Review from Lecture 17 & 18

- Overview of the ds_set implementation
- `begin`, `find`, `destroy_tree`, `insert`
- In-order, pre-order, and post-order traversal; Breadth-first and depth-first tree search

```
template <class T>
void breadth_first_print (TreeNode<T> *p) {
    if (p != NULL) {
        std::list<TreeNode<T>*> current_level;
        current_level.push_back(p);
        while (current_level.size() != 0) {
            std::list<TreeNode<T>*> next_level;
            for (std::list<TreeNode<T>*>::iterator itr = current_level.begin();
                 itr != current_level.end(); itr++) {
                std::cout << (*itr)->value;
                if ((*itr)->left != NULL) { next_level.push_back((*itr)->left); }
                if ((*itr)->right != NULL) { next_level.push_back((*itr)->right); }
            }
            current_level = next_level;
        }
    }
}
```

- Iterator implementation. Finding the in order successor to a node: add parent pointers or add a list/vector/stack of pointers to the iterator.

Today’s Lecture

- Last piece of ds_set: removing an item, `erase`
- Tree height, longest-shortest paths, breadth-first search
- To support increment/decrement: Copy tree, Insert, and Erase with parent pointers
- Limitations of our ds_set implementation, brief intro to red-black trees

19.1 ds_set Warmup/Review Exercises

- Draw a diagram of a possible memory layout for a ds_set containing the numbers 16, 2, 8, 11, and 5. Is there only one valid memory layout for this data as a ds_set? Why?

- In what order should a forward iterator visit the data? Draw an abstract table representation of this data. (This is the “user of STL map” diagram of the data, which omits details of TreeNode memory layout).
19.2 Erase

First we need to find the node to remove. Once it is found, the actual removal is easy if the node has no children or only one child. *Draw picture of each case!*

| no children      | only a left child (with potentially a big subtree) | only a right child (with potentially a big subtree) |

It is harder if there are two children:

- Find the node with the greatest value in the left subtree or the node with the smallest value in the right subtree.
- The value in this node may be safely moved into the current node because of the tree ordering.
- Then we recursively apply erase to remove that node — which is guaranteed to have at most one child.

**Exercise:** Write a recursive version of erase.

**Exercise:** How does the order that nodes are deleted affect the tree structure? Starting with a mostly balanced tree, give an erase ordering that yields an unbalanced tree.
19.3 Height and Height Calculation Algorithm

- The height of a node in a tree is the length of the longest path down the tree from that node to a leaf node. The height of a leaf is 1. We will think of the height of a null pointer as 0.

- The height of the tree is the height of the root node, and therefore if the tree is empty the height will be 0.

**Exercise:** Write a simple recursive algorithm to calculate the height of a tree.

- What is the best/average/worst-case running time of this algorithm? What is the best/average/worst-case memory usage of this algorithm? Give a specific example tree that illustrates each case.

19.4 Shortest Paths to Leaf Node

- Now let’s write a function to instead calculate the shortest path to a NULL child pointer.

- What is the running time of this algorithm? Can we do better? **Hint:** How does a breadth-first vs. depth-first algorithm for this problem compare?
19.5 Tree Iterator Increment/Decrement - Implementation Choices

- The increment operator should change the iterator’s pointer to point to the next TreeNode in an in-order traversal — the “in-order successor” — while the decrement operator should change the iterator’s pointer to point to the “in-order predecessor”.

- Unlike the situation with lists and vectors, these predecessors and successors are not necessarily “nearby” (either in physical memory or by following a link) in the tree, as examples we draw in class will illustrate.

- There are two common solution approaches:
  - Each node stores a parent pointer. Only the root node has a null parent pointer. [method 1]
  - Each iterator maintains a stack of pointers representing the path down the tree to the current node. [method 2]

- If we choose the parent pointer method, we’ll need to rewrite the `insert` and `erase` member functions to correctly adjust parent pointers.

- Although iterator increment looks expensive in the worst case for a single application of `operator++`, it is fairly easy to show that iterating through a tree storing n nodes requires $O(n)$ operations overall.

Exercise: [method 1] Write a fragment of code that given a node, finds the in-order successor using parent pointers. Be sure to draw a picture to help you understand!

Exercise: [method 2] Write a fragment of code that given a tree iterator containing a pointer to the node and a stack of pointers representing path from root to node, finds the in-order successor (without using parent pointers).

Either version can be extended to complete the implementation of increment/decrement for the `ds_set` tree iterators.

Exercise: What are the advantages & disadvantages of each method?

19.6 Erase (now with parent pointers)

- If we choose to use parent pointers, we need to add to the Node representation, and re-implement several `ds_set` member functions.

- Study the new version of `copy_tree`, with parent pointers.

- Study the new version of `insert`, with parent pointers.

- Exercise: Rewrite `erase`, now with parent pointers.

19.7 Limitations of Our BST Implementation

- The efficiency of the main insert, find and erase algorithms depends on the height of the tree.

- The best-case and average-case heights of a binary search tree storing n nodes are both $O(\log n)$. The worst-case, which often can happen in practice, is $O(n)$.

- Developing more sophisticated algorithms to avoid the worst-case behavior will be covered in Introduction to Algorithms. One elegant extension to binary search tree is described below...
19.8 Red-Black Trees

In addition to the binary search tree properties, the following red-black tree properties are maintained throughout all modifications to the data structure:

1. Each node is either red or black.
2. The NULL child pointers are black.
3. Both children of every red node are black. Thus, the parent of a red node must also be black.
4. All paths from a particular node to a NULL child pointer contain the same number of black nodes.

What tree does our ds.set implementation produce if we insert the numbers 1-14 in order? The tree at the right is the result using a red-black tree. Notice how the tree is still quite balanced. Visit these links for an animation of the sequential insertion and re-balancing:

https://www.cs.usfca.edu/~galles/visualization/RedBlack.html
http://www.youtube.com/watch?v=vDHFF4wjWYU&noredirect=1

• What is the best/average/worst case height of a red-black tree with \( n \) nodes?

• What is the best/average/worst case shortest-path from root to leaf node in a red-black tree with \( n \) nodes?

19.9 Exercise [ /6

Fill in the tree on the right with the integers 1-7 to make a binary search tree. Also, color each node “red” or “black” so that the tree also fulfills the requirements of a Red-Black tree.

Draw two other red-black binary search trees with the values 1-7.

Note: Red-Black Trees are just one algorithm for self-balancing binary search tree. Others include: AVL trees, Splay Trees, (& more!).
template <class T> class TreeNode {
public:
  TreeNode() : left(NULL), right(NULL), parent(NULL) {}  
  TreeNode(const T& init) : value(init), left(NULL), right(NULL), parent(NULL) {}  
  T value;  
  TreeNode* left;  
  TreeNode* right;  
  TreeNode* parent; // to allow implementation of iterator increment & decrement
};

template <class T> class ds_set {
public:
  int size() const { return size_; }  
  bool operator==(const ds_set<T>& old) const { return (old.root_ == this->root_); }  
  // FIND, INSERT & ERASE
  iterator find(const T& key_value) { return find(key_value, root_); }  
  std::pair<iterator,bool> insert(T const& key_value) { return insert(key_value, root_, NULL); }  
  int erase(T const& key_value) { return erase(key_value, root_); }  
  // ITERATORS
  iterator begin() const { if (!root_) return iterator(NULL, this);  
    TreeNode<T>* p = root_;  
    while (p->left) p = p->left;  
    return iterator(p, this); }  
  iterator end() const { return iterator(NULL, this); }  
private:  
  // REPRESENTATION
  TreeNode<T>* root_;  
  int size_;  
  // PRIVATE HELPER FUNCTIONS
  int ds_set(T const& old) : size_(old.size_) { root_=this->copy_tree(old.root_, NULL); }  
  // to allow implementation of iterator increment & decrement
};

typedef tree_iterator<T> iterator;
friend class tree_iterator<T>;

TreeNode<T>* copy_tree(TreeNode<T>* old_root, TreeNode<T>* the_parent) {
  if (old_root == NULL) return NULL;  
  TreeNode<T>* p = new TreeNode<T>(old_root->value);      p->parent = the_parent;      this->size_++;  
  insert(key_value, p->left, p);  
  return std::pair<iterator,bool>(iterator(p, this), true);  
  }  
  if (key_value < p->value) return insert(key_value, p->right, p);  
  return std::pair<iterator,bool>(iterator(p, this), false);  
}

int size() const { return size_; }  
bool operator==(const ds_set<T>& old) const { return (old.root_ == this->root_); }  
// FIND, INSERT & ERASE
iterator find(const T& key_value) { return find(key_value, root_); }  
std::pair<iterator,bool> insert(T const& key_value) { return insert(key_value, root_, NULL); }  
int erase(T const& key_value) { return erase(key_value, root_); }  
// ITERATORS
iterator begin() const { if (!root_) return iterator(NULL, this);  
  TreeNode<T>* p = root_;  
  while (p->left) p = p->left;  
  return iterator(p, this); }  
iterator end() const { return iterator(NULL, this); }  
private:  
  // REPRESENTATION
  TreeNode<T>* root_;  
  int size_;  
  // PRIVATE HELPER FUNCTIONS
  int ds_set(T const& old) : size_(old.size_) { root_=this->copy_tree(old.root_, NULL); }  
  // to allow implementation of iterator increment & decrement
};

typedef tree_iterator<T> iterator;
friend class tree_iterator<T>;

TreeNode<T>* copy_tree(TreeNode<T>* old_root, TreeNode<T>* the_parent) {
  if (old_root == NULL) return NULL;  
  TreeNode<T>* p = new TreeNode<T>(old_root->value);      p->parent = the_parent;      this->size_++;  
  insert(key_value, p->left, p);  
  return std::pair<iterator,bool>(iterator(p, this), true);  
  }  
  if (key_value < p->value) return insert(key_value, p->right, p);  
  return std::pair<iterator,bool>(iterator(p, this), false);  
}

int size() const { return size_; }  
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  int ds_set(T const& old) : size_(old.size_) { root_=this->copy_tree(old.root_, NULL); }  
  // to allow implementation of iterator increment & decrement
}  