

(1)

## Sample Solution

## Homework #3

Q 2.13 a, b (DG)

$$1) a. B_3 = 1, B_5 = 2, B_7 = 5$$

Any full binary tree must have odd number of vertices  $B_{2k} = 0$  for any  $k$ .

b. Decompose the tree into two subtrees rooted at the two children of the root. The total number of full binary trees is the summation of the product of numbers of full binary trees from the two children.

$$B_n = \sum_{i=1}^{n-2} B_i B_{n-1-i}$$

2) Question 2.14 (DG)

Assume the array can be sorted. Sort the array in  $O(n \log n)$  time. Then, in  $O(n)$  time, scan the array and copy the elements to a new array while skipping the duplicates that are adjacent to each other.

3) Question 2.17

The observation is that if  $A[m] > m$ , then  $A[i] = i$  is not possible for  $i \geq m$ . This is because array contains distinct integers. Using divide and conquer, we split the array in halves and compare the middle element with its index to guide the search.

(a) We start with middle element  $A[\frac{n}{2}]$  and compare it with  $\frac{m}{2}$

(i) If  $A[\frac{n}{2}] = \frac{m}{2}$  succeeds.

(recursively)

(2) If  $A[\frac{n}{2}] > \frac{m}{2}$  discard the second half and search in the first

(3) If  $A[\frac{n}{2}] < \frac{m}{2}$  discard the first half and recursively search in the

second half of the array.

If we get down to a single element and this element is not equal to its index it fails.

Recurrence relation :  $T(n) = T(\frac{n}{2}) + O(1)$

$$T(n) = O(\log n)$$

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(4) Spread rumour fast.

$$1 \text{ to } 24. \quad \Delta_k = \frac{1 - (-2)^k}{3}, \quad \Delta_1 = 1, \quad \Delta_2 = -1, \quad \Delta_3 = 3, \quad \Delta_4 = -5,$$

$$\Delta_5 = 11$$

Day 1:  $\Delta_k = 1$   $(1,2), (3,4), (5,6), (7,8), (9,10), (11,12), (13,14), (15,16), (17,18),$   
 $(19,20), (21,22), (23,24)$

Day 2:  $\Delta_k = -1$   $(1,24), (3,2), (5,4), (7,6), (9,8), (11,10), (13,12), (15,14), (17,16)$   
 $(19,18), (21,20), (23,22)$

Day 3:  $\Delta_k = 3$   $(1,4), (3,6), (5,8), (7,10), (9,12), (11,14), (13,16), (15,18), (17,20)$   
 $(19,22), (21,24), (23,2)$

Day 4:  $\Delta_k = -5$   $(1,20), (3,22), (5,24), (7,2), (9,4), (11,6), (13,8), (15,10)$   
 $(17,12), (19,14), (21,16), (23,18)$

Day 5:  $\Delta_k = 11$   $(1,12), (3,14), (5,16), (7,18), (9,20), (11,22), (13,24),$   
 $(15,2), (17,4), (19,6), (21,8), (23,10)$

Question 2.25 (DG)

(5)  $Z = \text{pwrb} \text{ bin}(n/2)$

(a) Recurrence relation  $T_1(n) = T_1\left(\frac{n}{2}\right) + O(n^\alpha)$

$T_1(n) = O(n^\alpha)$  using master theorem

(b) fastmultiply ( $\text{dec2bin}(x_L), \text{pw2bin}(\frac{n}{2})$ ) +  $\text{dec2bin}(x_R)$

Recurrence Relation  $T_2(n) = 2T_2\left(\frac{n}{2}\right) + T_1\left(\frac{n}{2}\right) + O(n^\alpha) + O(n)$

$T_2(n) = O(n^\alpha)$  using Master Theorem.

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3.7.a. Do a DFS on  $G$ . with prenumber of vertex is taken in modulo 2. If there is an edge ~~b~~ (back edge)  $(u,v)$  such that prenumber of  $u$  modulo 2 is the same as prenumber of  $v$  modulo 2, then the graph is not bipartite, else it is bipartite.

b. only if is easy since an odd cycle cannot be colored with 2 colors. In the DFS of part 1, if  $(u,v)$  is a back ~~a~~ edge, and monochromatic (prenumber of  $u \mod 2$  is same as prenumber  $v \mod 2$ ) - we know the tree edges length have to be even (as  $u$  and  $v$  have the same prenumber) and with the back edge, cycle length is odd.

c. 3 Colors are sufficient (and necessary). Delete the edge which causes the odd cycle. The rest of the bipartite graph is can be colored with two colors. Now color one end of the edge deleted with the third color.

3.8 Construct a directed graph  $G = (V, E)$ . Node is of the form  $(a_{10}, a_7, a_4)$   $0 \leq a_{10} \leq 10, 0 \leq a_7 \leq 7, 0 \leq a_4 \leq 4$ .  $a_{10} + a_7 + a_4 = 11$ . An edge  $(a_{10}^i, a_7^i, a_4^i) \rightarrow (a_{10}^j, a_7^j, a_4^j)$  if they differ ~~two~~ in exactly two coordinates. In the two coordinates they differ, either before they have to be empty and after they have to be full the size of the previous one.

a. We want the final node to be  $(*, 2, *)$  or  $(*, *, 2)$

b. use the DFS to get the solution

$$(0, 7, 4) \rightarrow (4, 7, 0) \rightarrow (4, 3, 4) \rightarrow (8, 3, 0) \rightarrow (8, 0, 3)$$

$$\rightarrow (1, 7, 3) \rightarrow (1, 6, 4) \rightarrow (5, 6, 0) \rightarrow (5, 2, 4)$$

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3.9. degree of a node is the number of nodes adjacent to it. So in linear time, we can compute the degree of each node. Using the following algorithm, we compute two degrees of each node.

$\text{twodegree}(v) = 0$  for all  $v$ .  $\text{visited}(v) = \text{false}$

$\text{explore}(G_1, u)$

```
{
    visited[u] = true
    previsit(u)
    for each edge  $(\overset{u,v}{\overrightarrow{e}}) \in E$ 
        twodegree(v) = twodegree(v) + degree(u)
        if not visited(v)
            explore(G_1, v)
    postvisit(u);
}
```

3.11. An edge  $e$  is not in a cycle iff it is a bridge.

There is an unique path from  $u$  to  $v$ .

Delete  $(e)$  from  $G_1$ . Call the resulting graph  $G'$ .

If  $G'$  is not connected then output (" $e$  is a bridge - using DFS")  
 else output ("no cycle passes through  $e$ ")  
 ("there is a cycle passes through  $e$ ")

3.12  $\{u, v\}$  is an edge,  $\text{post}(u) < \text{post}(v)$ , ~~means~~ means  
 $\Rightarrow u$  was finished exploring before  $v$  was finished exploring  
 means  $u, v$  has to be a <sup>back</sup> tree edge. So  $v$  is an ancestor of  $u$ .