

# Functional Programming:

Lists, Pattern Matching, Recursive Programming (CTM  
Sections 1.1-1.7, 3.2, 3.4.1-3.4.2,4.7.2)

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# Introduction to Oz

- An introduction to programming concepts
- Declarative variables
- Structured data (example: lists)
- Functions over lists
- Correctness and complexity

# Variables

- Variables are short-cuts for values, they cannot be assigned more than once

**declare**

V = 9999\*9999

{Browse V\*V}

- Variable identifiers: is what you type
- Store variable: is part of the memory system
- The **declare** statement creates a store variable and assigns its memory address to the identifier 'V' in the environment

# Functions

- Compute the factorial function:
- Start with the mathematical definition

$$n! = 1 \times 2 \times \dots \times (n-1) \times n$$

declare

fun {Fact N}

  if N==0 then 1 else N\*{Fact N-1} end

end

$$0! = 1$$

$$n! = n \times (n-1)! \text{ if } n > 0$$

- Fact is declared in the environment
- Try large factorial {Browse {Fact 100}}

# Factorial in Haskell

`factorial :: Integer -> Integer`

`factorial 0 = 1`

`factorial n | n > 0 = n * factorial (n-1)`

# Composing functions

- Combinations of  $r$  items taken from  $n$ .
- The number of subsets of size  $r$  taken from a set of size  $n$

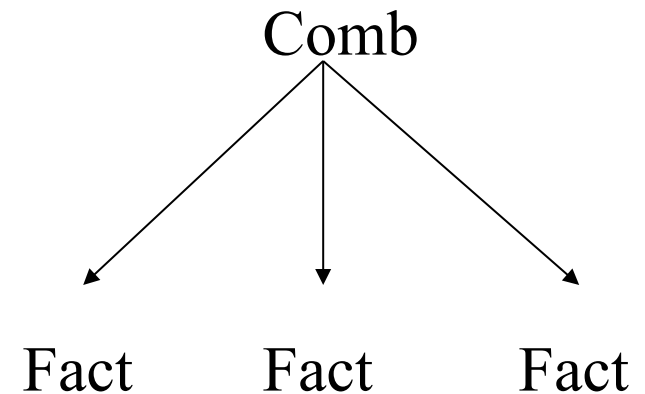
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

declare

fun {Comb N R}

{Fact N} div ({Fact R}\*{Fact N-R})

end



- Example of functional abstraction

# Structured data (lists)

- Calculate Pascal triangle
- Write a function that calculates the nth row as one structured value
- A list is a sequence of elements:  
[1 4 6 4 1]
- The empty list is written nil
- Lists are created by means of "[]" (cons)

```
      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
```

`declare`

`H=1`

`T = [2 3 4 5]`

`{Browse H|T} % This will show [1 2 3 4 5]`

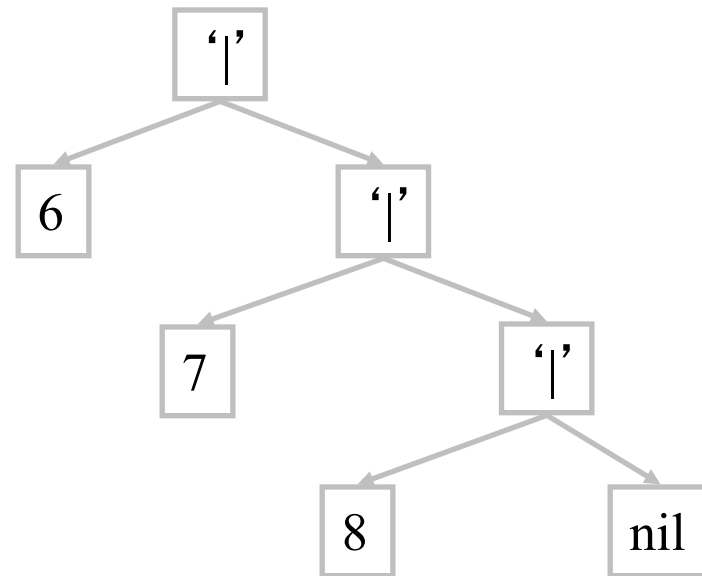
# Lists (2)

- Taking lists apart (selecting components)
- A cons has two components: a head, and a tail

`declare L = [5 6 7 8]`

L.1 gives 5

L.2 give [6 7 8]





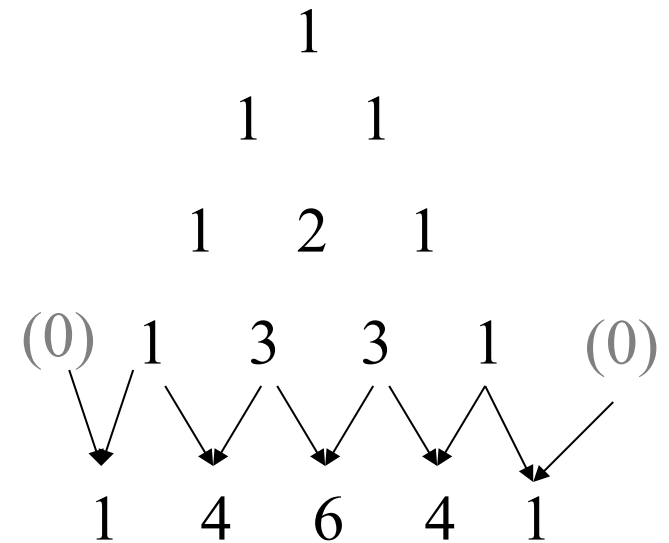
# Pattern matching

- Another way to take a list apart is by use of pattern matching with a case instruction

```
case L of H|T then {Browse H} {Browse T}
           else {Browse 'empty list' }
end
```

# Functions over lists

- Compute the function {Pascal N}
  - Takes an integer N, and returns the Nth row of a Pascal triangle as a list
1. For row 1, the result is [1]
  2. For row N, shift to left row N-1 and shift to the right row N-1
  3. Align and add the shifted rows element-wise to get row N



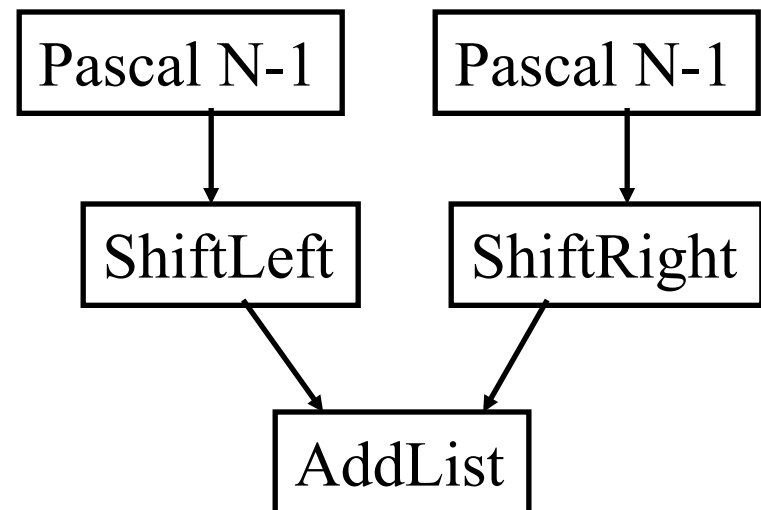
Shift right [0 1 3 3 1]

Shift left [1 3 3 1 0]

# Functions over lists (2)

```
declare
fun {Pascal N}
  if N==1 then [1]
  else
    {AddList
     {ShiftLeft {Pascal N-1}}
     {ShiftRight {Pascal N-1}}}}
  end
end
```

Pascal N



# Functions over lists (3)

```
fun {ShiftLeft L}
  case L of H|T then
    H{ShiftLeft T}
  else [0]
  end
end

fun {ShiftRight L} 0|L end
```

```
fun {AddList L1 L2}
  case L1 of H1|T1 then
    case L2 of H2|T2 then
      H1+H2{AddList T1 T2}
    end
  else nil end
end
```

# Top-down program development

- Understand how to solve the problem by hand
- Try to solve the task by decomposing it to simpler tasks
- Devise the main function (main task) in terms of suitable auxiliary functions (subtasks) that simplify the solution (ShiftLeft, ShiftRight and AddList)
- Complete the solution by writing the auxiliary functions
- Test your program bottom-up: auxiliary functions first.

# Is your program correct?

- “A program is correct when it does what we would like it to do”
- In general we need to reason about the program:
- **Semantics for the language**: a precise model of the operations of the programming language
- **Program specification**: a definition of the output in terms of the input (usually a mathematical function or relation)
- Use mathematical techniques to reason about the program, using programming language semantics

# Mathematical induction

- Select one or more inputs to the function
- Show the program is correct for the *simple cases* (base cases)
- Show that if the program is correct for a *given case*, it is then correct for the *next case*.
- For natural numbers, the base case is either 0 or 1, and for any number  $n$  the next case is  $n+1$
- For lists, the base case is `nil`, or a list with one or a few elements, and for any list `T` the next case is `H|T`

# Correctness of factorial

```
fun {Fact N}  
  if N==0 then 1 else N*{Fact N-1} end  
end
```

$$\underbrace{1 \times 2 \times \dots \times (n-1)}_{\text{Fact}(n-1)} \times n$$

- Base Case  $N=0$ : {Fact 0} returns 1
- Inductive Case  $N>0$ : {Fact N} returns  $N*\{\text{Fact } N-1\}$  assume {Fact  $N-1$ } is correct, from the spec we see that {Fact N} is  $N*\{\text{Fact } N-1\}$



# Complexity

- Pascal runs very slow, try {Pascal 24}
- {Pascal 20} calls: {Pascal 19} twice, {Pascal 18} four times, {Pascal 17} eight times, ..., {Pascal 1}  $2^{19}$  times
- Execution time of a program up to a constant factor is called the program's *time complexity*.
- Time complexity of {Pascal N} is proportional to  $2^N$  (exponential)
- Programs with exponential time complexity are impractical

```
declare
fun {Pascal N}
  if N==1 then [1]
  else
    {AddList
      {ShiftLeft {Pascal N-1}}
      {ShiftRight {Pascal N-1}}}
  end
end
```

# Faster Pascal

- Introduce a local variable L
- Compute {FastPascal N-1} only once
- Try with 30 rows.
- FastPascal is called N times, each time a list on the average of size N/2 is processed
- The time complexity is proportional to  $N^2$  (polynomial)
- Low order polynomial programs are practical.

```
fun {FastPascal N}
  if N==1 then [1]
  else
    local L in
      L={FastPascal N-1}
      {AddList {ShiftLeft L} {ShiftRight L}}
    end
  end
end
```

# Iterative computation

- An iterative computation is one whose execution stack is bounded by a constant, independent of the length of the computation
- Iterative computation starts with an initial state  $S_0$ , and transforms the state in a number of steps until a final state  $S_{\text{final}}$  is reached:

$$S_0 \longrightarrow S_1 \longrightarrow \dots \longrightarrow S_{\text{final}}$$

# The general scheme

```
fun {Iterate  $S_i$ }  
  if {IsDone  $S_i$ } then  $S_i$   
  else  $S_{i+1}$  in  
     $S_{i+1} = \{Transform\ S_i\}$   
    {Iterate  $S_{i+1}$ }  
  end  
end
```

- *IsDone* and *Transform* are problem dependent

# The computation model

- STACK : [ R={Iterate  $S_0$ } ]
- STACK : [  $S_1 = \{Transform\ S_0\}$ ,  
R={Iterate  $S_1$ } ]
- STACK : [ R={Iterate  $S_i$ } ]
- STACK : [  $S_{i+1} = \{Transform\ S_i\}$ ,  
R={Iterate  $S_{i+1}$ } ]
- STACK : [ R={Iterate  $S_{i+1}$ } ]

# Newton's method for the square root of a positive real number

- Given a real number  $x$ , start with a guess  $g$ , and improve this guess iteratively until it is accurate enough
- The improved guess  $g'$  is the average of  $g$  and  $x/g$ :

$$g' = (g + x / g) / 2$$

$$\varepsilon = g - \sqrt{x}$$

$$\varepsilon' = g' - \sqrt{x}$$

For  $g'$  to be a better guess than  $g$ :  $\varepsilon' < \varepsilon$

$$\varepsilon' = g' - \sqrt{x} = (g + x / g) / 2 - \sqrt{x} = \varepsilon^2 / 2g$$

$$\text{i.e. } \varepsilon^2 / 2g < \varepsilon, \quad \varepsilon / 2g < 1$$

$$\text{i.e. } \varepsilon < 2g, \quad g - \sqrt{x} < 2g, \quad 0 < g + \sqrt{x}$$

# Newton's method for the square root of a positive real number

- Given a real number  $x$ , start with a guess  $g$ , and improve this guess iteratively until it is accurate enough
- The improved guess  $g'$  is the average of  $g$  and  $x/g$ :
- Accurate enough is defined as:

$$|x - g^2| / x < 0.00001$$

# SqrtIter

```
fun {SqrtIter Guess X}
  if {GoodEnough Guess X} then Guess
  else
    Guess1 = {Improve Guess X} in
    {SqrtIter Guess1 X}
  end
end
```

- Compare to the general scheme:
  - The state is the pair `Guess` and `X`
  - *IsDone* is implemented by the procedure `GoodEnough`
  - *Transform* is implemented by the procedure `Improve`



# The program version 1

```
fun {Sqrt X}
  Guess = 1.0
in {SqrtIter Guess X}
end
fun {SqrtIter Guess X}
  if {GoodEnough Guess X} then
    Guess
  else
    {SqrtIter {Improve Guess X} X}
  end
end
end
```

```
fun {Improve Guess X}
  (Guess + X/Guess)/2.0
end
fun {GoodEnough Guess X}
  {Abs X - Guess*Guess}/X < 0.00001
end
```

# Using local procedures

- The main procedure Sqrt uses the helper procedures SqrtIter, GoodEnough, Improve, and Abs
- SqrtIter is only needed inside Sqrt
- GoodEnough and Improve are only needed inside SqrtIter
- Abs (absolute value) is a general utility
- The general idea is that helper procedures should not be visible globally, but only locally

# Sqrt version 2

```
local
  fun {SqrtIter Guess X}
    if {GoodEnough Guess X} then Guess
    else {SqrtIter {Improve Guess X} X} end
  end
  fun {Improve Guess X}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough Guess X}
    {Abs X - Guess*Guess}/X < 0.000001
  end
in
  fun {Sqrt X}
    Guess = 1.0
  in {SqrtIter Guess X} end
end
```

# Sqrt version 3

- Define GoodEnough and Improve inside SqrtIter

local

```
fun {SqrtIter Guess X}
  fun {Improve}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough}
    {Abs X - Guess*Guess}/X < 0.000001
  end
```

in

```
  if {GoodEnough} then Guess
  else {SqrtIter {Improve} X} end
```

end

in fun {Sqrt X}

```
  Guess = 1.0 in
  {SqrtIter Guess X}
```

end

end

# Sqrt version 3

- Define GoodEnough and Improve inside SqrtIter

local

```
fun {SqrtIter Guess X}
  fun {Improve}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough}
    {Abs X - Guess*Guess}/X < 0.000001
  end
```

in

```
  if {GoodEnough} then Guess
  else {SqrtIter {Improve} X} end
```

end

in fun {Sqrt X}

```
  Guess = 1.0 in
```

```
  {SqrtIter Guess X}
```

end

end

The program has a single drawback: on each iteration two procedure values are created, one for Improve and one for GoodEnough

# Sqrt final version

```
fun {Sqrt X}
  fun {Improve Guess}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough Guess}
    {Abs X - Guess*Guess}/X < 0.000001
  end
  fun {SqrtIter Guess}
    if {GoodEnough Guess} then Guess
    else {SqrtIter {Improve Guess}} end
  end
  Guess = 1.0
in {SqrtIter Guess}
end
```

The final version is  
a compromise between  
abstraction and efficiency

# From a general scheme to a control abstraction (1)

```
fun {Iterate  $S_i$ }  
  if {IsDone  $S_i$ } then  $S_i$   
  else  $S_{i+1}$  in  
     $S_{i+1} = \{Transform\ S_i\}$   
    {Iterate  $S_{i+1}$ }  
  end  
end
```

- *IsDone* and *Transform* are problem dependent

## From a general scheme to a control abstraction (2)

```
fun {Iterate S IsDone Transform}
  if {IsDone S} then S
  else S1 in
    S1 = {Transform S}
    {Iterate S1 IsDone Transform}
  end
end
```

```
fun {Iterate  $S_i$ }
  if {IsDone  $S_i$ } then  $S_i$ 
  else  $S_{i+1}$  in
     $S_{i+1} = \{Transform S_i\}$ 
    {Iterate  $S_{i+1}$ }
  end
end
```



# Sqrt using the Iterate abstraction

```
fun {Sqrt X}
  fun {Improve Guess}
    (Guess + X/Guess)/2.0
  end
  fun {GoodEnough Guess}
    {Abs X - Guess*Guess}/X < 0.000001
  end
  Guess = 1.0
in
  {Iterate Guess GoodEnough Improve}
end
```

# Sqrt using the control abstraction

```
fun {Sqrt X}  
  {Iterate  
    1.0  
    fun {$ G} {Abs X - G*G}/X < 0.000001 end  
    fun {$ G} (G + X/G)/2.0 end  
  }  
end
```

Iterate could become a linguistic abstraction

# Sqrt in Haskell

```
let sqrt x = head (dropWhile (not . goodEnough) sqrtGuesses)
```

```
  where
```

```
    goodEnough guess = (abs (x - guess*guess))/x < 0.00001
```

```
    improve guess = (guess + x/guess)/2.0
```

```
    sqrtGuesses = 1:(map improve sqrtGuesses)
```

# Exercises

12. Prove the correctness of `AddList` and `ShiftLeft`.
13. Prove that the alternative version of Pascal triangle (not using `ShiftLeft`) is correct. Make `AddList` and `OpList` commutative.
14. Modify the Pascal function to use local functions for `AddList`, `ShiftLeft`, `ShiftRight`. Think about the abstraction and efficiency tradeoffs.
15. CTM Exercise 3.10.2 (page 230)
16. CTM Exercise 3.10.3 (page 230)
17. Develop a control abstraction for iterating over a list of elements.