Fuzzy Sets

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(adapted from slides by R. Jang)

Fuzzy Sets: Outline

Introduction
Basic definitions and terminology
Set-theoretic operations
MF formulation and parameterization
- MFs of one and two dimensions
- Derivatives of parameterized MFs

More on fuzzy union, intersection, and complement
- Fuzzy complement
- Fuzzy intersection and union
- Parameterized T-norm and T-conorm

Motivation

-Treat vague (uncertain) concepts or information

-Use knowledge expressed linguistically

-Perform non-linear mapping from input to output described precisely mathematically

Probability vs. Fuzziness

Randomness:

uncertainty described by tendency (frequency) of a random variable to take on a value in a specified region

Interpretations: frequency -> willingness to accept bet (subjective probability)

Fuzziness:

degree to which the element satisfies properties characterized by a fuzzy set.

Interpretations: Possibility -> similarity -> desirability

Boolean Algebra (Run through)

Assign binary truth value to statements

A	statement
1	true
0	false

A	¬ A
1	0
0	1

Combine statements using AND and OR operators

A	B	AvB
0	0	0
0	1	1
1	0	1
1	1	1

A	B	A^ B
0	0	0
0	1	0
1	0	0
1	1	1

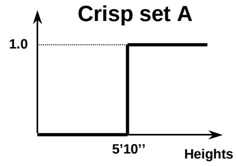
Fuzzy Sets

Binary Logic vs. Fuzzy Logic:

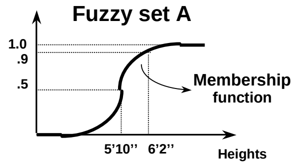
Sets with crisp and fuzzy boundaries, respectively

A = Set of tall people

Crisp set A

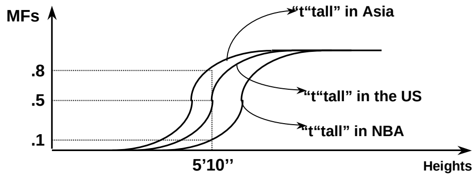


Fuzzy set A



Membership Functions (MFs)

Characteristics of MFs:
- Subjective measures
- Not probability functions



Fuzzy Sets

Formal definition:

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X, \mu_A(x): X \mapsto [0,1]\}$$

Fuzzy set — Membership function (MF) — Universe or universe of discourse

A fuzzy set is totally characterized by a membership function (MF).

Fuzzy Sets with Discrete Universes

Fuzzy set C = "desirable city to live in"

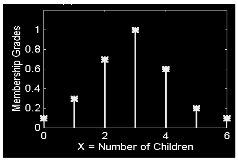
X = {SF, Boston, Troy} (discrete and nonordered)

C = {(SF, 0.9), (Boston, 0.8), (Troy, 0.6)}

Fuzzy set A = "sensible number of children"

X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



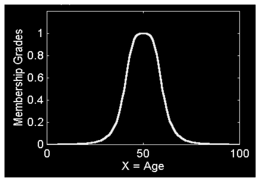
Fuzzy Sets with Cont. Universes

Fuzzy set B = "about 50 years old"

X = Set of positive real numbers (continuous)

B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^2}$$



Alternative Notation

A fuzzy set A can be alternatively denoted as follows:

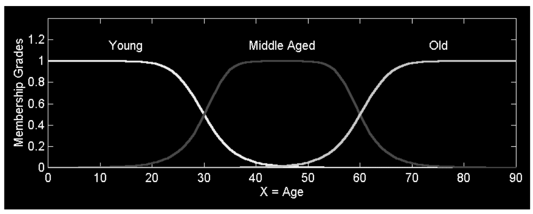
X is discrete ⇨ $A = \sum_{x_i \in X} \mu_A(x_i) / x_i$

X is continuous ⇨ $A = \int_X \mu_A(x) / x$

Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

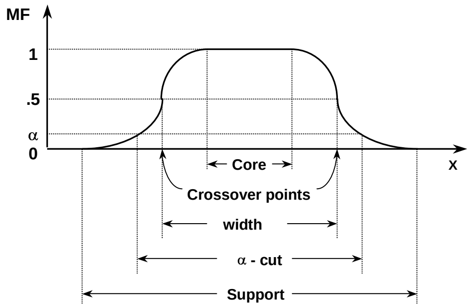
Fuzzy Partition

Fuzzy partitions formed by the linguistic values "young", "middle aged", and "old":



Soft Computing: Fuzzy Sets

MF Terminology



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More Terminology and Definitions

Normality

core non-empty

Fuzzy singleton

support one single point

Fuzzy numbers

fuzzy set on real line R that satisfies convexity and normalit

Symmetricity

$\mu_A(c+x) = \mu_A(c-x)$ for all $x \in X$

Open left or right, closed

$\lim_{x \to -\infty} \mu_A(x) = 1$

$\lim_{x \to +\infty} \mu_A(x) = 0$

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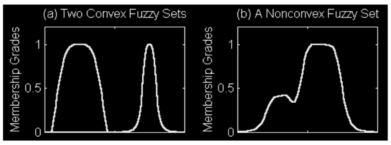
Convexity of Fuzzy Sets

A fuzzy set *A* is convex if for any λ in [0, 1],

$$\mu_A(\lambda x_1 + (1-\lambda) x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, *A* is convex if all its α-cuts are convex.



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Set-Theoretic Operations

Subset:

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x) \quad \text{for all x}$$

Complement:

$$\bar{A} = X - A \Leftrightarrow \mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

Union:

$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

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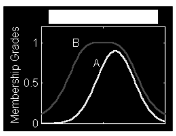
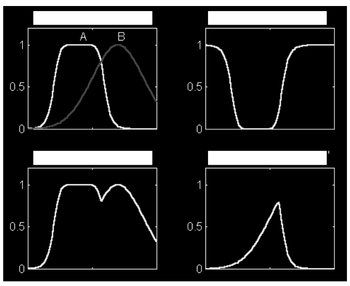
Set-Theoretic Operations





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Membership Function (MF) Formulation

Triangular MF: $trimf(x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$

Trapezoidal MF: $trapmf(x; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$

Gaussian MF: $gaussmf(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$

Generalized bell MF: $gbellmf(x; a, b, c) = \dfrac{1}{1 + \left|\frac{x-c}{b}\right|^{2b}}$

bellmanu.m

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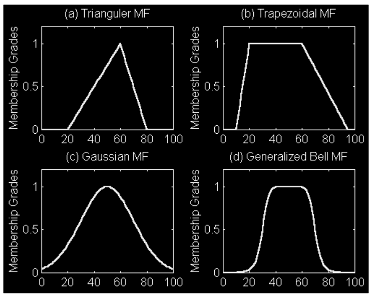
MF Formulation



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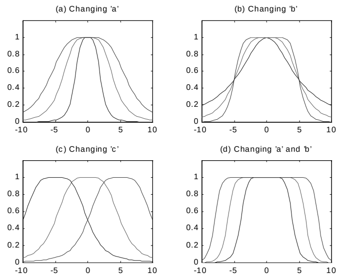
Manipulating parameter of the generalized bell function



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siganim.m

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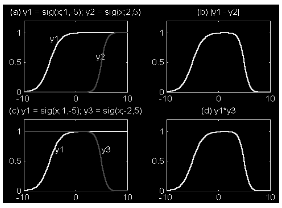
MF Formulation

Sigmoidal MF: $sigmf(x;a,b,c) = \frac{1}{1+e^{-a(x-c)}}$

Extensions:

Abs. difference of two sig. MF

Product of two sig. MF



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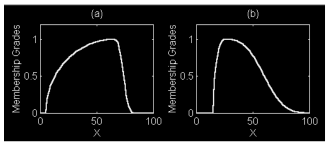
MF Formulation

L-R MF:

$$LR(x;c,\alpha,\beta) = \begin{cases} F_L\left(\frac{c-x}{\alpha}\right), & x < c \\ F_R\left(\frac{x-c}{\beta}\right), & x \ge c \end{cases}$$

Example: $F_L(x) = \sqrt{\max(0, 1-x^2)}$ $F_R(x) = \exp(-|x|^3)$

c=65
a=60
b=10



c=25
a=10
b=40

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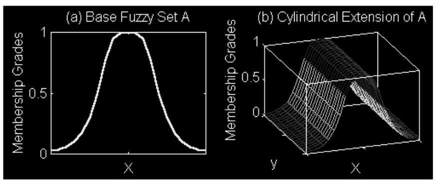
Cylindrical Extension

Base set A **Cylindrical Ext. of A**



$$c(A) = \int_{X \times Y} \mu_A(x)/(x,y)$$

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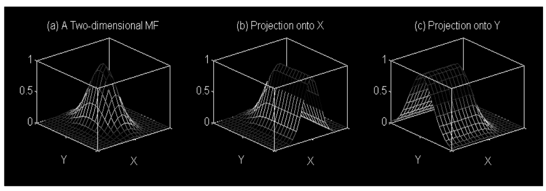
2D MF Projection

Two-dimensional MF **Projection onto X** **Projection onto Y**



$\mu_R(x,y)$ $\mu_A(x) = \max_y \mu_R(x,y)$ $\mu_B(y) = \max_x \mu_R(x,y)$

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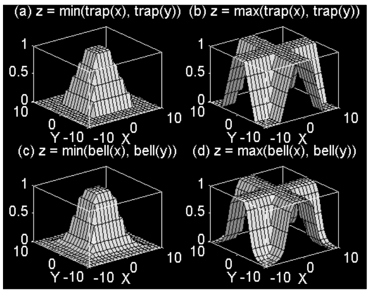
2D MFs



Fuzzy Complement

General requirements:

- **Boundary: N(0)=1 and N(1) = 0**
- **Monotonicity: N(a) > N(b) if a < b**
- **Involution: N(N(a) = a**

Two types of fuzzy complements:

- **Sugeno's complement:**

$$N_s(a) = \frac{1-a}{1+sa}$$

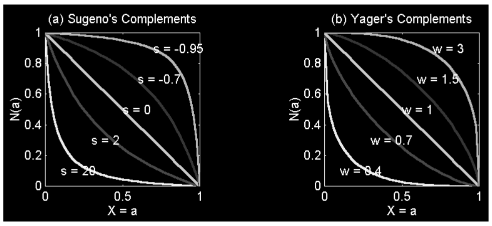
- **Yager's complement:**

$$N_w(a) = (1-a^w)^{1/w}$$

Fuzzy Complement

Sugeno's complement: $N_s(a) = \frac{1-a}{1+sa}$

Yager's complement: $N_w(a) = (1-a^w)^{1/w}$



negation.m

Fuzzy Intersection: T-norm

Basic requirements:

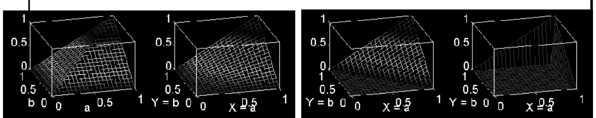
- **Boundary: T(0, 0) = 0, T(a, 1) = T(1, a) = a**
- **Monotonicity: T(a, b) < T(c, d) if a < c and b < d**
- **Commutativity: T(a, b) = T(b, a)**
- **Associativity: T(a, T(b, c)) = T(T(a, b), c)**

Four examples (page 37):

- **Minimum: T_m(a, b)**
- **Algebraic product: T_a(a, b)**
- **Bounded product: T_b(a, b)**
- **Drastic product: T_d(a, b)**

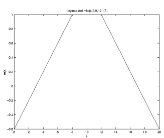
T-norm Operator

Minimum: T_m(a, b)	Algebraic product: T_{ap}(a, b)	Bounded product: T_{bp}(a, b)	Drastic product: T_{dp}(a, b)
$T_{min}(a,b) = \min(a,b) = a \wedge b$	$T_{ap}(a,b) = ab$	$T_{bp}(a,b) = 0 \vee (a+b-1)$	$T_{dp}(a,b) = \begin{cases} a, & \text{if } b=1 \\ b, & \text{if } a=1 \\ 0, & \text{if } a,b<1 \end{cases}$

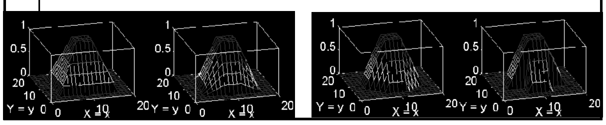


Variation of a=μ_A(x) and b=μ_B(y)

$$a = trapezoid(x,3,8,12,17)$$
$$b = trapezoid(y,3,8,12,17)$$



Minimum: T_m(a, b)	Algebraic product: T_{ap}(a, b)	Bounded product: T_{bp}(a, b)	Drastic product: T_d(a, b)



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Fuzzy Union: T-conorm or S-norm

Basic requirements:
- Boundary: S(1, 1) = 1, S(a, 0) = S(0, a) = a
- Monotonicity: S(a, b) < S(c, d) if a < c and b < d
- Commutativity: S(a, b) = S(b, a)
- Associativity: S(a, S(b, c)) = S(S(a, b), c)

Four examples (page 38):
- Maximum: $S_m(a, b)$
- Algebraic sum: $S_a(a, b)$
- Bounded sum: $S_b(a, b)$
- Drastic sum: $S_d(a, b)$

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Soft Computing: Fuzzy Sets

T-conorm or S-norm

Maximum: $S_m(a, b)$

$S_{min}(a,b) = \max(a,b) = a \vee b$

Algebraic sum: $S_a(a, b)$

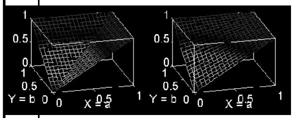
$S_{as}(a,b) = a + b - ab$

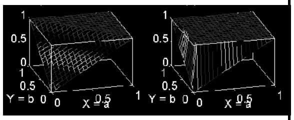
Bounded sum: $S_b(a, b)$

$S_{bs}(a,b) = 1 \wedge (a + b)$

Drastic sum: $S_d(a, b)$

$$S_{ds}(a,b) = \begin{cases} a, & \text{if } b = 0 \\ b, & \text{if } a = 0 \\ 1, & \text{if } a, b > 0 \end{cases}$$





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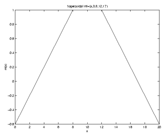
Variation of a=μ_A(x) and b=μ_B(y)

$a = trapezoid(x,3,8,12,17)$

$b = trapezoid(y,3,8,12,17)$

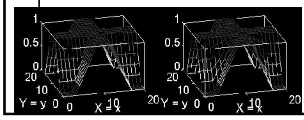


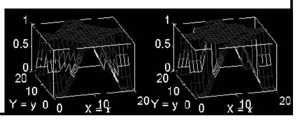
Maximum: $S_m(a, b)$

Algebraic sum: $S_a(a, b)$

Bounded sum: $S_b(a, b)$

Drastic sum: $S_d(a, b)$





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Generalized DeMorgan's Law

T-norms and T-conorms are duals which support the generalization of DeMorgan's law:
- T(a, b) = N(S(N(a), N(b)))
- S(a, b) = N(T(N(a), N(b)))

$T_m(a, b)$ ⟺ $S_m(a, b)$

$T_a(a, b)$ ⟺ $S_a(a, b)$

$T_b(a, b)$ ⟺ $S_b(a, b)$

$T_d(a, b)$ ⟺ $S_d(a, b)$

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Parameterized T-norm and S-norm

Parameterized T-norms and dual T-conorms have been proposed by several researchers:
- Yager
- Schweizer and Sklar
- Dubois and Prade
- Hamacher
- Frank
- Sugeno
- Dombi

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Soft Computing: Fuzzy Sets

Summary

Basic definitions and terminology
- Fuzzy Sets (ordered pairs of variable and MF value)

MF formulation and parameterization
- MFs of one and two dimensions

Set-theoretic operations
- Union, intersection, etc.
- Generalization of intersection (AND): T-norm, e.g., "min"
- Generalization of union (OR): T-conorm, e.g., "max"

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last slide

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