Programming Languages
(CSCI 4430/6430)
Part 1: Functional Programming: Summary

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Other programming languages

**Imperative**
- Algol (Naur 1958)
- Cobol (Hopper 1959)
- BASIC (Kennedy and Kurtz 1964)
- Pascal (Wirth 1970)
- C (Kernighan and Ritchie 1971)
- Ada (Whitaker 1979)

**Functional**
- ML (Milner 1973)
- Scheme (Sussman and Steele 1975)
- Haskell (Hughes et al 1987)

**Object-Oriented**
- Smalltalk (Kay 1980)
- C++ (Stroustrup 1983)
- Eiffel (Meyer 1985)
- Java (Gosling 1994)
- C# (Hejlsberg 2000)

**Actor-Oriented**
- Act (Lieberman 1981)
- ABCL (Yonezawa 1988)
- Actalk (Briot 1989)
- Erlang (Armstrong 1990)
- E (Miller et al 1998)
- SALSA (Varela and Agha 1999)

**Scripting**
- Python (van Rossum 1985)
- Perl (Wall 1987)
- Tcl (Ousterhout 1988)
- Lua (Ierusalimschy et al 1994)
- JavaScript (Eich 1995)
- PHP (Lerdorf 1995)
- Ruby (Matsumoto 1995)
Language syntax

• Defines what are the legal programs, i.e. programs that can be executed by a machine (interpreter)
• Syntax is defined by grammar rules
• A grammar defines how to make ‘sentences’ out of ‘words’
• For programming languages: sentences are called statements (commands, expressions)
• For programming languages: words are called tokens
• Grammar rules are used to describe both tokens and statements
Language Semantics

- Semantics defines what a program does when it executes
- Semantics should be simple and yet allows reasoning about programs (correctness, execution time, and memory use)
The syntax of a $\lambda$-calculus expression is as follows:

$$
e ::= v \quad \text{variable} \\
| \quad \lambda v. e \quad \text{functional abstraction} \\
| \quad (e e) \quad \text{function application}
$$

The semantics of a $\lambda$-calculus expression is called beta-reduction:

$$(\lambda x. E \ M) \Rightarrow E\{M/x\}$$

where we alpha-rename the lambda abstraction $E$ if necessary to avoid capturing free variables in $M$. 
Alpha renaming is used to prevent capturing free occurrences of variables when beta-reducing a lambda calculus expression.

In the following, we rename $x$ to $z$, (or any other fresh variable):

$$(\lambda x. (y x) x) \xrightarrow{\alpha} (\lambda z. (y z) x)$$

Only bound variables can be renamed. No free variables can be captured (become bound) in the process. For example, we cannot alpha-rename $x$ to $y$. 
**β-reduction**

\[
(\lambda x. E M) \xrightarrow{\beta} E\{M/x\}
\]

Beta-reduction may require alpha renaming to prevent capturing free variable occurrences. For example:

\[
(\lambda x. \lambda y. (x y) (y w))
\]

\[
\xrightarrow{\alpha} (\lambda x. \lambda z. (x z) (y w))
\]

\[
\xrightarrow{\beta} \lambda z. ((y w) z)
\]

Where the free \( y \) remains free.
\[ \eta \text{-conversion} \]

\[ \lambda x. (E \ x) \overset{\eta}{\rightarrow} E \]

if \( x \) is not free in \( E \).

For example:

\[ (\lambda x. \lambda y. (x \ y) \ (y \ w)) \]
\[ \overset{\alpha}{\rightarrow} (\lambda x. \lambda z. (x \ z) \ (y \ w)) \]
\[ \overset{\beta}{\rightarrow} \lambda z. ((y \ w) \ z) \]
\[ \overset{\eta}{\rightarrow} (y \ w) \]
Currying

The lambda calculus can only represent functions of one variable. It turns out that one-variable functions are sufficient to represent multiple-variable functions, using a strategy called currying.

E.g., given the mathematical function: \( h(x,y) = x+y \)
of type \( h: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \)

We can represent \( h \) as \( h' \) of type: \( h': \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \)
Such that
\[
h(x,y) = h'(x)(y) = x+y
\]
For example,
\[
h'(2) = g, \text{ where } g(y) = 2+y
\]

We say that \( h' \) is the curried version of \( h \).
Function Composition in Lambda Calculus

S: \( \lambda x. (s \ x) \)  (Square)

I: \( \lambda x. (i \ x) \)  (Increment)

C: \( \lambda f. \lambda g. \lambda x. (f \ (g \ x)) \)  (Function Composition)

(((C S) I)

Recall semantics rule:

\((\lambda x. E \ M) \Rightarrow E\{M/x}\)
Order of Evaluation in the Lambda Calculus

Does the order of evaluation change the final result?

Consider:

\[ \lambda x. (\lambda x. (s x) (\lambda x. (i x) x)) \]

There are two possible evaluation orders:

1. \[ \begin{align*}
\lambda x. (\lambda x. (s x) (\lambda x. (i x) x)) \\
\Rightarrow \lambda x. (\lambda x. (s x) (i x)) \\
\Rightarrow \lambda x. (s (i x))
\end{align*} \]

2. \[ \begin{align*}
\lambda x. (\lambda x. (s x) (\lambda x. (i x) x)) \\
\Rightarrow \lambda x. (s (\lambda x. (i x) x)) \\
\Rightarrow \lambda x. (s (i x))
\end{align*} \]

Is the final result always the same?

Recall semantics rule:

\[ (\lambda x. E M) \Rightarrow E[M/x] \]

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Church-Rosser Theorem

If a lambda calculus expression can be evaluated in two different ways and both ways terminate, both ways will yield the same result.

Also called the *diamond* or *confluence* property.

Furthermore, if there is a way for an expression evaluation to terminate, using normal order will cause termination.
Order of Evaluation and Termination

Consider:

\[(\lambda x. y (\lambda x. (x x) \lambda x. (x x)))\]

There are two possible evaluation orders:

\[(\lambda x. y (\lambda x. (x x) \lambda x. (x x))) \Rightarrow (\lambda x. y (\lambda x. (x x) \lambda x. (x x)))\]

and:

\[(\lambda x. y (\lambda x. (x x) \lambda x. (x x))) \Rightarrow y\]

Recall semantics rule:

\[(\lambda x. E M) \Rightarrow E\{M/x\}\]

Applicative Order

Normal Order

In this example, normal order terminates whereas applicative order does not.

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Free and Bound Variables

The lambda functional abstraction is the only syntactic construct that *binds* variables. That is, in an expression of the form:

$$\lambda v. e$$

we say that free occurrences of variable $v$ in expression $e$ are *bound*. All other variable occurrences are said to be *free*.

E.g.,

$$((\lambda x. \lambda y. (x y) (y w))$$

Bound Variables  Free Variables
A lambda calculus expression with no free variables is called a combinator. For example:

I: \( \lambda x.x \) (Identity)
App: \( \lambda f.\lambda x.(f\ x) \) (Application)
C: \( \lambda f.\lambda g.\lambda x.(f\ (g\ x)) \) (Composition)
L: \( (\lambda x.(x\ x)\ \lambda x.(x\ x)) \) (Loop)
Cur: \( \lambda f.\lambda x.\lambda y.((f\ x)\ y) \) (Currying)
Seq: \( \lambda x.\lambda y.((\lambda z.\ y\ x) \ x) \) (Sequencing--normal order)
ASeq: \( \lambda x.\lambda y.((y\ x) \ x) \) (Sequencing--applicative order)

where \( y \) denotes a thunk, i.e., a lambda abstraction wrapping the second expression to evaluate.

The meaning of a combinator is always the same independently of its context.
The currying combinator can be written in Oz as follows:

\[
\begin{align*}
\text{fun} \{\$ F\} \\
& \quad \text{fun} \{\$ X\} \\
& \quad \quad \text{fun} \{\$ Y\} \\
& \quad \quad \quad \{F \times Y\} \\
& \quad \quad \text{end} \\
& \quad \text{end} \\
& \text{end}
\end{align*}
\]

It takes a function of two arguments, \(F\), and returns its curried version, e.g.,

\[
\{\{\text{Curry Plus} \ 2\} \ 3\} \Rightarrow 5
\]
Recursion Combinator (Y or rec)

$X$ can be defined as $(Yf)$, where $Y$ is the recursion combinator.

\[
Y : \lambda f. (\lambda x. (f (\lambda y. ((x x) y))) \\
    \lambda x. (f (\lambda y. ((x x) y)))
\]

You get from the normal order to the applicative order recursion combinator by $\eta$-expansion ($\eta$-conversion from right to left).
Natural Numbers in Lambda Calculus

|0|: $\lambda x.x$ (Zero)
|1|: $\lambda x.\lambda x.x$ (One)

...  

|n+1|: $\lambda x.|n|$ (N+1)

s: $\lambda n.\lambda x.n$ (Successor)

\[
\begin{align*}
(s \ 0) \\
(\lambda n.\lambda x.n \ \lambda x.x) \\
\Rightarrow \lambda x.\lambda x.x
\end{align*}
\]

Recall semantics rule:

\[
(\lambda x. E \ M) \ \Rightarrow \ E\{M/x\}
\]
Booleans and Branching ($if$) in $\lambda$ Calculus

$$|true|: \lambda x.\lambda y.x \quad \text{(True)}$$

$$|false|: \lambda x.\lambda y.y \quad \text{(False)}$$

$$|if|: \lambda b.\lambda t.\lambda e.((b t) e) \quad \text{(If)}$$

Recall semantics rule:

$$((\lambda x. E \ M) \Rightarrow E\{M/x\})$$

$$(\lambda b.\lambda t.\lambda e.((b t) e) \\lambda x.\lambda y.x) \ a \ b)$$

$$\Rightarrow ((\lambda t.\lambda e.((\lambda x.\lambda y.x \ t) e) a) \ b)$$

$$\Rightarrow (\lambda e.((\lambda x.\lambda y.x \ a) e) \ b)$$

$$\Rightarrow ((\lambda x.\lambda y.x \ a) \ b)$$

$$\Rightarrow (\lambda y.a \ b)$$

$$\Rightarrow a$$
Church Numerals

|0|: \( \lambda f. \lambda x. x \)  (Zero)

|1|: \( \lambda f. \lambda x. (f \ x) \)  (One)

\ldots

|n|: \( \lambda f. \lambda x. (f \ldots (f \ x) \ldots) \)  (N applications of f to x)

s: \( \lambda n. \lambda f. \lambda x. (f ((n \ f) \ x)) \)  (Successor)

Recall semantics rule:

\((\lambda x. E \ M) \Rightarrow E\{M/x\}\)

\((s \ 0)\)

\((\lambda n. \lambda f. \lambda x. (f ((n \ f) \ x)) \ \lambda f. \lambda x. x)\)

\(\Rightarrow \lambda f. \lambda x. (f ((\lambda f. \lambda x. x \ f) \ x))\)

\(\Rightarrow \lambda f. \lambda x. (f (\lambda x. x \ x))\)

\(\Rightarrow \lambda f. \lambda x. (f \ x)\)
Church Numerals: isZero?

isZero?: \( \lambda n.((n \lambda x.\text{false}) \text{true}) \)  (Is \( n = 0 \)?)

\[
\begin{align*}
\text{(isZero? 0)}
\Rightarrow & (\lambda f.\lambda x.x) \\
\Rightarrow & (\lambda f.\lambda x.\text{false}) \text{true}) \\
\Rightarrow & (\lambda x.\text{true}) \\
\Rightarrow & \text{true}
\end{align*}
\]

\[
\begin{align*}
\text{(isZero? 1)}
\Rightarrow & (\lambda f.\lambda x.(f \ x)) \\
\Rightarrow & (\lambda x.\lambda x.\text{false} \ x) \text{true}) \\
\Rightarrow & (\lambda x.\text{false} \ x) \text{true}) \\
\Rightarrow & (\lambda x.\text{false} \ x) \text{true}) \\
\Rightarrow & \text{false}
\end{align*}
\]
Functions

• Compute the factorial function: $n! = 1 \times 2 \times \cdots \times (n-1) \times n$
• Start with the mathematical definition

\[
\text{declare}
\text{fun \{Fact N\}}
\text{if N==0 then 1 else N*\{Fact N-1\} end}
\text{end}
\]
• Fact is declared in the environment
• Try large factorial \{Browse \{Fact 100\}\}

0! = 1

$n! = n \times (n-1)!$ if $n > 0$
Factorial in Haskell

factorial :: Integer -> Integer
factorial  0       = 1
factorial  n | n > 0   = n * factorial (n-1)
Structured data (lists)

- Calculate Pascal triangle
- Write a function that calculates the nth row as one structured value
- A list is a sequence of elements: 
  \[1 4 6 4 1\]
- The empty list is written nil
- Lists are created by means of "|" (cons)

```declare
H=1
T = [2 3 4 5]
{Browse H|T}  % This will show [1 2 3 4 5]
```
Pattern matching

• Another way to take a list apart is by use of pattern matching with a case instruction

\[
\text{case } L \text{ of } H|T \text{ then } \{\text{Browse } H\} \{\text{Browse } T\}
\]
\[
\text{else } \{\text{Browse ‘empty list’ }\}
\]
\[
\text{end}
\]
Functions over lists

- Compute the function \{Pascal N\}
- Takes an integer N, and returns the Nth row of a Pascal triangle as a list

1. For row 1, the result is \([1]\)
2. For row N, shift to left row N-1 and shift to the right row N-1
3. Align and add the shifted rows element-wise to get row N

\[
\begin{array}{cccc}
1 & 1 & & \\
1 & 2 & 1 & \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 3 & 3 & 1 \\
\end{array}
\]

Shift right \([0 \ 1 \ 3 \ 3 \ 1]\)

Shift left \([1 \ 3 \ 3 \ 1 \ 0]\)
Functions over lists

```plaintext
dec {Pascal N} =
  if N==1 then [1]
  else {AddList {ShiftLeft {Pascal N-1}}{ShiftRight {Pascal N-1}}}
ed
```
Functions over lists (2)

fun {ShiftLeft L}
  case L of H|T then
    H|{ShiftLeft T}
  else [0]
  end
end

fun {ShiftRight L}  0|L end

fun {AddList L1 L2}
  case L1 of H1|T1 then
    case L2 of H2|T2 then
      H1+H2|{AddList T1 T2}
    end
  else nil end
end
Pattern matching in Haskell

• Another way to take a list apart is by use of pattern matching with a case instruction:

```haskell
case l of (h:t) -> h:t
     []   -> []
end
```

• Or more typically as part of a function definition:

```haskell
id (h:t) -> h:t
id []   -> []
```
Functions over lists in Haskell

--- Pascal triangle row

\[
pascal :: \text{Integer} \rightarrow \text{[Integer]}
pascal 1 = [1]
pascal n = \text{addList} (\text{shiftLeft} (pascal (n-1)))
\]

\[
\quad (\text{shiftRight} (pascal (n-1)))
\]

where

\[
\text{shiftLeft} [] = [0]
\]
\[
\text{shiftLeft} (h:t) = h:\text{shiftLeft} \ t
\]

\[
\text{shiftRight} \_ = 0:\_ \\
\text{addList} \_ \_ = \_
\]
\[
\text{addList} (h1:t1) (h2:t2) = (h1+h2):\text{addList} \ t1 \ t2
\]
Mathematical induction

• Select one or more inputs to the function
• Show the program is correct for the *simple cases* (base cases)
• Show that if the program is correct for a *given case*, it is then correct for the *next case*.
• For natural numbers, the base case is either 0 or 1, and for any number $n$ the next case is $n+1$
• For lists, the base case is nil, or a list with one or a few elements, and for any list $T$ the next case is $H|T$
Correctness of factorial

fun {Fact N}
    if N==0 then 1 else N*{Fact N-1} end
end

• Base Case N=0: {Fact 0} returns 1
• Inductive Case N>0: {Fact N} returns N*{Fact N-1} assume
{Fact N-1} is correct, from the spec we see that {Fact N} is
N*{Fact N-1}
Iterative computation

• An iterative computation is one whose execution stack is bounded by a constant, independent of the length of the computation.

• Iterative computation starts with an initial state $S_0$, and transforms the state in a number of steps until a final state $S_{\text{final}}$ is reached:

$$S_0 \rightarrow S_1 \rightarrow \ldots \rightarrow S_{\text{final}}$$
The general scheme

fun \{Iterate \: S_i\}
  
  if \{IsDone \: S_i\} then \: S_i
  
  else \: S_{i+1} in
    
    S_{i+1} = \{Transform \: S_i\}
    
    \{Iterate \: S_{i+1}\}
    
  end

end

• \textit{IsDone} and \textit{Transform} are problem dependent
From a general scheme to a control abstraction (2)

fun {Iterate S IsDone Transform}
  if {IsDone S} then S
  else S1 in
    S1 = {Transform S}
    {Iterate S1 IsDone Transform}
  end
end

fun {Iterate S_i}
  if {IsDone S_i} then S_i
  else S_{i+1} in
    S_{i+1} = {Transform S_i}
    {Iterate S_{i+1}}
  end
end
Sqrt using the control abstraction

fun {Sqrt X}
    {Iterate
        1.0
        fun {$ G} {Abs X - G*G}/X < 0.000001 end
        fun {$ G} (G + X/G)/2.0 end
    }
end

Iterate could become a linguistic abstraction
Sqrt in Haskell

```haskell
let sqrt x = head (dropWhile (not . goodEnough) sqrtGuesses)
  where
    goodEnough guess = (abs (x – guess*guess))/x < 0.00001
    improve guess = (guess + x/guess)/2.0
    sqrtGuesses = 1:(map improve sqrtGuesses)
```

This sqrt example uses infinite lists enabled by lazy evaluation, and the map control abstraction.
Higher-order programming

- **Higher-order programming** = the set of programming techniques that are possible with procedure values (lexically-scoped closures)

- **Basic operations**
  - **Procedural abstraction**: creating procedure values with lexical scoping
  - **Genericity**: procedure values as arguments
  - **Instantiation**: procedure values as return values
  - **Embedding**: procedure values in data structures

- **Higher-order programming** is the foundation of component-based programming and object-oriented programming
Procedural abstraction

• Procedural abstraction is the ability to convert any statement into a procedure value
  – A procedure value is usually called a closure, or more precisely, a lexically-scoped closure
  – A procedure value is a pair: it combines the procedure code with the environment where the procedure was created (the contextual environment)

• Basic scheme:
  – Consider any statement <s>
  – Convert it into a procedure value: \( P = \text{proc } \{\} <s> \text{ end} \)
  – Executing \( \{P\} \) has exactly the same effect as executing \( <s> \)
Procedure values

- Constructing a procedure value in the store is not simple because a procedure may have external references

```plaintext
local P Q in
    P = proc { $ ... } { Q ... } end
    Q = proc { $ ... } { Browse hello } end
local Q in
    Q = proc { $ ... } { Browse hi } end
    { P ... }
end
end
```
Procedure values (2)

local P Q in
  P = proc { ... } { Q ... } end
  Q = proc { ... } { Browse hello } end
local Q in
  Q = proc { ... } { Browse hi } end end
end

P → x_1
proc { ... } { Q ... } end
Q → x_2
Browse → x_0
proc { ... } { Browse hello } end

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Genericity

- Replace specific entities (zero 0 and addition +) by function arguments
- The same routine can do the sum, the product, the logical or, etc.

```plaintext
fun {SumList L}
    case L
    of   nil then 0
         []   X|L2 then X+{SumList L2}
    end
end

fun {FoldR L F U}
    case L
    of   nil then U
         []   X|L2 then {F X {FoldR L2 F U}}
    end
end
```
Genericity in Haskell

- Replace specific entities (zero 0 and addition +) by function arguments
- The same routine can do the sum, the product, the logical or, etc.

\[
\text{sumlist} :: (\text{Num } a) \Rightarrow [a] \rightarrow a \\
\text{sumlist} [] = 0 \\
\text{sumlist} (h:t) = h + \text{sumlist } t
\]

\[
\text{foldr'} :: (a \rightarrow b \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b \\
\text{foldr'} \_ u [] = u \\
\text{foldr'} f u (h:t) = f \ h \ (\text{foldr'} f u \ t)
\]
Instantiation

- Instantiation is when a procedure returns a procedure value as its result.
- Calling `{FoldFactory fun {A B} A+B end 0}` returns a function that behaves identically to `SumList`, which is an « instance » of a folding function.

C. Varela; Adapted w/permission from S. Haridi and P. Van Roy
Embedding

• Embedding is when procedure values are put in data structures

• Embedding has many uses:
  – Modules: a module is a record that groups together a set of related operations
  – Software components: a software component is a generic function that takes a set of modules as its arguments and returns a new module. It can be seen as specifying a module in terms of the modules it needs.
  – Delayed evaluation (also called explicit lazy evaluation): build just a small part of a data structure, with functions at the extremities that can be called to build more. The consumer can control explicitly how much of the data structure is built.
Control Abstractions

fun {FoldL Xs F U}
  case Xs
  of nil then U
  [] X|Xr then {FoldL Xr F {F X U}}
  end
end
end

What does this program do?
{Browse {FoldL [1 2 3]
  fun ${X Y} X|Y end nil}}
FoldL in Haskell

foldl' :: (b->a->b) -> b -> [a] -> b
foldl' _ u [] = u
foldl' f u (h:t) = foldl' f (f u h) t

Notice the unit u is of type b, and the function f is of type b->a->b.
List-based techniques

fun {Map Xs F}
    case Xs
    of nil then nil
    [] X|Xr then
        {F X}|{Map Xr F}
    end
end

fun {Filter Xs P}
    case Xs
    of nil then nil
    [] X|Xr andthen {P X} then
        X|{Filter Xr P}
    [] X|Xr then {Filter Xr P}
    end
end
Map in Haskell

map' :: (a -> b) -> [a] -> [b]
map' _ [] = []
map' f (h:t) = f h:map' f t

_ means that the argument is not used (read “don’t care”).
map’ is to distinguish it from the Prelude map function.
Filter in Haskell

\[
\begin{align*}
\text{filter}' &:: (a \rightarrow \text{Bool}) \rightarrow [a] \rightarrow [a] \\
\text{filter}' \_ \ [\] & = [] \\
\text{filter}' \ p \ (h:t) & = \text{if } p \ h \ \text{then } h: \text{filter}' \ p \ t \\
& \quad \text{else } \text{filter}' \ p \ t
\end{align*}
\]
Filter as FoldR application

fun {Filter L P}
  {FoldR L fun {$ H T}
   if {P H} then
     H|T
   else T end
   end nil}
end

filter'' :: (a-> Bool) -> [a] -> [a]
filter'' p l = foldr
  (\ h t -> if p h
     then h:t
     else t) [] l
Lazy evaluation

• The functions written so far are evaluated eagerly (as soon as they are called)
• Another way is lazy evaluation where a computation is done only when the results is needed

• Calculates the infinite list:
  0 | 1 | 2 | 3 | ...

```
declare
fun lazy {Ints N}
  N|{Ints N+1}
end
```
Lazy evaluation (2)

• Write a function that computes as many rows of Pascal’s triangle as needed
• We do not know how many beforehand
• A function is lazy if it is evaluated only when its result is needed
• The function PascalList is evaluated when needed

```haskell
fun lazy {PascalList Row}
  Row | {PascalList}
       {AddList}
       {ShiftLeft Row}
       {ShiftRight Row}}}
end
```
Larger Example: The Sieve of Eratosthenes

- Produces prime numbers
- It takes a stream 2...N, peals off 2 from the rest of the stream
- Delivers the rest to the next sieve
Lazy Sieve

\[
\text{fun} \ \text{lazy} \ \{\text{Sieve} \ Xs\} \\
\quad X | Xr = Xs \ \text{in} \\
\quad X \mid \{\text{Sieve} \ \{\text{LFilter} \\
\quad \quad Xr \\
\quad \quad \quad \text{fun} \ \{\$ Y\} \ Y \mod X \neq 0 \ \text{end} \\
\quad \quad \}\} \\
\end{array}
\]

\[
\text{fun} \ \{\text{Primes}\} \ \{\text{Sieve} \ \{\text{Ints 2}\}\} \ \text{end}
\]
Lazy Filter

For the Sieve program we need a lazy filter

```haskell
fun lazy {LFilter Xs F}
    case Xs
    of nil then nil
    [] X|Xr then
        if {F X} then X|{LFilter Xr F} else {LFilter Xr F} end
    end
end
```
Primes in Haskell

```haskell
ints :: (Num a) => a -> [a]
ints n = n : ints (n+1)

sieve :: (Integral a) => [a] -> [a]
sieve (x:xr) = x:sieve (filter (\y -> (y `mod` x /= 0)) xr)

primes :: (Integral a) => [a]
brimes = sieve (ints 2)
```

Functions in Haskell are lazy by default. You can use `take 20 primes` to get the first 20 elements of the list.
List Comprehensions

- Abstraction provided in lazy functional languages that allows writing higher level set-like expressions
- In our context we produce lazy lists instead of sets
- The mathematical set expression
  - \( \{x*y \mid 1 \leq x \leq 10, 1 \leq y \leq x\} \)
- Equivalent List comprehension expression is
  - \([X*Y \mid X = 1..10 ; Y = 1..X]\)
- Example:
  - \([1*1 \; 2*1 \; 2*2 \; 3*1 \; 3*2 \; 3*3 \; ... \; 10*10]\)
List Comprehensions

• The general form is
• \[ f(x, y, \ldots, z) \mid x \leftarrow \text{gen}(a_1, \ldots, a_n) ; \text{guard}(x, \ldots) \]
  \[ y \leftarrow \text{gen}(x, a_1, \ldots, a_n) ; \text{guard}(y, x, \ldots) \]
  \[ \ldots \]
  \[ \]
• No linguistic support in Mozart/Oz, but can be easily expressed
Example 1

- \( z = [x \# x \mid x \leftarrow \text{from}(1,10)] \)
- \( Z = \{ \text{LMap} \ \{ \text{LFrom} 1 \ 10 \} \ \text{fun} \{ X \} \ X \# X \ \text{end} \} \)

- \( z = [x \# y \mid x \leftarrow \text{from}(1,10), \ y \leftarrow \text{from}(1,x)] \)
- \( Z = \{ \text{LFlatten} \)
  
  \( \{ \text{LMap} \ \{ \text{LFrom} 1 \ 10 \} \)
  
  \( \text{fun} \{ X \} \ \{ \text{LMap} \ \{ \text{LFrom} 1 \ X \} \)
  
  \( \text{fun} \ \{ Y \} \ \ X \# Y \ \text{end} \)
  
  \( \}\)
  
  \( \}\)
  
  \( \}\)
Example 2

• \( z = [x\#y \mid x \leftarrow \text{from}(1,10), y \leftarrow \text{from}(1,x), x+y\leq 10] \)

• \( Z = \{ \text{LFilter} \}
  \{ \text{LFflatten} \}
  \{ \text{LMap \{LFrom 1 10\}} \}
  \text{fun \{$ X\} \{ \text{LMap \{LFrom 1 \ X\}} \}}
  \text{fun \{$ Y\} \ X\#Y \text{ end} \}}
  \text{end}
\}
\}
\}
\text{fun \{$ X\#Y\} \ X+Y=\lt 10 \text{ end} \} \}

C. Varela; Adapted from S. Haridi and P. Van Roy
List Comprehensions in Haskell

\[
\text{lc1} = [(x,y) \mid x \leftarrow [1..10], y \leftarrow [1..x]]
\]

\[
\text{lc2} = \text{filter} \ (\lambda (x,y)\rightarrow(x+y\leq 10)) \ \text{lc1}
\]

\[
\text{lc3} = [(x,y) \mid x \leftarrow [1..10], y \leftarrow [1..x], x+y\leq 10]
\]

Haskell provides syntactic support for list comprehensions. List comprehensions are implemented using a built-in list monad.
Quicksort using list comprehensions

quicksort :: (Ord a) => [a] -> [a]
quicksort [] = []
quicksort (h:t) = quicksort [x | x <- t, x < h] ++
    [h] ++
quicksort [x | x <-t, x >= h]
Types of typing

- Languages can be *weakly typed*
  - Internal representation of types can be manipulated by a program
    - e.g., a string in C is an array of characters ending in ‘\0’.

- *Strongly typed* programming languages can be further subdivided into:
  - *[Dynamically typed]* languages
    - Variables can be bound to entities of any type, so in general the type is only known at *run-time*, e.g., Oz, SALSA.
  - *[Statically typed]* languages
    - Variable types are known at *compile-time*, e.g., C++, Java.
Type Checking and Inference

• Type checking is the process of ensuring a program is well-typed.
  – One strategy often used is *abstract interpretation*:
    • The principle of getting partial information about the answers from partial information about the inputs
    • Programmer supplies types of variables and type-checker deduces types of other expressions for consistency

• Type inference frees programmers from annotating variable types: types are inferred from variable usage, e.g. ML, Haskell.
Abstract data types

• A datatype is a set of values and an associated set of operations
• A datatype is abstract only if it is completely described by its set of operations regardless of its implementation
• This means that it is possible to change the implementation of the datatype without changing its use
• The datatype is thus described by a set of procedures
• These operations are the only thing that a user of the abstraction can assume
Example: A Stack

- Assume we want to define a new datatype \(\langle\text{stack } T\rangle\) whose elements are of any type \(T\)
  
  \[
  \begin{align*}
  \text{fun } \{\text{NewStack}\} &: \langle\text{Stack } T\rangle \\
  \text{fun } \{\text{Push } \langle\text{Stack } T\rangle \langle T\rangle \} &: \langle\text{Stack } T\rangle \\
  \text{fun } \{\text{Pop } \langle\text{Stack } T\rangle \langle T\rangle \} &: \langle\text{Stack } T\rangle \\
  \text{fun } \{\text{IsEmpty } \langle\text{Stack } T\rangle \} &: \langle\text{Bool}\rangle
  \end{align*}
  \]

- These operations normally satisfy certain laws:
  
  \[
  \{\text{IsEmpty } \{\text{NewStack}\}\} = \text{true}
  \]
  
  for any \(E\) and \(S0, S1=\{\text{Push } S0 E\}\) and \(S0 = \{\text{Pop } S1 E\}\) hold
  
  \[
  \{\text{Pop } \{\text{NewStack}\} E\} \text{ raises error}
  \]
Stack (another implementation)

fun {NewStack} nil end
fun {Push S E} E|S end
fun {Pop S E} case S of X|S1 then E = X S1 end end
fun {IsEmpty S} S==nil end

fun {NewStack} emptyStack end
fun {Push S E} stack(E S) end
fun {Pop S E} case S of stack(X S1) then E = X S1 end end
fun {IsEmpty S} S==emptyStack end
data Stack a = Empty | Stack a (Stack a)

newStack :: Stack a
newStack = Empty

push :: Stack a -> a -> Stack a
push s e = Stack e s

pop :: Stack a -> (Stack a, a)
pop (Stack e s) = (s, e)

isempty :: Stack a -> Bool
isempty Empty = True
isempty (Stack _ _) = False
Secure abstract data types: A secure stack

With the wrapper & unwrapper we can build a secure stack

local Wrap Unwrap in
  {NewWrapper Wrap Unwrap}
  fun {NewStack} {Wrap nil} end
  fun {Push S E} {Wrap E|{Unwrap S}} end
  fun {Pop S E}
    case {Unwrap S} of X|S1 then
      E=X {Wrap S1} end
    end
  fun {IsEmpty S} {Unwrap S}==nil end
end

proc {NewWrapper ?Wrap ?Unwrap}
  Key={NewName}
in
  fun {Wrap X}
    fun ${ K}
      if K==Key then X end
    end
  fun {Unwrap C}
    {C Key}
  end
end
Stack abstract data type as a module in Haskell

module StackADT (Stack,newStack,push,pop,isEmpty) where

data Stack a = Empty | Stack a (Stack a)
newStack = Empty

...  

• Modules can then be imported by other modules, e.g.:

module Main (main) where
import StackADT ( Stack, newStack,push,pop,isEmpty )

main = do print (push (push newStack 1) 2)
Declarative operations (1)

• An operation is *declarative* if whenever it is called with the same arguments, it returns the same results independent of any other computation state.

• A declarative operation is:
  – *Independent* (depends only on its arguments, nothing else)
  – *Stateless* (no internal state is remembered between calls)
  – *Deterministic* (call with same operations always give same results)

• Declarative operations can be composed together to yield other declarative components
  – All basic operations of the declarative model are declarative and combining them always gives declarative components
Why declarative components (1)

- There are two reasons why they are important:
- *(Programming in the large)* A declarative component can be written, tested, and proved correct independent of other components and of its own past history.
  - The complexity (reasoning complexity) of a program composed of declarative components is the sum of the complexity of the components
  - In general the reasoning complexity of programs that are composed of nondeclarative components explodes because of the intimate interaction between components
- *(Programming in the small)* Programs written in the declarative model are much easier to reason about than programs written in more expressive models (e.g., an object-oriented model).
  - Simple algebraic and logical reasoning techniques can be used
Monads

• Purely functional programming is declarative in nature: whenever a function is called with the same arguments, it returns the same results independent of any other computation state.

• How to model the real world (that may have context dependences, state, nondeterminism) in a purely functional programming language?
  – Context dependences: e.g., does file exist in expected directory?
  – State: e.g., is there money in the bank account?
  – Nondeterminism: e.g., does bank account deposit happen before or after interest accrual?

• Monads to the rescue!
The Monad class defines two basic operations:

```haskell
class Monad m where

    (>>=) :: m a -> (a -> m b) -> m b -- bind
    return :: a -> m a
    fail :: String -> m a

    m >>= k = m >>= \_ -> k
```

- The `>>=` infix operation binds two monadic values, while the `return` operation injects a value into the monad (container).
- Example monadic classes are `IO`, `lists` (`[]`) and `Maybe`. 
do syntactic sugar

- In the IO class, \( x >>= y \), performs two actions sequentially (like the Seq combinator in the lambda-calculus) passing the result of the first into the second.
- Chains of monadic operations can use do:
  
  \[
  \text{do } e_1 ; e_2 = e_1 >> e_2 \\
  \text{do } p \leftarrow e_1; e_2 = e_1 >>= \lambda p \rightarrow e_2
  \]

- Pattern match can fail, so the full translation is:
  
  \[
  \text{do } p \leftarrow e_1; e_2 = e_1 >>= (\lambda v \rightarrow \text{case of } p \rightarrow e_2 \_ \rightarrow \text{fail “s”})
  \]

- Failure in IO monad produces an error, whereas failure in the List monad produces the empty list.
Monad class laws

• All instances of the Monad class should respect the following laws:

\[
\begin{align*}
\text{return } a \gg= k & = k \ a \\
m \gg= \text{return} & = m \\
x s \gg= \text{return} \ . \ f & = \text{fmap} \ f \ xs \\
m \gg= (\lambda x \rightarrow k \ x \gg= h) & = (m \gg= k) \gg= h
\end{align*}
\]

• These laws ensure that we can bind together monadic values with \( \gg= \) and inject values into the monad (container) using \text{return} in consistent ways.

• The MonadPlus class includes an \text{mzero} element and an \text{mplus} operation. For lists, \text{mzero} is the empty list (\([\ ]\)), and the \text{mplus} operation is list concatenation (\([+])\).
List comprehensions with monads

\[ \text{lc1} = [(x,y) \mid x \leftarrow [1..10], y \leftarrow [1..x]] \]

\[ \text{lc1'} = \text{do } x \leftarrow [1..10] \]
\[ \quad y \leftarrow [1..x] \]
\[ \quad \text{return } (x,y) \]

\[ \text{lc1''} = [1..10] >>= (\ x \rightarrow 
\quad [1..x] >>= (\ y \rightarrow 
\quad \text{return } (x,y))) \]

List comprehensions are implemented using a built-in list monad. Binding \((l >>= f)\) applies the function \(f\) to all the elements of the list \(l\) and concatenates the results. The return function creates a singleton list.
List comprehensions with monads (2)

\[ \text{lc3} = [(x,y) \mid x \leftarrow [1..10], y \leftarrow [1..x], x+y \leq 10] \]

\[ \text{lc3'} = \text{do } x \leftarrow [1..10] \]
\[ \quad y \leftarrow [1..x] \]
\[ \quad \text{True} \leftarrow \text{return } (x+y \leq 10) \]
\[ \quad \text{return } (x,y) \]

\[ \text{lc3''} = [1..10] \ggg (\lambda x \rightarrow [1..x] \ggg (\lambda y \rightarrow \text{return } (x+y \leq 10) \ggg (\lambda b \rightarrow \text{case } b \text{ of } \text{True} \rightarrow \text{return } (x,y); _ \rightarrow \text{fail } ""))))) \]

Guards in list comprehensions assume that \texttt{fail} in the List monad returns an empty list.
Monads summary

• Monads enable keeping track of imperative features (state) in a way that is modular with purely functional components.
  – For example, fib remains functional, yet the R monad enables us to keep a count of instructions separately.

• Input/output, list comprehensions, and optional values (Maybe class) are built-in monads in Haskell.

• Monads are useful to modularly define semantics of domain-specific languages.