Review from Lecture 21

- Queues and Stacks, What’s a Priority Queue?
- A Priority Queue as a Heap, percolate_up and percolate_down
- A Heap as a Vector, Building a Heap, Heap Sort
- HW8 Contest Results

Today’s Class

- Merging heaps are the motivation for leftist heaps
- Mathematical background & Basic algorithms

22.1 Leftist Heaps — Overview

- Our goal is to be able to merge two heaps in \(O(\log n)\) time, where \(n\) is the number of values stored in the larger of the two heaps.
  - Merging two binary heaps requires \(O(n)\) time
- Leftist heaps are binary trees where we deliberately attempt to eliminate any balance.
- Leftists heaps are implemented explicitly as trees.

22.2 Leftist Heaps — Mathematical Background

- **Definition:** The null path length (NPL) of a tree node is the length of the shortest path to a node with 0 children or 1 child. The NPL of a leaf is 0. The NPL of a NULL pointer is -1.
- **Definition:** A leftist tree is a binary tree where at each node the null path length of the left child is greater than or equal to the null path length of the right child.
- **Definition:** The right path of a node (e.g. the root) is obtained by following right children until a NULL child is reached.
  - In a leftist tree, the right path of a node is at least as short as any other path to a NULL child.
- **Theorem:** A leftist tree with \(r > 0\) nodes on its right path has at least \(2^r - 1\) nodes.
  - This can be proven by induction on \(r\).
- **Corollary:** A leftist tree with \(n\) nodes has a right path length of at most \(|\log(n + 1)| = O(\log n)\) nodes.
- **Definition:** A leftist heap is a leftist tree where the value stored at any node is less than or equal to the value stored at either of its children.

22.3 Leftist Heap Operations

- The insert and delete_min operations will depend on the merge operation.
- Here is the fundamental idea behind the merge operation. Given two leftist heaps, with \(h1\) and \(h2\) pointers to their root nodes, and suppose \(h1->value <= h2->value\). Recursively merge \(h1->right\) with \(h2\), making the resulting heap \(h1->right\).
- When the leftist property is violated at a tree node involved in the merge, the left and right children of this node are swapped. This is enough to guarantee the leftist property of the resulting tree.
- **Merge** requires \(O(\log n + \log m)\) time, where \(m\) and \(n\) are the numbers of nodes stored in the two heaps, because it works on the right path at all times.
22.4 Merge Code

template <class T>
class LeftNode {
  public:
    LeftNode() : npl(0), left(0), right(0) {}
    LeftNode(const T& init) : value(init), npl(0), left(0), right(0) {
      T value;
      int npl; // the null-path length
      LeftNode* left;
      LeftNode* right;
    }
};

Here are the two functions used to implement leftist heap merge operations. Function merge is the driver. Function
merge1 does most of the work. These functions call each other recursively.

template <class Etype>
LeftNode<Etype>* merge(LeftNode<Etype> *H1, LeftNode<Etype> *H2) {
  if (!h1)
    return h2;
  else if (!h2)
    return h1;
  else if (h2->value > h1->value)
    return merge1(h1, h2);
  else
    return merge1(h2, h1);
}

template <class Etype>
LeftNode<Etype>* merge1(LeftNode<Etype> *h1, LeftNode<Etype> *h2) {
  if (h1->left == NULL)
    h1->left = h2;
  else {
    h1->right = merge(h1->right, h2);
    if (h1->left->npl < h1->right->npl)
      swap(h1->left, h1->right);
    h1->npl = h1->right->npl + 1;
  }
  return h1;
}

22.5 Exercises

1. Explain how merge can be used to implement insert and delete_min, and then write code to do so.

2. Show the state of a leftist heap at the end of:

   insert 1, 2, 3, 4, 5, 6
   delete_min
   insert 7, 8
   delete_min
   delete_min