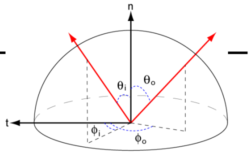
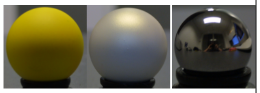
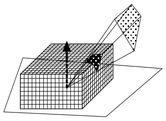
The Rendering Equation
& Monte Carlo Ray Tracing

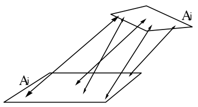
Last Time?

- Local Illumination
  - – BRDF
  - – Ideal Diffuse Reflectance
  - – Ideal Specular Reflectance
  - – The Phong Model
- Radiosity Equation/Matrix
- Calculating the Form Factors







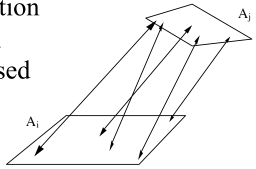


From Last Time

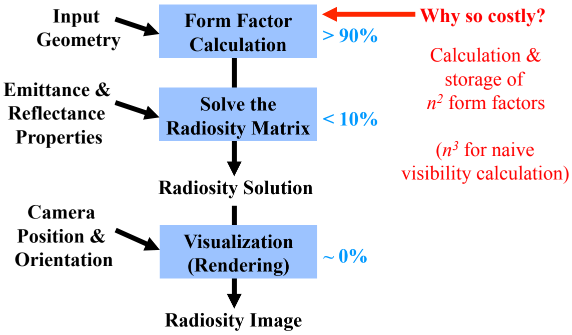
- Computing Form Factors
- Advanced Radiosity
  - – Progressive Radiosity
  - – Adaptive Subdivision
  - – Discontinuity Meshing
  - – Hierarchical Radiosity

Form Factor from Ray Casting

- Cast *n* rays between the two patches
  - – Compute visibility (what fraction of rays do not hit an occluder)
  - – Integrate the point-to-point form factor
- Permits the computation of the patch-to-patch form factor, as opposed to point-to-patch



Stages in a Radiosity Solution



Input Geometry → **Form Factor Calculation** > 90%

Emittance & Reflectance Properties → **Solve the Radiosity Matrix** < 10%

Radiosity Solution

Camera Position & Orientation → **Visualization (Rendering)** ~ 0%

Radiosity Image

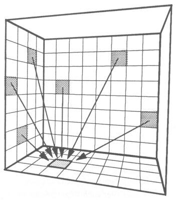
Why so costly?

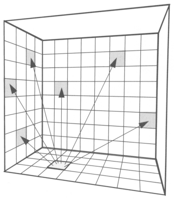
Calculation & storage of n^2 form factors

(n^3 for naive visibility calculation)

Progressive Refinement

- Goal: Provide frequent and timely updates to the user during computation
- Key Idea: Update the entire image at every iteration, rather than a single patch
- How? Instead of summing the light received by one patch, distribute the radiance of the patch with the most *undistributed radiance*.

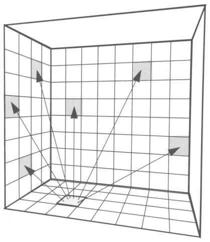




Reordering the Solution for PR

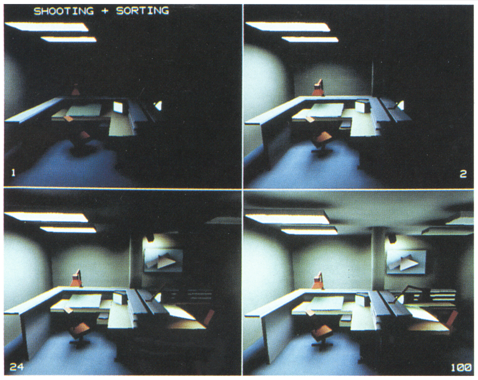
Shooting: the radiosity of all patches is updated for each iteration:

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ \vdots \\ B_n \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ \vdots \\ B_n \end{bmatrix} + \begin{bmatrix} \cdots & \rho_1 F_{1i} & \cdots \\ \cdots & \rho_2 F_{2i} & \\ & & \\ \cdots & \rho_n F_{ni} & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ B_i \\ \vdots \end{bmatrix}$$

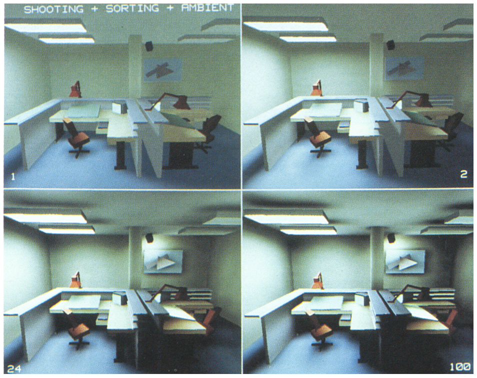


This method is fundamentally a Southwell relaxation

Progressive Refinement w/out Ambient Term

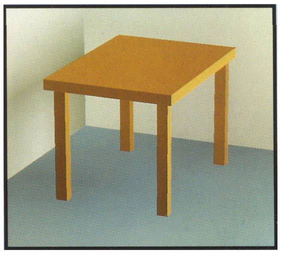


Progressive Refinement with Ambient Term



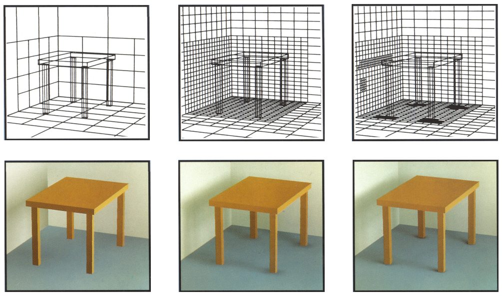
Increasing the Accuracy of the Solution

What's wrong with this picture?



- Image quality is a function of patch size
- Compute a solution on a uniform initial mesh, then refine the mesh in areas that exceed some error tolerance:
  - shadow boundaries
  - other areas with a high radiosity gradient

Adaptive Subdivision of Patches



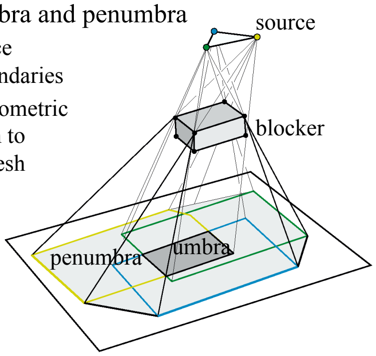
Coarse patch solution (145 patches)

Improved solution (1021 subpatches)

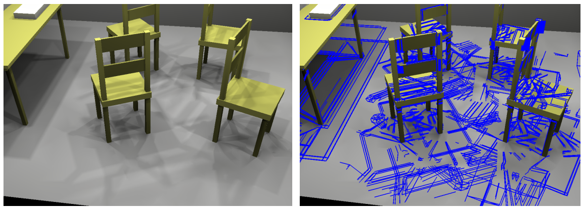
Adaptive subdivision (1306 subpatches)

Discontinuity Meshing

- Limits of umbra and penumbra
  - Captures nice shadow boundaries
  - Complex geometric computation to construct mesh



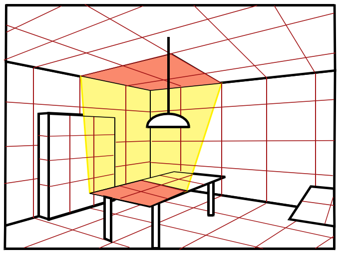
Discontinuity Meshing



"Fast and Accurate Hierarchical
Radiosity Using Global Visibility"
Durand, Drettakis, & Puech 1999

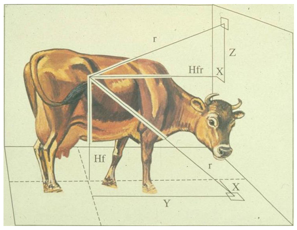
Hierarchical Radiosity

- Group elements when the light exchange is not important
  - Breaks the quadratic complexity
  - Control non trivial, memory cost



Practical Problems with Radiosity

- Meshing
  - memory
  - robustness
- Form factors
  - computation



Cow-cow form factor?

- Diffuse limitation
  - extension to specular takes too much memory

Questions?

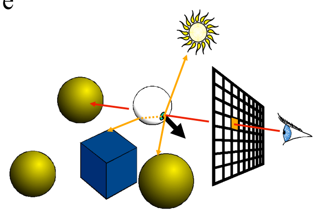
Lightscape `http://www.lightscape.com`

Today

- Does Ray Tracing Simulate Physics?
- The Rendering Equation
- Monte-Carlo Integration
- Sampling
- Monte-Carlo Ray Tracing vs. Path Tracing

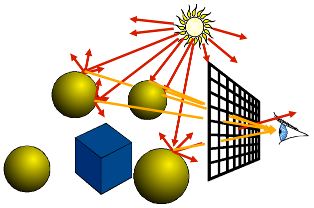
Does Ray Tracing Simulate Physics?

- No…. traditional ray tracing is also called "*backward*" *ray tracing*
- In reality, photons actually travel from the light to the eye



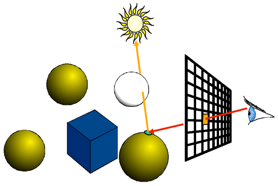
Forward Ray Tracing

- Start from the light source
  - But very, very low probability to reach the eye
- What can we do about it?
  - Always send a ray to the eye…. still not efficient

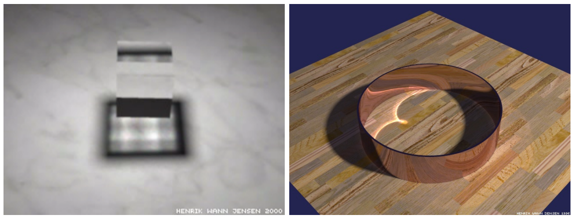


Transparent Shadows?

- What to do if the shadow ray sent to the light source intersects a transparent object?
  - Pretend it's opaque?
  - Multiply by transparency color?
    (ignores refraction & does not produce caustics)
- Unfortunately, ray tracing is full of dirty tricks



Is this Traditional Ray Tracing?

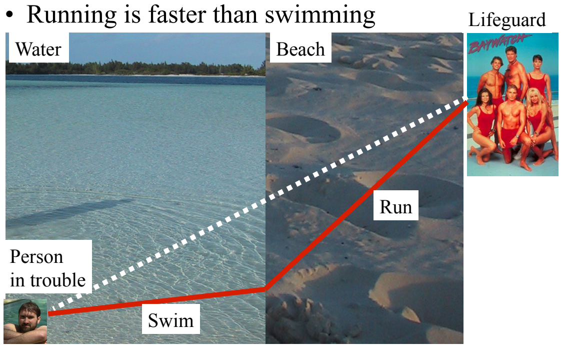


Images by Henrik Wann Jensen

- No, Refraction and complex reflection for illumination are not handled properly in traditional (backward) ray tracing

Refraction and the Lifeguard Problem

- Running is faster than swimming



Today

- Does Ray Tracing Simulate Physics?
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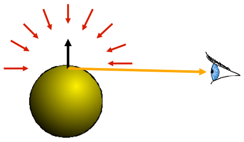
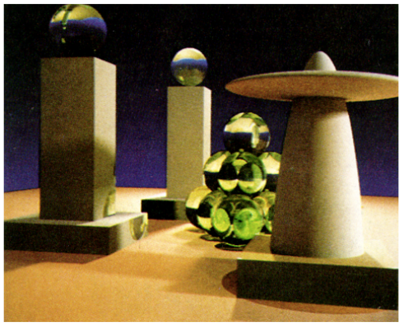
The Rendering Equation

- Clean mathematical framework for light-transport simulation
- At each point, outgoing light in one direction is the integral of incoming light in all directions multiplied by reflectance property

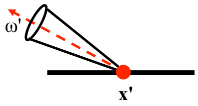


Reading for Today:

- "The Rendering Equation", Kajiya, SIGGRAPH 1986



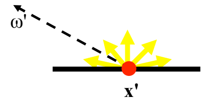
The Rendering Equation



$$L(x',\omega') = E(x',\omega') + \int \rho_{x'}(\omega,\omega') L(x,\omega) G(x,x') V(x,x')\, dA$$

L (x',ω') is the radiance from a point on a surface in a given direction ω'

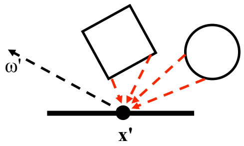
The Rendering Equation



$$L(x',\omega') = E(x',\omega') + \int \rho_{x'}(\omega,\omega') L(x,\omega) G(x,x') V(x,x')\, dA$$

E(x',ω') is the emitted radiance from a point: E is non-zero only if x' is emissive (a light *source*)

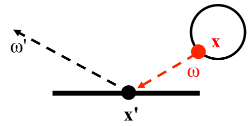
The Rendering Equation



$$L(x',\omega') = E(x',\omega') + \underbrace{\int \rho_{x'}(\omega,\omega') L(x,\omega) G(x,x') V(x,x')\, dA}$$

Sum the contribution from all of the other surfaces in the scene

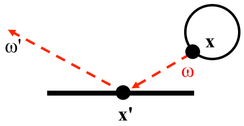
The Rendering Equation



$$L(x',\omega') = E(x',\omega') + \int \rho_{x'}(\omega,\omega') L(x,\omega) G(x,x') V(x,x')\, dA$$

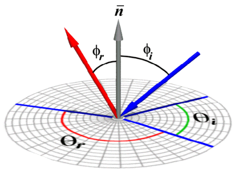
For each x, compute L(x, ω), the radiance at point x in the direction ω (from x to x')

The Rendering Equation

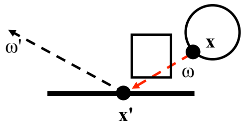


$$L(x',\omega') = E(x',\omega') + \int \rho_{x'}(\omega,\omega') L(x,\omega) G(x,x') V(x,x')\, dA$$

scale the contribution by $\rho_{x'}(\omega,\omega')$, the reflectivity (BRDF) of the surface at x'



The Rendering Equation



$L(x',\omega') = E(x',\omega') + \int \rho_{x'}(\omega,\omega')L(x,\omega)G(x,x')V(x,x')\ dA$

For each x, compute V(x,x'), the visibility between x and x': 1 when the surfaces are unobstructed along the direction ω, 0 otherwise

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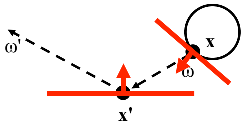
The Rendering Equation

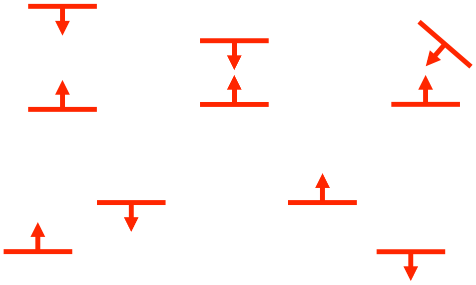


$L(x',\omega') = E(x',\omega') + \int \rho_{x'}(\omega,\omega')L(x,\omega)G(x,x')V(x,x')\ dA$

For each x, compute G(x, x'), which describes the on the geometric relationship between the two surfaces at x and x’

Intuition about G(x,x')?

- Which arrangement of two surfaces will yield the greatest transfer of light energy? Why?



Rendering Equation ➔ Radiosity

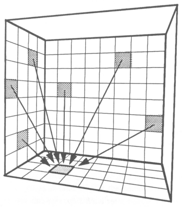
$L(x',\omega') = E(x',\omega') + \int \rho_{x'}(\omega,\omega')L(x,\omega)G(x,x')V(x,x')\ dA$

Radiosity assumption: perfectly diffuse surfaces (not directional)

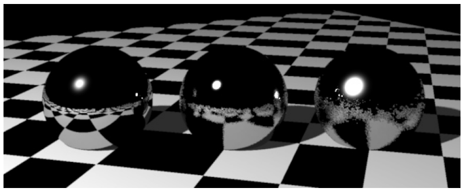
$B_{x'} = E_{x'} + \rho_{x'} \int B_x\ G(x,x')V(x,x')$

discretize

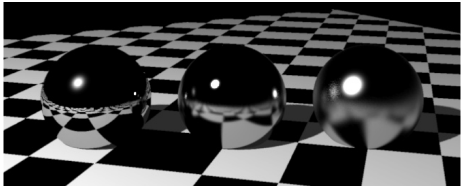
$$B_i = E_i + \rho_i \sum_{j=1}^{n} F_{ij} B_j$$



Questions?



1 glossy sample per pixel



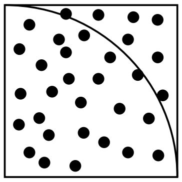
256 glossy samples per pixel

Today

- Does Ray Tracing Simulate Physics?
- The Rendering Equation
- Monte-Carlo Integration
  - Probabilities and Variance
  - Analysis of Monte-Carlo Integration
- Sampling
- Monte-Carlo Ray Tracing vs. Path Tracing

Monte-Carlo Computation of π

- Take a random point (x,y) in unit square
- Test if it is inside the ¼ disc
  - Is $x^2 + y^2 < 1$?
- Probability of being inside disc?
  - area of ¼ unit circle / area of unit square $= \pi / 4$



- $\pi \approx 4$ * number inside disc / total number
- The error depends on the number or trials

Convergence & Error

- Let’s compute 0.5 by flipping a coin:
  - 1 flip: 0 or 1
    → average error = 0.5
  - 2 flips: 0, 0.5, 0.5 or 1
    → average error = 0. 25
  - 4 flips: 0 (*1),0.25 (*4), 0.5 (*6), 0.75(*4), 1(*1)
    → average error = 0.1875
- Unfortunately, doubling the number of samples does not double accuracy

Review of (Discrete) Probability

- Random variable can take discrete values x_i
- Probability p_i for each x_i

  $0 < p_i < 1, \ \Sigma p_i = 1$
- Expected value $E(x) = \sum_{i=1}^{n} p_i x_i$
- Expected value of function of random variable
  - $f(x_i)$ is also a random variable

$$E[f(x)] = \sum_{i=1}^{n} p_i f(x_i)$$

Variance & Standard Deviation

- Variance σ^2: deviation from expected value
- Expected value of square difference

$$\sigma^2 = E[(x - E[x])^2] = \sum_i (x_i - E[x])^2 p_i$$

- Also

$$\sigma^2 = E[x^2] - (E[x])^2$$

- Standard deviation σ:
  square root of variance (notion of error, RMS)

Monte Carlo Integration

- Turn integral into finite sum
- Use *n* random samples
- As *n* increases…
  - Expected value remains the same
  - Variance decreases by *n*
  - Standard deviation (error) decreases by $\frac{1}{\sqrt{n}}$
- Thus, converges with $\frac{1}{\sqrt{n}}$

Advantages of MC Integration

- Few restrictions on the integrand
  - Doesn’t need to be continuous, smooth, ...
  - Only need to be able to evaluate at a point
- Extends to high-dimensional problems
  - Same convergence
- Conceptually straightforward
- Efficient for solving at just a few points

Disadvantages of MC Integration

- Noisy
- Slow convergence
- Good implementation is hard
  - Debugging code
  - Debugging math
  - Choosing appropriate techniques
- Punctual technique, no notion of smoothness of function (e.g., between neighboring pixels)

Questions?

- "A Theoretical Framework for Physically Based Rendering", Lafortune and Willems, Computer Graphics Forum, 1994.

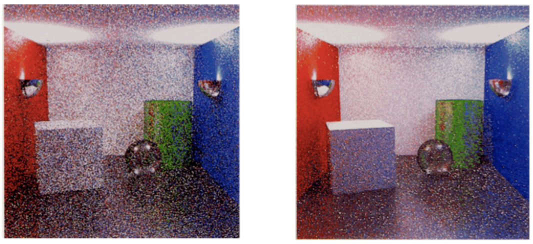


Figure B: *An indirectly illuminated scene rendered using path tracing and bidirectional path tracing respectively. The latter method results in visibly less noise for the same amount of work.*

Today

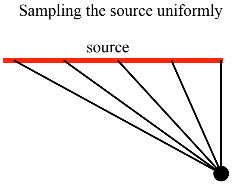
- Does Ray Tracing Simulate Physics?
- The Rendering Equation
- Monte-Carlo Integration
- Sampling
  - Stratified Sampling
  - Importance Sampling
- Monte-Carlo Ray Tracing vs. Path Tracing

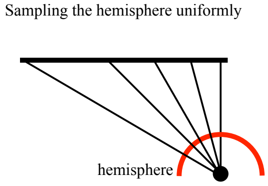
Domains of Integration

- Pixel, lens (Euclidean 2D domain)
- Time (1D)
- Hemisphere
  - Work needed to ensure *uniform* probability

Example: Light Source

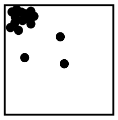
- We can integrate over surface *or* over angle
- But we must be careful to get probabilities and integration measure right!



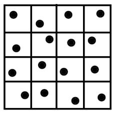


Stratified Sampling

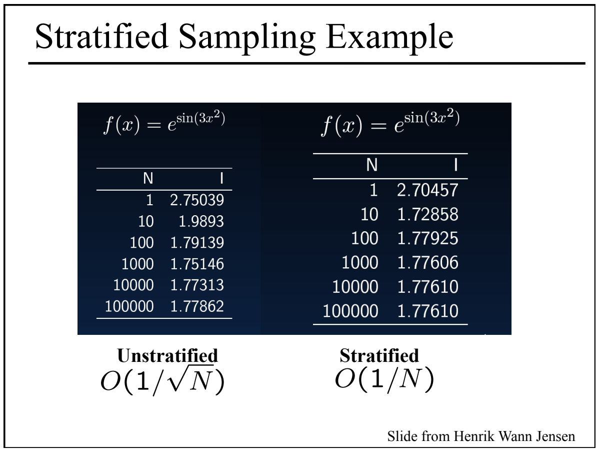
- With uniform sampling, we can get unlucky
  - E.g. all samples in a corner



- To prevent it, subdivide domain Ω into non-overlapping regions Ω_i
  - Each region is called a stratum



- Take one random samples per Ω_i



Stratified Sampling Example

$f(x) = e^{\sin(3x^2)}$

N	I
1	2.75039
10	1.9893
100	1.79139
1000	1.75146
10000	1.77313
100000	1.77862

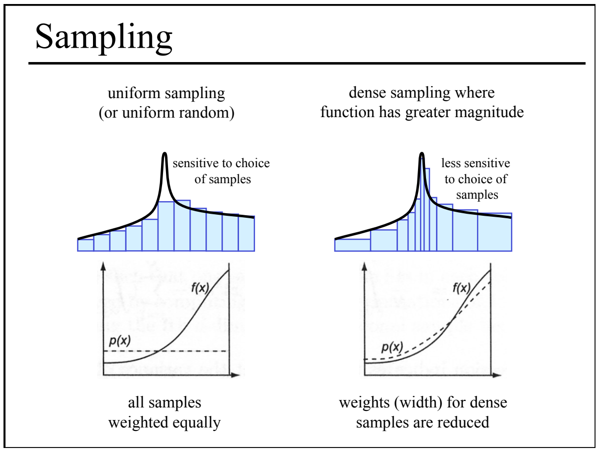
Unstratified
$O(1/\sqrt{N})$

$f(x) = e^{\sin(3x^2)}$

N	I
1	2.70457
10	1.72858
100	1.77925
1000	1.77606
10000	1.77610
100000	1.77610

Stratified
$O(1/N)$

Slide from Henrik Wann Jensen



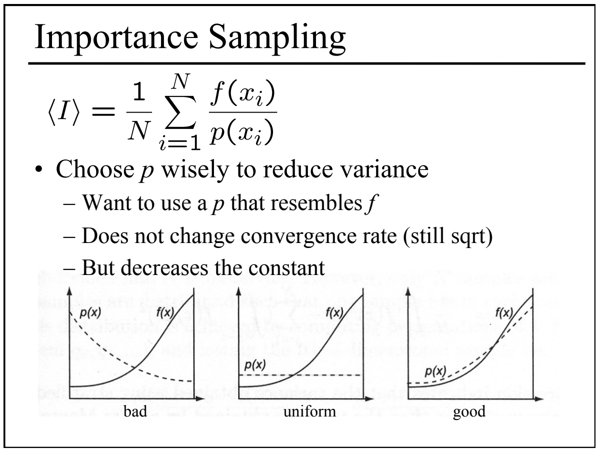
Sampling

uniform sampling
(or uniform random)

dense sampling where
function has greater magnitude

all samples
weighted equally

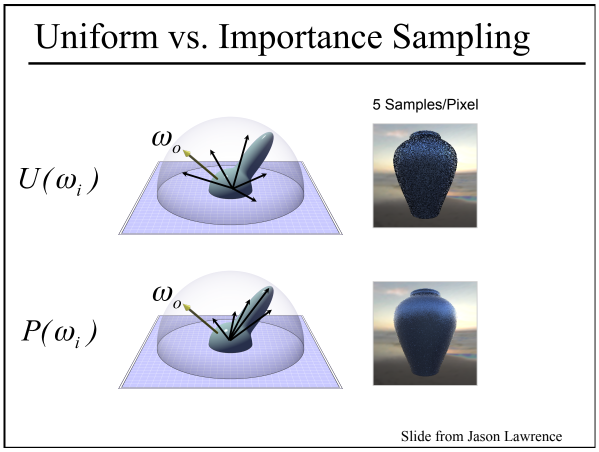
weights (width) for dense
samples are reduced



Importance Sampling

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}$$

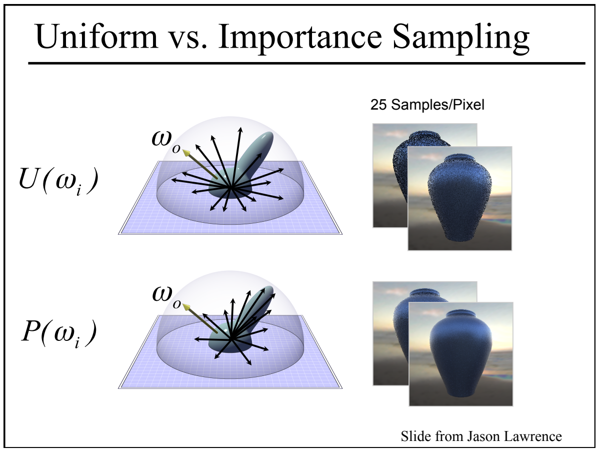
- Choose *p* wisely to reduce variance
  - Want to use a *p* that resembles *f*
  - Does not change convergence rate (still sqrt)
  - But decreases the constant



Uniform vs. Importance Sampling

5 Samples/Pixel

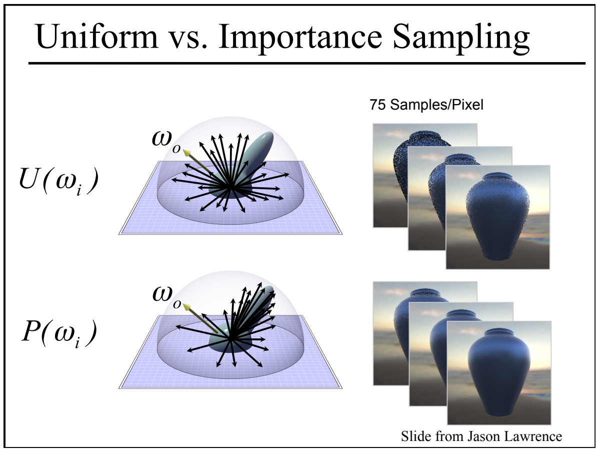
Slide from Jason Lawrence



Uniform vs. Importance Sampling

25 Samples/Pixel

Slide from Jason Lawrence

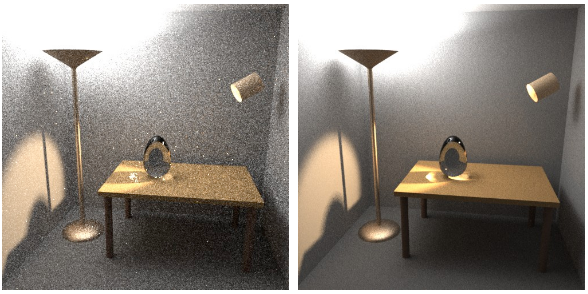


Uniform vs. Importance Sampling

75 Samples/Pixel

Slide from Jason Lawrence

Questions?



Naïve sampling strategy

Optimal sampling strategy

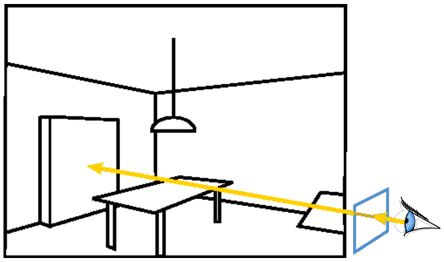
Veach & Guibas "Optimally Combining Sampling Techniques for Monte Carlo Rendering" SIGGRAPH 95

Today

- Does Ray Tracing Simulate Physics?
- The Rendering Equation
- Monte-Carlo Integration
- Sampling
- Monte-Carlo Ray Tracing & Path Tracing

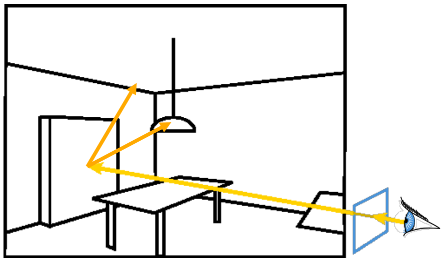
Ray Casting

- Cast a ray from the eye through each pixel



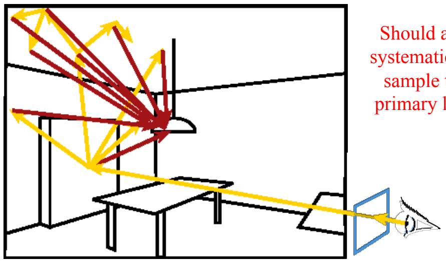
Ray Tracing

- Cast a ray from the eye through each pixel
- Trace secondary rays (light, reflection, refraction)



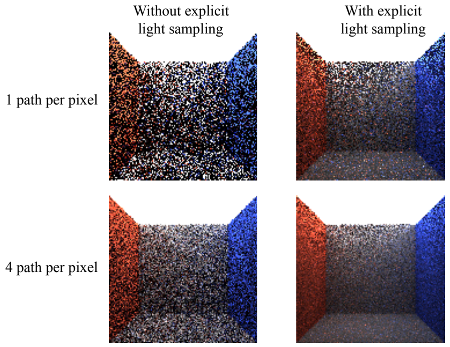
Monte-Carlo Ray Tracing

- Cast a ray from the eye through each pixel
- Cast random rays to accumulate radiance contribution
  - Recurse to solve the Rendering Equation



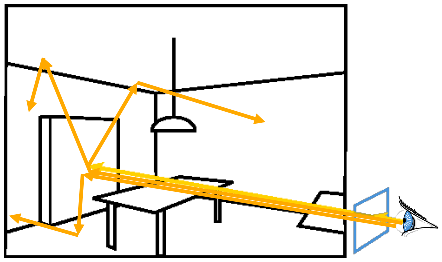
Should also systematically sample the primary light

Importance of Sampling the Light

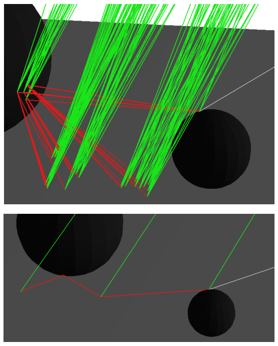


Monte Carlo Path Tracing

- Trace only one secondary ray per recursion
- But send many primary rays per pixel (performs antialiasing as well)



Ray Tracing vs Path Tracing



2 bounces
5 glossy samples
5 shadow samples

How many rays cast per pixel?

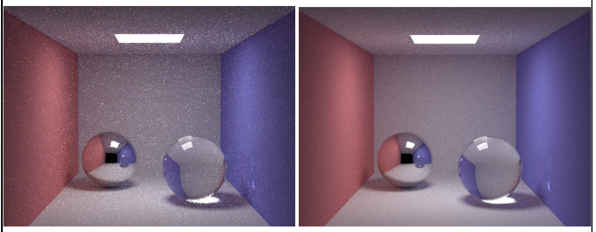
1main ray + 5 shadow rays +
5 glossy rays + 5x5 shadow rays +
5*5 glossy rays + 5x5x5 shadow rays
= 186 rays

How many 3 bounce paths can we trace per pixel for the same cost?

186 rays / 8 ray casts per path
= ~23 paths

Which will probably have less error?

Questions?

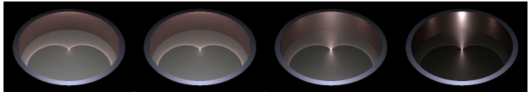


10 paths/pixel **100 paths/pixel**

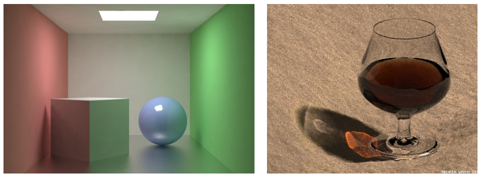
Images from Henrik Wann Jensen

Readings for Friday (3/22) *pick one*:

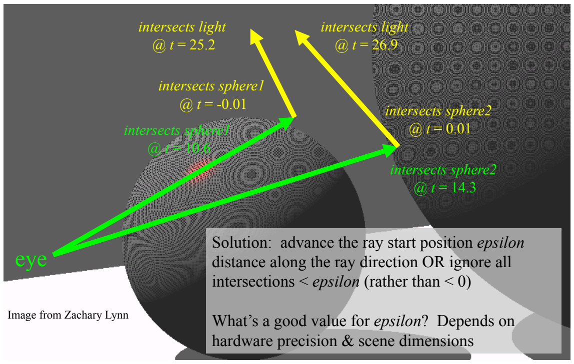
- "Rendering Caustics on Non-Lambertian Surfaces", Henrik Wann Jensen, *Graphics Interface* 1996.



- "Global Illumination using Photon Maps", Henrik Wann Jensen, *Rendering Techniques* 1996.



Raytracing & Epsilon



Solution: advance the ray start position *epsilon* distance along the ray direction OR ignore all intersections < *epsilon* (rather than < 0)

What's a good value for *epsilon*? Depends on hardware precision & scene dimensions

Image from Zachary Lynn