Declarative Concurrency (VRH 4)

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> Adapted with permission from Seif Haridi KTH Peter Van Roy

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Review of concurrent programming

- · There are four basic approaches:
 - Sequential programming (no concurrency)
 - Declarative concurrency (streams in a functional language, Oz)
- Message passing with active objects (Erlang, SALSA)
- Atomic actions on shared state (Java)
- The atomic action approach is the most difficult, yet it is the one you will probably be most exposed to!
- · But, if you have the choice, which approach to use?
 - Use the simplest approach that does the job: sequential if that is
 ok, else declarative concurrency if there is no observable
 nondeterminism, else message passing if you can get away with it.

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Concurrency

- Some programs are best written as a set of activities that run independently (concurrent programs)
- Concurrency is essential for interaction with the external environment
- Examples includes GUI (Graphical User Interfaces), operating systems, web services
- Also programs that are written independently but interact only when needed (client-server, peer-to-peer applications)
- This lecture is about declarative concurrency, programs with no observable nondeterminism, the result is a function
- Independent procedures that execute on their pace and may communicate through shared dataflow variables

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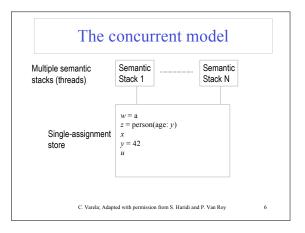
Overview

- Programming with threads
 - The model is augmented with threads
 - Programming techniques: stream communication, order-determining concurrency, coroutines, concurrent composition
- · Lazy execution
 - demand-driven computations, lazy streams, and list comprehensions
- · Soft real-time programming

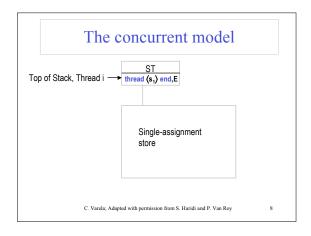
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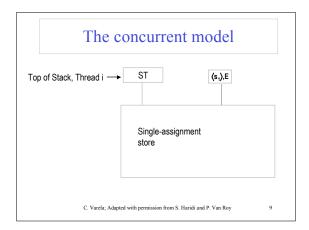
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The sequential model Statements are executed sequentially from a single semantic stack Single-assignment xSingle-assignment x x y = 42C. Varela; Adapted with permission from S. Haridi and P. Van Roy



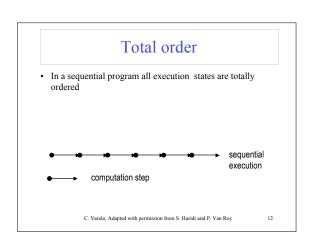
Concurrent declarative model The following defines the syntax of a statement, $\langle s \rangle$ denotes a statement $\langle s \rangle ::= skip \qquad empty statement variable-variable binding variable-value binding variable-value binding sequential composition declaration proc <math>\langle k \rangle \langle y_1 \rangle \dots \langle y_n \rangle \} \langle s_1 \rangle$ end $| f(x) \text{ then } \langle s_1 \rangle \text{ else } \langle s_2 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \} \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \} \langle s_1 \rangle \text{ end}$ $| f(x) \text{ then } \langle s_1 \rangle \text{ else } \langle s_2 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \} \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{ end}$ $| f(x) \langle y_1 \rangle \dots \langle y_n \rangle \langle s_1 \rangle \text{$





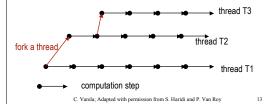
Basic concepts The model allows multiple statements to execute "at the same time". Imagine that these threads really execute in parallel, each has its own processor, but share the same memory Reading and writing different variables can be done simultaneously by different threads, as well as reading the same variable Writing the same variable is done sequentially The above view is in fact equivalent to an interleaving execution: a totally ordered sequence of computation steps, where threads take turns doing one or more steps in sequence

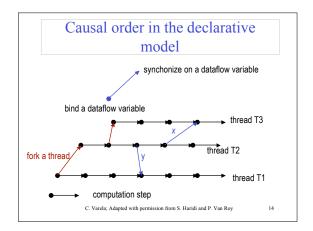
Causal order In a sequential program all execution states are totally ordered In a concurrent program all execution states of a given thread are totally ordered The execution state of the concurrent program as a whole is partially ordered C. Varela; Adapted with permission from S. Haridi and P. Van Roy 11



Causal order in the declarative model · In a concurrent program all execution states of a given

- thread are totally ordered
- · The execution state of the concurrent program is partially





Nondeterminism

- · An execution is nondeterministic if there is a computation step in which there is a choice what to do next
- Nondeterminism appears naturally when there is concurrent access to shared state

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Example of nondeterminism Thread 1 Thread 2 store y = 5time time The thread that binds x first will continue, the other thread will raise an exception C. Varela; Adapted with permission from S. Haridi and P. Van Roy

Nondeterminism

- · In the concurrent declarative model when there is only one binder for each dataflow variable, the nondeterminism is not observable on the store (i.e. the store develops to the same final results)
- · This means for correctness we can ignore the concurrency

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Scheduling

- The choice of which thread to execute next and for how long is done by a part of the system called the scheduler
- A thread is runnable if its next statement to execute is not blocked on a dataflow variable, otherwise the thread is suspended
- A scheduler is fair if it does not starve a runnable thread, i.e. all runnable threads eventually execute
- Fair scheduling makes it easy to reason about programs and program composition
- Otherwise some correct program (in isolation) may never get processing time when composed with other programs

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The semantics

• In the sequential model we had: (ST, σ)

ST is a stack of semantic statements σ is the single assignment store

• In the concurrent model we have:

 (MST, σ)

MST is a (multi)set of stacks of semantic statements σ is the single assignment store

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The initial execution state

statement

({ [((s), \emptyset)] }, Ø)

stack

store

multiset

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Execution (the scheduler)

- At each step, one <u>runnable</u> semantic stack is *selected* from MST (the multiset of stacks), call it ST, s.t. MST = ST ∪ MST'
- Assume the current store is σ , one computation step is done that transforms ST to ST' and σ to σ '
- The total computation state is transformed from (MST, σ) to (ST' \cup MST', σ ')
- Which stack is selected, and how many steps are taken is the task of the scheduler, a good scheduler should be fair, i.e., each runnable 'thread' will eventually be selected
- The computation stops when there are no runnable stacks

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Example of runnable threads

```
proc {Loop P N}
if N > 0 then
{P} {Loop P N-1}
else skip end
end
thread {Loop
proc {$} {Show 1} end
thread {Loop
proc {$} {Show 2} end
1000}
end
thread {Loop
proc {$} {Show 2} end
1000}
end
```

- This program will interleave the execution of two threads, one printing 1, and the other printing 2
- We assume a fair scheduler

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Dataflow computation

- Threads suspend on data unavailability in dataflow variables
- The {Delay X} primitive makes the thread suspends for X milliseconds, after that, the thread is runnable

```
declare X

{Browse X}

local Y in

thread {Delay 1000} Y = 10*10 end

X = Y + 100*100

end
```

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Illustrating dataflow computation

```
declare X0 X1 X2 X3
{Browse [X0 X1 X2 X3]}
thread
    Y0 Y1 Y2 Y3
in
    {Browse [Y0 Y1 Y2 Y3]}
    Y0 = X0 + 1
    Y1 = X1 + Y0
    Y2 = X2 + Y1
    Y3 = X3 + Y2
    {Browse completed}
end
```

- Enter incrementally the values of X0 to X3
- When X0 is bound the thread will compute Y0=X0+1, and will suspend again until X1 is bound

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Concurrent Map fun {Map Xs F} · This will fork a thread for each individual element in the input case Xs of nil then nil Each thread will run only if [] X|Xr then both the element X and the procedure F is known thread {F X} end|{Map Xr F}

```
Concurrent Map Function
fun {Map Xs F}
case Xs
of nil then nil
[] X|Xr then thread {F X} end |{Map Xr F}
• What this looks like in the kernel language:
proc {Map Xs F Rs}
case Xs
of nil then Rs = nil
    XXrthen R Rrin
Rs = R|Rr
      thread R = {F X} end
{Map Xr F Rr}
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```

How does it work?

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· If we enter the following statements: declare FXYZ {Browse thread {Map X F} end}

end

- · A thread executing Map is created.
- · It will suspend immediately in the case-statement because
- If we thereafter enter the following statements: X = 1|2|Yfun {F X} X*X end
- The main thread will traverse the list creating two threads for the first two arguments of the list

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How does it work?

The main thread will traverse the list creating two threads for the first two arguments of the list:

```
thread {F 1} end, and thread {F 2} end,
```

After entering:

Y = 3|ZZ = nil

the program will complete the computation of the main thread and the newly created thread thread $\{F~3\}$ end, resulting in the final list [1 4 9].

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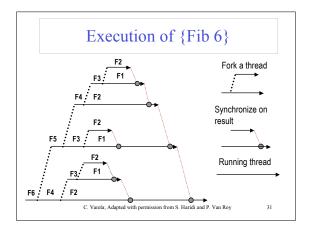
Simple concurrency with dataflow

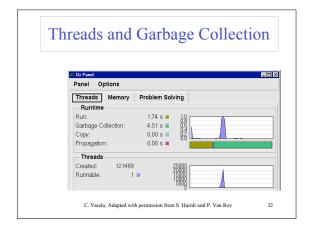
end

```
• Declarative programs can be fun {Fib X}
                                    if X=<2 then 1
  easily made concurrent
                                    else
· Just use the thread statement
                                     thread (Fib X-1) end + (Fib X-2)
   where concurrency is needed
                                    end
```

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```
Understanding why
fun {Fib X}
 if X=<2 then 1
 else F1 F2 in
  F1) = thread (Fib X-1) end
   F2 = {Fib X-2}
                                 Dataflow dependency
└►(F1) + F2
  end
end
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```





Streams

- A stream is a sequence of messages
- · A stream is a First-In First-Out (FIFO) channel
- The producer augments the stream with new messages, and the consumer reads the messages, one by one.



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Stream Communication I

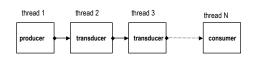
- The data-flow property of Oz easily enables writing threads that communicate through streams in a producerconsumer pattern.
- A stream is a list that is created incrementally by one thread (the producer) and subsequently consumed by one or more threads (the consumers).
- · The consumers consume the same elements of the stream.

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Stream Communication II

- Producer, produces incrementally the elements
- Transducer(s), transform(s) the elements of the stream
- Consumer, accumulates the results



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Stream communication patterns

- The producer, transducers, and the consumer can, in general, be described by certain program patterns
- · We show various patterns

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fun {Producer State} if {More State} then X = {Produce State} in X | {Produce State} in X | {Producer {Transform State}} else nil end end • The definition of More, Produce, and Transform is problem dependent • State could be multiple arguments • The above definition is not a complete program! C. Varela; Adapted with permission from S. Haridi and P. Van Roy 37

Example Producer fun {Generate N Limit} fun (Producer State) if {More State} then if N=<Limit then X = {Produce State} in N | {Generate N+1 Limit} X | {Producer { Transform State}} else nil end else nil end end · The State is the two arguments N and Limit - The predicate More is the condition N=<Limit· The predicate Produce is the identity function on N • The Transform function $(N,Limit) \Rightarrow (N+1,Limit)$ C. Varela; Adapted with permission from S. Haridi and P. Van Roy

```
fun {Consumer State InStream}
case InStream
of nil then {Final State}
[] X | RestInStream then
NextState = {Consume X State} in
{Consumer NextState RestInStream}
end
end

• Final and Consume are problem dependent
```

```
Example Consumer
                                    fun {Consumer State InStream}
fun {Sum A Xs}
                                      case InStream
 case Xs
                                      of nil then {Final State}
 of nil then A
                                     [] X | RestInStream then
                                       NextState = {Consume X State} in
 [] X|Xr then {Sum A+X Xr}
                                       {Consumer NextState RestInStream}
 end
· The State is A
· Final is just the identity function on State
• Consume takes X and State \Rightarrow X + State
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```

```
fun {Transducer Pattern 1

fun {Transducer State InStream}
case InStream
of nil then nil
[] X | RestinStream then
NextState#TX = {Transform X State}
TX | {Transducer NextState RestinStream}
end
end

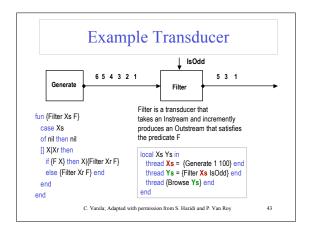
• A transducer keeps its state in State, receives messages on InStream and sends messages on OutStream

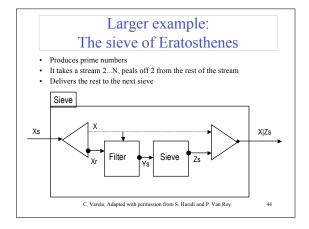
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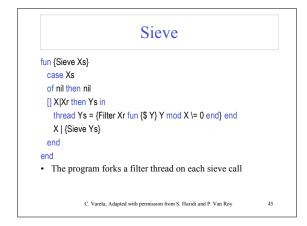
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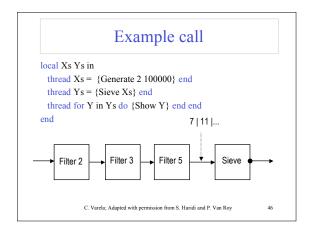
```
fun {Transducer Pattern 2

fun {Transducer State InStream}
case InStream
of nil then nil
| X | RestInStream then
if {Test X#State} then
NextState | Tx = (Transform X State)
TX | {Transducer NextState RestInStream}
else {Transducer State RestInStream} end
end
end
end
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```









Solutions

There are three alternatives:

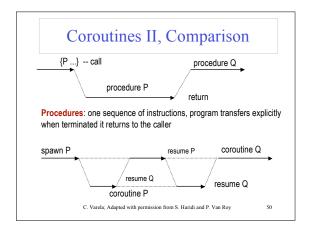
- 1. Play with the speed of the different threads, i.e. play with the scheduler to make the producer slower
- 2. Create a bounded buffer, say of size N, so that the producer waits automatically when the buffer is full
- 3. Use demand-driven approach, where the consumer activates the producer when it needs a new element (lazy evaluation)
- The last two approaches introduce the notion of flowcontrol between concurrent activities (very common)

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Coroutines I

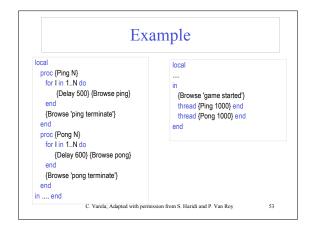
- Languages that do not support concurrent threads might instead support a notion called coroutining
- A coroutine is a nonpreemptive thread (sequence of instructions), there is no scheduler
- Switching between threads is the programmer's responsibility

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Coroutines II, Comparison {P ...} -- call procedure Q procedure P return Coroutines: New sequences of instructions, programs explicitly do all the scheduling, by spawn, suspend and resume spawn P resume Q coroutine P C. Varela, Adapted with permission from S. Haridi and P. Van Roy 51

Time In concurrent computation one would like to handle time proc {Time.delay T} - The running thread suspends for T milliseconds proc {Time.alarm T U} - Immediately creates its own thread, and binds U to unit after T milliseconds



Concurrent control abstraction • We have seen how threads are forked by 'thread ... end' • A natural question to ask is: how can we join threads? fork threads C. Varela; Adapted with permission from S. Haridi and P. Van Roy

Termination detection

- This is a special case of detecting termination of multiple threads, and making another thread wait on that event.
- The general scheme is quite easy because of dataflow variables:

```
thread \langle S1 \rangle X1 = unit end
thread \langle S2 \rangle X2 = X1 end
thread \langle Sn \rangle X_n = X_{n-1} end
{Wait Xn}
% Continue main thread
```

- When all threads terminate the variables $X_1 \dots X_N$ will be merged together labeling a single box that contains the value unit.
- $\{\text{Wait }X_{N}\}$ suspends the main thread until X_{N} is bound.

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Concurrent Composition

```
conc S<sub>1</sub> [] S<sub>2</sub> [] ... [] S<sub>n</sub> end
{Conc [proc{$} S1 end
             proc{$} S2 end
             proc{$} Sn end] }
```

- · Takes a single argument that is a list of nullary procedures.
- When it is executed, the procedures are forked concurrently. The next statement is executed only when all procedures in the list terminate.

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Conc

```
local
proc {Conc1 Ps I O}
case Ps of P|Pr then
M in
thread {P} M = I end
{Conc1 Pr M O}
[ nil then O = I
end
     end
    proc {Conc Ps}
X in {Conc1 Ps unit X}
       {Wait X}
```

This abstraction takes a list of zero-argument procedures and terminate after all these threads have terminated

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Example

```
local
proc {Ping N}
 for I in 1..N do
                                        in
     {Delay 500} {Browse ping}
                                         {Browse 'game started'}
                                         {Conc
 {Browse 'ping terminate'}
                                           [ proc {$} {Ping 1000} end
                                             proc {$} {Pong 1000} end ]}
end
proc {Pong N}
                                          {Browse 'game terminated'}
 for I in 1..N do
    {Delay 600} {Browse pong}
 {Browse 'pong terminate'}
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```

Futures

- A **future** is a read-only capability of a single-assignment variable. For example to create a future of the variable X we perform the operation!! to create a future Y: Y = !!X
- A thread trying to use the value of a future, e.g. using Y, will suspend until the variable of the future, e.g. X, gets bound.
- One way to execute a procedure lazily, i.e. in a demand-driven manner, is to use the operation {ByNeed +P ?F}
- ByNeed takes a zero-argument function P, and returns a future F. When a thread tries to access the value of F, the function $\{P\}$ is called, and its result is bound to F.
- This allows us to perform demand-driven computations in a straightforward

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Example

- declare Y
 {ByNeed fun {\$} 1 end Y}
 {Browse Y}
- we will observe that Y becomes a future, i.e. we will see Y<Future> in the
- If we try to access the value of Y, it will get bound to 1.
- One way to access Y is by perform the operation {Wait Y} which triggers the producing procedure.

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Thread Priority and Real Time

- · Try to run the program using the following statement:
 - {Sum 0 thread {Generate 0 100000000} end}
- · Switch on the panel and observe the memory behavior of the program.
- · You will quickly notice that this program does not behave well.
- The reason has to do with the asynchronous message passing. If the producer sends messages i.e. create new elements in the stream, in a faster rate than the consumer can consume, increasingly more buffering will be needed until the system starts to break down.
- One possible solution is to control experimentally the rate of thread execution so
 that the consumers get a larger time-slice than the producers do.

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Priorities

- · There are three priority levels
 - · high,
 - · medium, and
 - · low (the default)
- · A priority level determines how often a runnable thread is allocated a time slice.
- In Oz, a high priority thread cannot starve a low priority one. Priority determines only how large piece of the processor-cake a thread can get.
- Each thread has a unique name. To get the name of the current thread the procedure Thread.this/1 is called.
- Having a reference to a thread, by using its name, enables operations on threads such as:
 - · terminating a thread, or
 - · raising an exception in a thread.
- · Thread operations are defined the standard module Thread.

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Thread priority and thread control

fun {Thread.state T}
proc{Thread.injectException T E}
fun {Thread.this}

%% returns thread state
%% exception E injected into thread
%% returns 1st class reference to thread

proc{Thread.setPriority T P} %% P is high, medium or low proc{Thread.setThisPriority P} %% as above on current thread

fun{Property.get priorities}

%% get priority ratios

proc{Property.put priorities(high:H medium:M)}

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Thread Priorities

- Oz has three priority levels. The system procedure
 - {Property.put priorities p(medium:Y high:X)}
 - Sets the processor-time ratio to X:1 between high-priority threads and mediumpriority thread.
 - It also sets the processor-time ratio to \mathtt{Y} : 1 between medium-priority threads and low-priority threads. \mathtt{X} and \mathtt{Y} are integers.
 - Example:
 - {Property.put priorities p(high:10 medium:10)}
- Now let us make our producer-consumer program work. We give the producer low priority, and the consumer high. We also set the priority ratios to 10:1 and 10:1

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The program with priorities

```
local L in
{Property.put priorities p(high:10 medium:10)}
thread
{Thread.setThisPriority low}
L = {Generate 0 100000000}
end
thread
{Thread.setThisPriority high}
{Sum 0 L}
end
end
```

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Exercises

- 85. SALSA asynchronous message passing enables to tag messages with properties: priority, delay, and waitfor. Compare these mechanisms with Oz thread priorities, time delays and alarms, and futures.
- 86. How do SALSA tokens relate to Oz dataflow variables and futures?
- 87. What is the difference between multiple thread termination detection in Oz and join blocks in SALSA?
- 88. VRH Exercise 4.11.3 (page 339)
 - Compare the sequential and concurrent execution performance of equivalent SALSA programs.
- 89. VRH Exercise 4.11.5 (page 339)

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