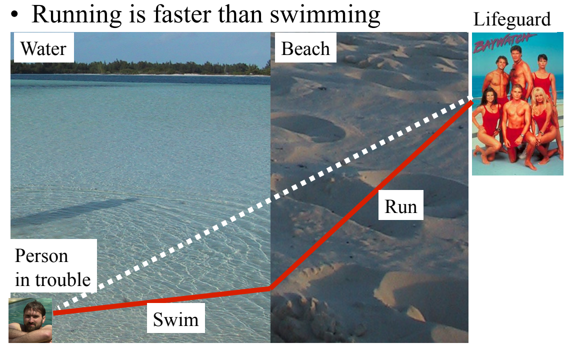


Images by Henrik Wann Jensen

- No, Refraction and complex reflection for illumination are not handled properly in traditional (backward) ray tracing

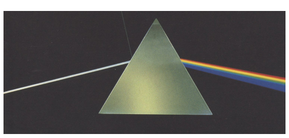
Refraction and the Lifeguard Problem

- Running is faster than swimming



What makes a Rainbow?

- Refraction is wavelength-dependent
  - Refraction increases as the wavelength of light decreases
  - violet and blue experience more bending than orange and red
- Usually ignored in graphics
- Rainbow is caused by
  refraction + internal reflection + refraction

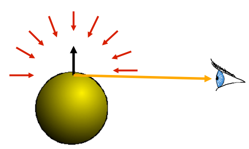


Pink Floyd, *The Dark Side of the Moon*

From "Color and Light in Nature" by Lynch and Livingstone

The Rendering Equation

- Clean mathematical framework for light-transport simulation
- At each point, outgoing light in one direction is the integral of incoming light in all directions multiplied by reflectance property

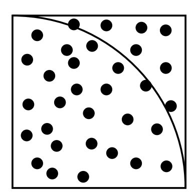


Today

- Does Ray Tracing Simulate Physics?
- Monte-Carlo Integration
  - Probabilities and Variance
  - Analysis of Monte-Carlo Integration
- Sampling
- Advanced Monte-Carlo Rendering

Monte-Carlo Computation of π

- Take a random point (x,y) in unit square
- Test if it is inside the ¼ disc
  - Is $x^2 + y^2 < 1$?
- Probability of being inside disc?
  - area of ¼ unit circle / area of unit square $= \pi / 4$



- $\pi \approx 4$ * number inside disc / total number
- The error depends on the number or trials

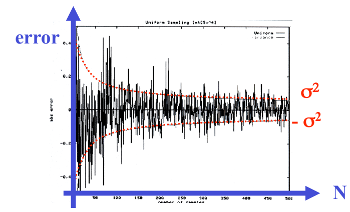
Convergence & Error

- Let's compute 0.5 by flipping a coin:
  - 1 flip: 0 or 1
    → average error = 0.5
  - 2 flips: 0, 0.5, 0.5 or 1
    → average error = 0. 25
  - 4 flips: 0 (*1),0.25 (*4), 0.5 (*6), 0.75(*4), 1(*1)
    → average error = 0.1875
- Unfortunately, doubling the number of samples does not double accuracy

Another Example:

$$I = \int_0^1 5x^4 dx$$

- We know it should be 1.0
- In practice with uniform samples:



Review of (Discrete) Probability

- Random variable can take discrete values x_i
- Probability p_i for each x_i

  $0 < p_i < 1, \ \Sigma p_i = 1$
- Expected value $E(x) = \sum_{i=1}^{n} p_i x_i$
- Expected value of function of random variable
  - $f(x_i)$ is also a random variable

$$E[f(x)] = \sum_{i=1}^{n} p_i f(x_i)$$

Variance & Standard Deviation

- Variance σ^2: deviation from expected value
- Expected value of square difference

$$\sigma^2 = E[(x - E[x])^2] = \sum_i (x_i - E[x])^2 p_i$$

- Also

$$\sigma^2 = E[x^2] - (E[x])^2$$

- Standard deviation σ:
  square root of variance (notion of error, RMS)

Monte Carlo Integration

- Turn integral into finite sum
- Use *n* random samples
- As *n* increases…
  - Expected value remains the same
  - Variance decreases by *n*
  - Standard deviation (error) decreases by $\frac{1}{\sqrt{n}}$
- Thus, converges with $\frac{1}{\sqrt{n}}$

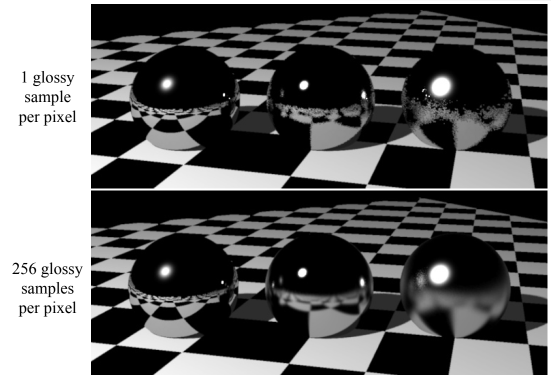
Advantages of MC Integration

- Few restrictions on the integrand
  - Doesn't need to be continuous, smooth, ...
  - Only need to be able to evaluate at a point
- Extends to high-dimensional problems
  - Same convergence
- Conceptually straightforward
- Efficient for solving at just a few points

Disadvantages of MC Integration

- Noisy
- Slow convergence
- Good implementation is hard
  - Debugging code
  - Debugging math
  - Choosing appropriate techniques
- Punctual technique, no notion of smoothness of function (e.g., between neighboring pixels)

Questions?



Today

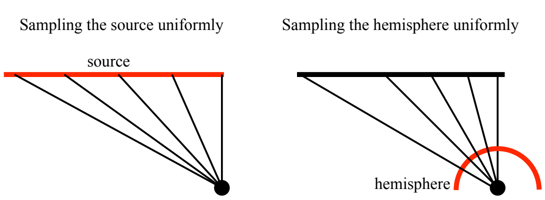
- Does Ray Tracing Simulate Physics?
- Monte-Carlo Integration
- Sampling
  - Stratified Sampling
  - Importance Sampling
- Advanced Monte-Carlo Rendering

Domains of Integration

- Pixel, lens (Euclidean 2D domain)
- Time (1D)
- Hemisphere
  - Work needed to ensure *uniform* probability

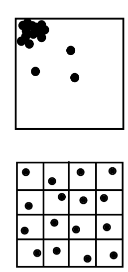
Example: Light Source

- We can integrate over surface *or* over angle
- But we must be careful to get probabilities and integration measure right!



Stratified Sampling

- With uniform sampling, we can get unlucky
  - E.g. all samples in a corner
- To prevent it, subdivide domain Ω into non-overlapping regions Ω_i
  - Each region is called a stratum
- Take one random samples per Ω_i



Example

- Borrowed from Henrik Wann Jensen

$f(x) = e^{\sin(3x^2)}$

N	I
1	2.75039
10	1.9893
100	1.79139
1000	1.75146
10000	1.77313
100000	1.77862

Unstratified
$O(1/\sqrt{N})$

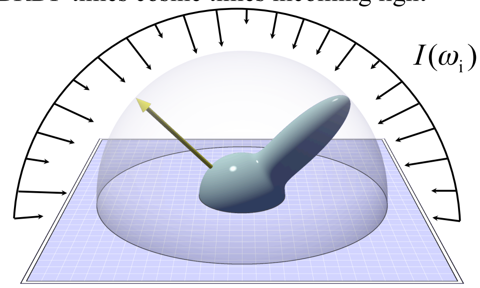
$f(x) = e^{\sin(3x^2)}$

N	I
1	2.70457
10	1.72858
100	1.77925
1000	1.77606
10000	1.77610
100000	1.77610

Stratified
$O(1/N)$

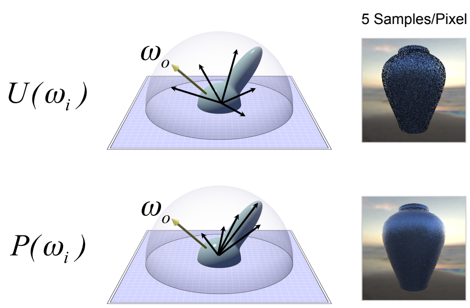
Glossy Rendering

- Integrate over hemisphere
- BRDF times cosine times incoming light



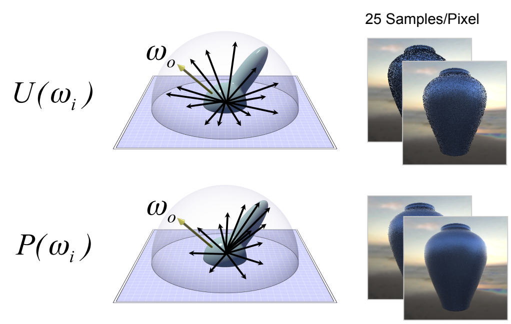
Slide from Jason Lawrence

Sampling a BRDF



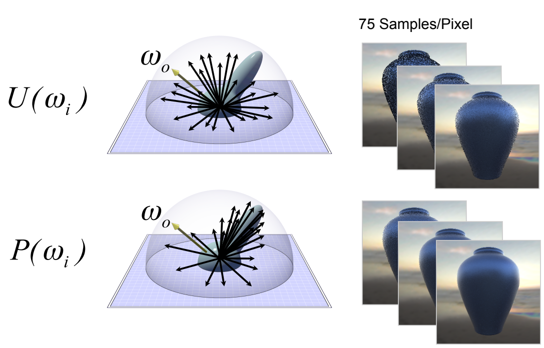
Slide from Jason Lawrence

Sampling a BRDF



Slide from Jason Lawrence

Sampling a BRDF

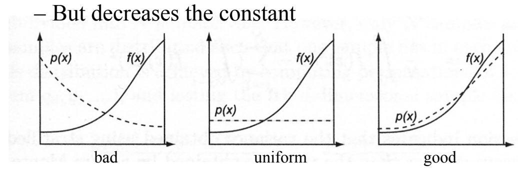


Slide from Jason Lawrence

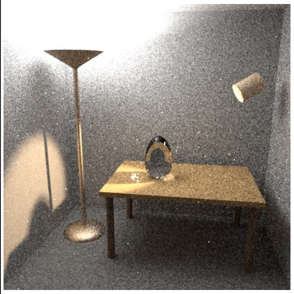
Importance Sampling

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

- Choose p wisely to reduce variance
  - p that resembles f
  - Does not change convergence rate (still sqrt)
  - But decreases the constant



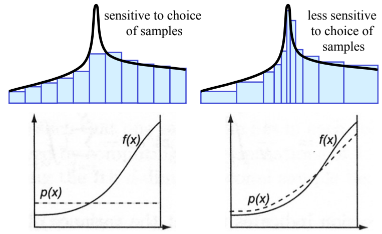
Veach & Guibas "Optimally Combining Sampling Techniques for Monte Carlo Rendering" SIGGRAPH 95



Naïve sampling strategy

Optimal sampling strategy

Sampling



uniform sampling
(or uniform random)

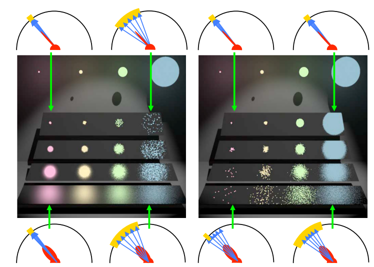
dense sampling
where function has
greater magnitude

To optimally combine samples from different distributions, *weight more highly the samples from the locally densest distribution*

all samples
weighted equally

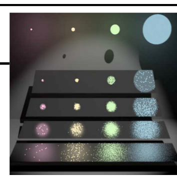
weights (width) for dense
samples are reduced

"Optimally combining sampling techniques for Monte Carlo rendering"
Veach & Guibas, SIGGRAPH 1995



Chosen uniformly within the cone of directions subtended by each light source
(4 samples / pixel)

Chosen with probability proportional to the BRDF
(4 samples / pixel)



Balance heuristic
(8 samples / pixel)

Power heuristic
(8 samples / pixel)

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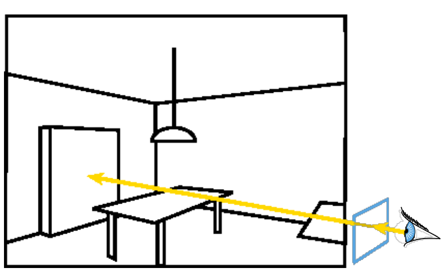
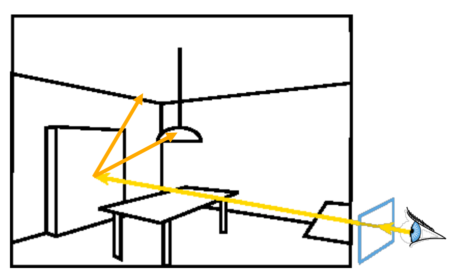
Ray Casting

- Cast a ray from the eye through each pixel



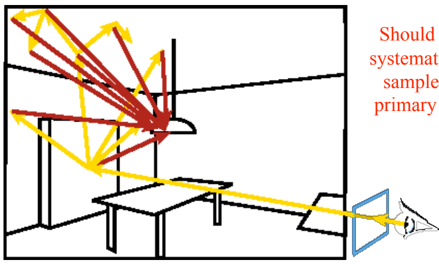
Ray Tracing

- Cast a ray from the eye through each pixel
- Trace secondary rays (light, reflection, refraction)



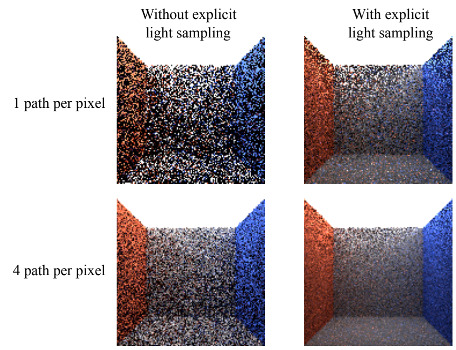
Monte-Carlo Ray Tracing

- Cast a ray from the eye through each pixel
- Cast random rays to accumulate radiance contribution
  - Recurse to solve the Rendering Equation



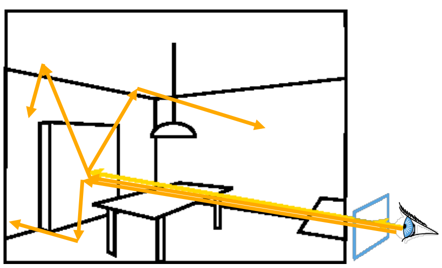
Should also systematically sample the primary light

Importance of Sampling the Light

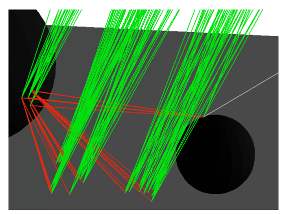


Monte Carlo Path Tracing

- Trace only one secondary ray per recursion
- But send many primary rays per pixel (performs antialiasing as well)



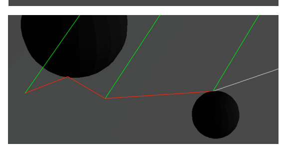
Ray Tracing vs Path Tracing



2 bounces
5 glossy samples
5 shadow samples

How many rays cast per pixel?

1main ray + 5 shadow rays +
5 glossy rays + 5x5 shadow rays +
5*5 glossy rays + 5x5x5 shadow rays
= 186 rays

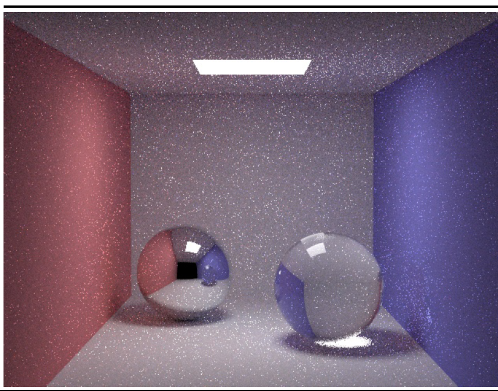


How many 3 bounce paths can we trace per pixel for the same cost?

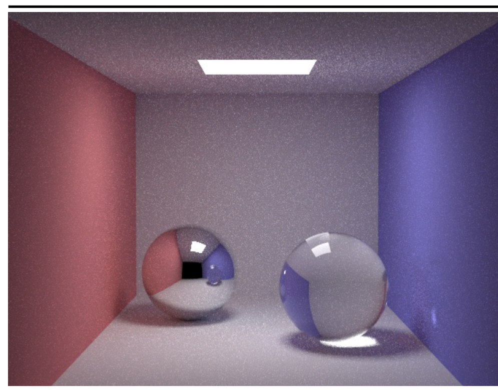
186 rays / 8 ray casts per path
= ~23 paths

Which will probably have less error?

Results: 10 paths/pixel



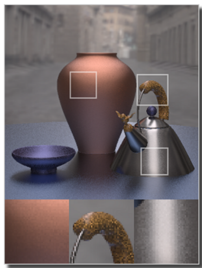
Results: 100 paths/pixel



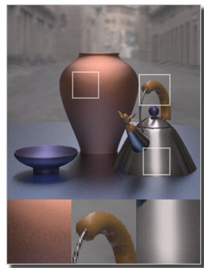
Reading for Today

"Efficient BRDF Importance Sampling Using a Factored Representation"
Lawrence, Rusinkiewicz, & Ramamoorthi, SIGGRAPH 2004

1200
Samples/Pixel



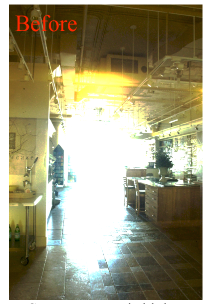
Traditional importance function



Lawrence et al.

Reading for Friday:

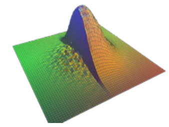
"Fast Bilateral Filtering for the Display of High-Dynamic Range Images",
Durand & Dorsey, SIGGRAPH 2002



Scene contrast is high
Display contrast is low

Contrast reduced
Details preserved

The Secret:



Bilateral filter