CSCI 4961/6961 Advanced robotics, Spring 2000
Course notes

Notes on friction cones

The friction cone is a graphical representation of the forces that may arise due to the normal force and a (Coulomb) friction force. It allows a simple geometric approach to analyzing static and quasistatic equilibrium problems with multiple contacts. The main advantage is that you can tell if an object is in equilibrium without calculating the exact forces.

The typical approach to physical analysis and simulation is to assume some contact mode, solve for the forces, and check to see whether the forces are consistent with the assumed contact mode. There are two contact modes that we will see:

- sticking: the two objects in contact do not move relative to each other. In this case, the force due to friction is:
$$|F_f| \leq \mu N \tag{1}$$
where N is the normal force, μ is the coefficient of friction, and F_f is the force due to friction. This equation describes a cone about the contact normal. Note that the angular half-width of the cone is $\tan^{-1}\mu$.

- sliding: the two objects in contact are moving relative to each other. In this case, the force due to friction is:
$$|F_f| = \mu N \tag{2}$$
The friction force will be tangent to the contact; its direction will be such that it opposes motion of the two bodies in contact. Note that in this case, the contact force must lie on one edge of the friction cone.

Resultant forces and equilibrium conditions

For the problems we'll address, an object in a gravitational field will have a number of contacts. For this object to be in equilibrium, the applied forces (typically from frictional contacts) must balance the gravitational force on an object. This is simply another way of saying that the total force and torque on an object must be zero if its linear and angular acceleration are zero.

To determine force equilibrium, we would normally add all the forces together (regardless of where they act upon the object). If they sum to zero (or the total applied force is equal and opposite to the gravitational force). Torque equilibrium would be determined by choosing a point about which to measure torque, and summing the torque contributed by each force ($T = \vec{r} \times \vec{F}$). Here, the point at which the force acts is important.

Since we are aiming for a simple graphical method, we don't really want to deal with lots of individual forces, particularly when it comes to finding the total torque on an object. Fortunately for us, there is a simple graphical method of combining two forces into a single force that has the same total force and torque as the original two forces.

This method takes advantage of the fact that you can move a force along its line of action without changing the torque (no matter about which point you measure torque). The procedure is as follows: draw the lines of action for the two forces (this is simply a line which the force lies on). Move both forces to the intersection point of the lines of action, and add the forces together. This new force can be combined with other forces, one by one, in the same manner.

Note that if two forces are colinear, they produce no torque, and the net force is simply their sum.

Analysis of planar frictional contact problems

For frictional contact problems, we do not know what the contact forces will be, so we can't use the above method directly. What we do know is that the contact force will lie in the friction cone. Recall that we simply need to know whether the contact forces balance the gravitational force.

Since we know all the possible forces at each contact (the forces that lie in the friction cone), we can figure out where we can get a resultant force. By extending the friction cones, we can find the area of intersection. For each point in this area, there is some line of action in each friction cone that passes through that point. Therefore, it is possible to get a resultant force at that point.

In order for this resultant force to balance the gravitional force, it must be in force and torque balance. To achieve torque balance, the resultant force must lie on the line of action of the gravitational force. To achieve force balance, the two forces must be equal and opposite.

If the line of action of the gravitional force passes through the area of intersection, the former condition can be satisfied. To satisfy the latter condition, you must determine whether a resultant force can arise in the proper direction (opposite gravity). As long as this is possible, then there exists a solution where the forces are equal and opposite. Although often trivial, this is an important step.

To determine whether a resultant force will be in the proper direction, you must take into account the direction of the force produced at each contact. (The normal force can only push the object, not pull it.)

Summary procedure

The procedure for analyzing planar frictional contact problems is:

- Assume some contact mode for each contact.
- Draw the friction cone at each contact — note that for sliding contacts, the contact force must lie on an edge of the friction cone (you must choose which edge).
- Extend the friction cones at each contact and find their intersection — for sliding contacts, you just extend the appropriate edge of the friction cone, not the entire friction cone.
- Check whether the line of action of the gravitational force passes through this region of intersection. If it does not, then the object cannot be in equilibrium under the assumed contact modes (and force direction for sliding contacts)
- Check whether a resultant force opposing gravity can arise on the line of action of gravity in the intersection region. If it can, then the object will be in equilibrium under the assumed contact mode (and force direction for sliding contacts).