CSCI 4961/6961 Advanced robotics, Spring 2000
Course notes: Feedback control

1 A simple example

Suppose we have a simple, one degree of freedom robotic manipulator powered by a linear electric actuator that produces a force (to move the manipulatior) that is proportional to the input current:

$$F(t) = K_m I(t) \tag{1}$$

where K_m is a constant that relates the input current to the force produced. This force moves the manipulator according to the following equation:

$$F(t) = m\ddot{q}(t) \tag{2}$$

where m is the mass of the manipulator and $q(t)$ is its position.

Suppose we use proportional feedback control to set the input current to this actuator. We use the control law:

$$I(t) = K_p(r(t) - q(t)) \tag{3}$$

where $r(t)$ is the desired (or reference) position and K_p is the proportional feedback gain. In other words, the current input to the system is set proportional to the error in the output position. Combining these equations, we get:

$$m\ddot{q}(t) + K_p K_m q(t) = K_p K_m r(t) \tag{4}$$

This is a second order differential equation in $q(t)$ which has constant coefficients.

Another variation of this system that we might consider is to add friction to the model. Suppose the force produced by the actuator were reduced because of viscous friction:

$$F(t) = K_m I(t) - B\dot{q}(t) \tag{5}$$

(where B is a viscous friction coefficient). With the proportional feedback law, this yields the differential equation:

$$m\ddot{q}(t) + B\dot{q}(t) + K_p K_m q(t) = K_p K_m r(t) \tag{6}$$

2 Solving second order differential equations

There are two types of solutions that we seek for differential equations: homogeneous solutions and particular solutions. Consider the second order differential equation:

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = f(t) \tag{7}$$

The function $f(t)$ is sometimes called a *forcing function*. In the examples above, the forcing function is the reference position. If the reference position changes, it forces the system modeled by the differential equation to respond. A solution $x_p(t)$ that satisfies this equation is called a particular solution.

We can also consider the differential equation:

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = 0 \tag{8}$$

This is the homogeneous equation which reflects the natural dynamics of the system (i.e. in the absence of a forcing function). A solution to this equation $x_h(t)$ (called a homogeneous solution) describes how the system responds given some initial conditions.

Because of the linearity of ordinary differential equations, we can combine the homogeneous and particular solutions:

$$x(t) = C_h x_h(t) + C_p x_p(t) \tag{9}$$

where C_h and C_p are constants; the resulting $x(t)$ is then a solution to the differential equation.

3 The harmonic oscillator

Consider a differential equation of the form:

$$\ddot{x}(t) + \omega_n^2 x(t) = f(t) \tag{10}$$

which is the well known harmonic oscillator. The homogeneous solutions to this differential equation are $\sin \omega_n t$ and $\cos \omega_n t$.

Typically, the forcing function used is a step function. We assume the system is at rest, i.e. $f(t) = 0$, $x(t) = 0$, and $\dot{x}(t) = 0$ for $t < 0$. Then for $t \geq 0$, $f(t) = 1$, and we consider the behavior of the system for $t \geq 0$.

For a constant forcing function $f(t) = 1$, the particular solution is $x_p(t) = 1$. Therefore, the solution to the differential equation is:

$$x(t) = C_1 \sin \omega_n t + C_2 \cos \omega_n t + 1 \tag{11}$$

We must solve for the constants C_1 and C_2 by using the initial conditions:

$$x(0) = C_2 + 1 = 0 \tag{12}$$

$$\dot{x}(0) = C_1 \omega_n = 0 \tag{13}$$

so $C_1 = 0$ and $C_2 = -1$ and the solution to this differential equation (for a unit step function at $t = 0$ and the given initial conditions) is:

$$x(t) = 1 - \cos \omega_n t \tag{14}$$

Note that this solution oscillates about $x(t) = 1$ forever.

4 The damped harmonic oscillator

The damped harmonic oscillator has the form:

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = f(t) \tag{15}$$

The homogeneous solutions to this differential equation depend upon the value of the damping ratio ζ. This classifies the homogeneous solution into one of three cases: underdamped, overdamped, and critically damped.

The solutions can be described in terms of the solutions to the characteristic equation of this ODE:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{16}$$

The roots of this equation are

$$s_1 = -\zeta\omega_n + i\omega_d \qquad s_2 = -\zeta\omega_n - i\omega_d \tag{17}$$

where ω_d is the damped natural frequency of the system:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{18}$$

If s_1 and s_2 are distinct, then the homogeneous solutions are $e^{s_1 t}$ and $e^{s_2 t}$. If $s = s_1 = s_2$, then the homogeneous solutions are e^{st} and te^{st}.

We'll look at the three cases in turn and then examine a particular solution.

4.1 Underdamped ($\zeta < 1$)

If $\zeta < 1$ then ω_d is real, and therefore s_1 and s_2 are complex. The combined homogeneous solution is then:

$$x_h(t) = C_1 e^{(-\zeta\omega_n + i\omega_d)t} + C_2 e^{(-\zeta\omega_n - i\omega_d)t} \tag{19}$$

This can be rearranged into the following form:

$$x_h(t) = e^{-\zeta\omega_n t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) \tag{20}$$

where we would solve for the constants B_1 and B_2 using the initial conditions.

4.2 Overdamped ($\zeta > 1$)

If $\zeta > 1$ then ω_d is imaginary, and therefore s_1 and s_2 are real. The homogeneous solution is then:

$$x(t) = C_1 e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \tag{21}$$

where again we would solve for the constants C_1 and C_2 using the initial conditions.

4.3 Critically damped ($\zeta = 1$)

If $\zeta = 1$ then $\omega_d = 0$, and $s_1 = s_2 = -\zeta\omega_n$. The homogeneous solution is:

$$x(t) = C_1 e^{-\zeta\omega_n t} + C_2 t e^{-\zeta\omega_n t} \tag{22}$$

where the constants C_1 and C_2 can be determined using the initial conditions.

4.4 An example

Suppose the system is underdamped, that the system is initially at rest with $x(t) = 0$ and $\dot{x}(t) = 0$, and $f(t)$ is a unit step function at $t = 0$ (i.e. $f(t) = 0$ for $t < 0$ and $f(t) = 1$ for $t \geq 0$).

Again, the particular solution is $x_p(t) = 1$. Recall that we are only dealing with $t \geq 0$. Our solution is:

$$x(t) = e^{-\zeta\omega_n t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t) + 1 \tag{23}$$

Now using our initial conditions, we get:

$$x(0) = B_1 + 1 = 0 \tag{24}$$

$$\dot{x}(0) = -\zeta\omega_n B_1 + \omega_d B_2 = 0 \tag{25}$$

so $B_1 = -1$ and $B_2 = \frac{-\zeta\omega_n}{\omega_d} = \frac{-\zeta}{\sqrt{1-\zeta^2}}$. The solution to this problem is then:

$$x(t) = 1 - e^{-\zeta\omega_n t}(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\omega_d t) \tag{26}$$