Home Work No. 1 Fall 2010 Solutions (problems 2 to 5) .

February 13, 2015

0.1 We give a detailed solution at the end

0.2 a) Proof By Induction of $F_n \geq 2^{0.5n}$:

Base case: F_6=8 $\geq 2^3$

Assume the hypothesis is true for $n \leq k$

Now for $n = k + 1$

$F_{k+1} = F_k + F_{k-1}$

$F_{k+1} \geq 2^{0.5k} + 2^{0.5(k-1)}$

$F_{k+1} \geq 2^{0.5(k-1)} (2^{0.5} + 1)$

$F_{k+1} \geq 2^{0.5(k-1)} (1.414 + 1)$

$F_{k+1} \geq 2^{0.5(k-1)} (2.414)$

$F_{k+1} \geq 2^{0.5(k-1)} 2^{2\times 0.5}$

$F_{k+1} \geq 2^{0.5(k+1)}$

b) Proof By Induction of $F_n \leq 2^{cn}$ Let $c = 0.9$

Base case $F_0 = 1 \leq 2^0$

Assume the hypothesis is true for $n \leq k$

Now for $n \ = \ k+1$

$F_{k+1} \ \ = F_k \ + F_{k-1}$

$F_{k+1} \ \leq \ 2^{0.9k} \ + \ 2^{0.9(k-1)}$

$F_{k+1} \ \leq \ 2^{0.9(k-1)} \ (2^{0.9}+1)$

$F_{k+1} \ \leq \ 2^{0.9(k-1)} \ (1.87+1)$

$F_{k+1} \ \leq \ 2^{0.9(k-1)} \ (2.87)$

$F_{k+1} \ \leq \ 2^{0.9(k-1)} \ 2^{2\times 0.9}$

$F_{k+1} \ \leq \ 2^{0.9(k+1)}$

3)
a) ${(2^{\log(n)})}^{\log(n)} = n^{\log(n)}$ as $2^{\log(n)} = n$

$n^{\log(n)} = O(n^n)$
Hence ${(2^{\log(n)})}^{\log(n)} = O(n^{n\log(n)})$

b) $7^{\log(n} = y$
Taking logs on both sides, we get
$\log(n)\log(7)$ =$\log(y)$

$\log(n)^{\log(7)}$ $=\log(y)$

$n^{\log(7)} = y$ taking anti-logairthms

Hence

$7^{\log(n)} = \Theta(n^{\log(7)}) = O(n^{\log(7)})$

c)

$2^{7log(n)} \;=\; 2^{log(n^7)}$

by taking logairthms and anti-logarithms similar to the above part, we get

$2^{7log(n)} \;=\; \Theta(n^7) = O(n^7)$

4)

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1) swap(100,20),swap(100,40),swap(40,3) - 3 swaps
2) swap(200,40),swap(100,30) - 2 swaps
3) 30,12,15,10 d<b<c<a
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5)

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Input: a<b, c< d.

If (a> c) swap(a,c)
if (b>d) swap(b,d)
if (b>c) swap(b,c)
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6. $\log(\log(n))$, $\log(n^{1.5})$, $n/\log(n)$, $n^{1.5}$,$n^{\log(n)}$,4^n,$n!$,2^{n^2}, 2^{2^n}

7 1. $sum = \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

$sum = \frac{(n+1)^3}{6} \; - \; \frac{n}{6} - \frac{1}{6}$

Running Time of the algorithm is $O(n^3)$ assuming sum takes O(1) time/

2. $O(mn)$ assuming multiplication. assuming multiplication takes O(1) time

0.1. Solution - Please do not take points off for not showing steps

(a)

$$\frac{f(n)}{g(n)} = \frac{n-100}{n-200} \leq 201 \therefore f(n) = O(g(n))$$
$$\frac{g(n)}{f(n)} = \frac{n-200}{n-100} \leq 1 \therefore f(n) = \Omega(g(n))$$
$$\therefore f(n) = \Theta(g(n))$$

(b) $f(n) = O(g(n))$, Rule 2

(c)

$$\frac{f(n)}{g(n)} = \frac{100n + \log n}{n + (\log n)^2} \leq 100 \therefore f(n) = O(g(n))$$
$$\frac{g(n)}{f(n)} = \frac{n + (\log n)^2}{100n + \log n} \leq 1 \therefore f(n) = \Omega(g(n))$$
$$\therefore f(n) = \Theta(g(n))$$

(d) $f(n) = \Theta(g(n))$, Rule 1

(e) $f(n) = \Theta(g(n))$, Rule 1

(f) $\log n^2 = 2\log n, f(n) = \Theta(g(n))$, Rule 1

(g) Rule 4 applies to $g(n)$, $f(n) = \Omega(g(n))$, Rule 2 ($1.01 > 1$)

(h) n^2 dominates in $f(n)$, n dominates in $g(n)$, $f(n) = \Omega(g(n))$

(i) $f(n) = \Omega(g(n))$, Rule 4

(j) take the log of $f(n)$ and $g(n)$ yeilds, $f(n) = \log n * \log\log n$ and $g(n) = \log n - \log\log n$, substituting $p = log(n)$

$$\frac{f(n)}{g(n)} = \frac{p * \log p}{p - \log p} \not\leq \infty$$
$$\frac{g(n)}{f(n)} = \frac{p - \log p}{p * \log p} \leq 1 \text{ for large } n \therefore f(n) = \Omega(g(n))$$

(k) $f(n) = \Omega(g(n))$, Rule 4

(l) $f(n) = O(g(n))$, Rule 3

(m) $f(n) = O(g(n))$, Rule 3

(n)

$$\frac{f(n)}{g(n)} = \frac{2^n}{2^{n+1}} \leq \frac{1}{2} \therefore f(n) = O(g(n))$$
$$\frac{g(n)}{f(n)} = \frac{2^(n+1)}{2^n} \leq 2 \therefore f(n) = \Omega(g(n))$$
$$\therefore f(n) = \Theta(g(n))$$

5

Figure 1: Problem 1

(o) $f(n) = \Omega(g(n))$, factorial dominates exponentials

(p) take the log of $f(n)$ and $g(n)$ yeilds, $f(n) = \log n * \log\log n$ and $g(n) = (\log_2 n)^2 \log 2$, substituting $p = log(n)$

$$\frac{f(n)}{g(n)} = \frac{p * \log p}{p^2 \log 2} \leq 1 \therefore f(n) = O(g(n))$$

$$\frac{g(n)}{f(n)} = \frac{p^2 \log 2}{p * \log p} \not\leq \infty$$

(q) the series evaluates with the highest term being n^{k+1}, $f(n) = \Theta(g(n))$

Figure 2: Problem 1 - continued