Lambda Calculus (PDCS 2)
alpha-renaming, beta reduction, eta conversion,
applicative and normal evaluation orders, Church-Rosser theorem, combinators, booleans

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Mathematical Functions

Take the mathematical function:

\[ f(x) = x^2 \]

Assume \( f \) is a function that maps integers to integers:

\[ f: \mathbb{Z} \rightarrow \mathbb{Z} \]

We apply the function \( f \) to numbers in its domain to obtain a number in its range, e.g.:

\[ f(-2) = 4 \]
Function Composition

Given the mathematical functions:

\[ f(x) = x^2, \ g(x) = x+1 \]

\( f \cdot g \) is the composition of \( f \) and \( g \):

\[ f \cdot g (x) = f(g(x)) \]

\[ f \cdot g (x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1 \]
\[ g \cdot f (x) = g(f(x)) = g(x^2) = x^2 + 1 \]

Function composition is therefore not commutative. Function composition is a \((\text{higher-order})\) function, in this example, with the following type:

\[ \cdot : (Z \to Z) \times (Z \to Z) \to (Z \to Z) \]
Lambda Calculus (Church and Kleene 1930’s)

A unified language to manipulate and reason about functions.

Given
\[ f(x) = x^2 \]

\[ \lambda x. x^2 \]
represents the same \( f \) function, except it is *anonymous*.

To represent the function evaluation \( f(2) = 4 \),
we use the following \( \lambda \)-calculus syntax:

\[ (\lambda x. x^2) \ 2 \Rightarrow 2^2 \Rightarrow 4 \]
Lambda Calculus Syntax and Semantics

The syntax of a $\lambda$-calculus expression is as follows:

$$
e ::= v \quad \text{variable} \\
| \quad \lambda v.e \quad \text{functional abstraction} \\
| \quad (e e) \quad \text{function application}
$$

The semantics of a $\lambda$-calculus expression is called beta-reduction:

$$(\lambda x.E \ M) \rightarrow E\{M/x\}$$

where we alpha-rename the lambda abstraction $E$ if necessary to avoid capturing free variables in $M$. 
Currying

The lambda calculus can only represent functions of one variable. It turns out that one-variable functions are sufficient to represent multiple-variable functions, using a strategy called currying.

E.g., given the mathematical function: \( h(x,y) = x+y \)
of type \( h: Z \times Z \rightarrow Z \)

We can represent \( h \) as \( h' \) of type: \( h': Z \rightarrow Z \rightarrow Z \)
Such that
\[
\begin{align*}
    h(x,y) &= h'(x)(y) = x+y \\
\end{align*}
\]
For example,
\[
    h'(2) = g, \text{ where } g(y) = 2+y
\]

We say that \( h' \) is the curried version of \( h \).
Function Composition in Lambda Calculus

S: \( \lambda x. (s \, x) \) (Square)
I: \( \lambda x. (i \, x) \) (Increment)

C: \( \lambda f. \lambda g. \lambda x. (f \, (g \, x)) \) (Function Composition)

Recall semantics rule:

\( (\lambda x. E \, M) \Rightarrow E\{M/x\} \)
Order of Evaluation in the Lambda Calculus

Does the order of evaluation change the final result?
Consider:

\[ \lambda x. (\lambda x. (s x) (\lambda x. (i x) x)) \]

There are two possible evaluation orders:

- **Applicative Order**:
  \[
  \lambda x. (\lambda x. (s x) (\lambda x. (i x) x)) \\
  \Rightarrow \lambda x. (\lambda x. (s x) (i x)) \\
  \Rightarrow \lambda x. (s (i x))
  \]

- **Normal Order**:
  \[
  \lambda x. (\lambda x. (s x) (\lambda x. (i x) x)) \\
  \Rightarrow \lambda x. (s (\lambda x. (i x) x)) \\
  \Rightarrow \lambda x. (s (i x))
  \]

Is the final result always the same?
Church-Rosser Theorem

If a lambda calculus expression can be evaluated in two different ways and both ways terminate, both ways will yield the same result.

Also called the *diamond* or *confluence* property.

Furthermore, if there is a way for an expression evaluation to terminate, using normal order will cause termination.
Consider:

\[(\lambda x. y (\lambda x. (x x) \lambda x. (x x)))\]

There are two possible evaluation orders:

1. \[(\lambda x. y (\lambda x. (x x) \lambda x. (x x))) \Rightarrow (\lambda x. y (\lambda x. (x x) \lambda x. (x x)))\]

   and:

2. \[(\lambda x. y (\lambda x. (x x) \lambda x. (x x))) \Rightarrow y\]

In this example, normal order terminates whereas applicative order does not.
Free and Bound Variables

The lambda functional abstraction is the only syntactic construct that *binds* variables. That is, in an expression of the form:

\[ \lambda v. e \]

we say that occurrences of variable \( v \) in expression \( e \) are *bound*. All other variable occurrences are said to be *free*.

E.g.,

\[ (\lambda x. \lambda y. (x \ y) \ (y \ w)) \]
**Why $\alpha$-renaming?**

Alpha renaming is used to prevent capturing free occurrences of variables when reducing a lambda calculus expression, e.g.,

$$ (\lambda x. \lambda y. (x y) (y \ w)) $$

$$ \Rightarrow \lambda y. ((y \ w) \ y) $$

This reduction **erroneously** captures the free occurrence of $y$.

A correct reduction first renames $y$ to $z$, (or any other *fresh* variable) e.g.,

$$ (\lambda x. \lambda y. (x y) (y \ w)) $$

$$ \Rightarrow (\lambda x. \lambda z. (x z) (y \ w)) $$

$$ \Rightarrow \lambda z. ((y \ w) \ z) $$

where $y$ remains *free*. 

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**α-renaming**

Alpha renaming is used to prevent capturing free occurrences of variables when beta-reducing a lambda calculus expression.

In the following, we rename \( x \) to \( z \), (or any other fresh variable):

\[
(\lambda x. (y x) x) \xrightarrow{\alpha} (\lambda z. (y z) x)
\]

Only bound variables can be renamed. No free variables can be captured (become bound) in the process. For example, we cannot alpha-rename \( x \) to \( y \).
Beta-reduction may require alpha renaming to prevent capturing free variable occurrences. For example:

\[(\lambda x. E \; M) \overset{\beta}{\rightarrow} E\{M/x\}\]

Where the free \(y\) remains free.
\[ \lambda x. (E \ x) \xrightarrow{\eta} E \]

if \( x \) is not free in \( E \).

For example:

\[
\begin{align*}
(\lambda x. \lambda y. (x y) (y w)) \\
\xrightarrow{\alpha} (\lambda x. \lambda z. (x z) (y w)) \\
\xrightarrow{\beta} \lambda z. ((y w) z) \\
\xrightarrow{\eta} (y w)
\end{align*}
\]
A lambda calculus expression with no free variables is called a **combinator**. For example:

**I:** \( \lambda x. x \) (Identity)

**App:** \( \lambda f. \lambda x. (f \ x) \) (Application)

**C:** \( \lambda f. \lambda g. \lambda x. (f \ (g \ x)) \) (Composition)

**L:** \( \lambda x. (x \ x) \lambda x. (x \ x) \) (Loop)

**Cur:** \( \lambda f. \lambda x. \lambda y. ((f \ x) \ y) \) (Currying)

**Seq:** \( \lambda x. \lambda y. (\lambda z. y \ x) \) (Sequencing--normal order)

**ASeq:** \( \lambda x. \lambda y. (y \ x) \) (Sequencing--applicative order)

where \( y \) denotes a **thunk**, *i.e.*, a lambda abstraction wrapping the second expression to evaluate.

The meaning of a combinator is always the same independently of its context.
Functional programming languages have a syntactic form for lambda abstractions. For example the identity combinator:

\( \lambda x. x \)

can be written in Oz as follows:

```oz
fun {{ X} X} end
```

in Haskell as follows:

```haskell
\ x -> x
```

and in Scheme as follows:

```scheme
(lambda(x) x)
```
Currying Combinator in Oz

The currying combinator can be written in Oz as follows:

```oz
fun {$ F}
    fun {$ X}
        fun {$ Y}
            {F X Y}
        end
    end
end
```

It takes a function of two arguments, F, and returns its curried version, e.g.,

```
{{Curry Plus} 2} 3} ⇒ 5
```
Booleans and Branching (*if*) in λ Calculus

|true|: \( \lambda x. \lambda y. x \)  
\(|false|: \lambda x. \lambda y. y \)  

|if|: \( \lambda b. \lambda t. \lambda e. (b \ t \ e) \)  

Recall semantics rule:

\[ (\lambda x. E \ M) \Rightarrow E\{M/x\} \]

\(((\text{if } \text{true}) \ a) \ b) \Rightarrow (\lambda e. ((\lambda x. \lambda y. x) a) b) \Rightarrow ((\lambda x. \lambda y. x a) b) \Rightarrow (\lambda y. a b) \Rightarrow a \]
Exercises

1. PDCS Exercise 2.11.1 (page 31).
2. PDCS Exercise 2.11.2 (page 31).
3. PDCS Exercise 2.11.5 (page 31).
4. PDCS Exercise 2.11.6 (page 31).
5. Define Compose in Haskell. Demonstrate the use of curried Compose using an example.
6. PDCS Exercise 2.11.7 (page 31).
7. PDCS Exercise 2.11.9 (page 31).
8. PDCS Exercise 2.11.12 (page 31). Test your representation of booleans in Haskell.