

Lambda Calculus

alpha-renaming, beta reduction, applicative and normal evaluation orders, Church-Rosser theorem, combinators

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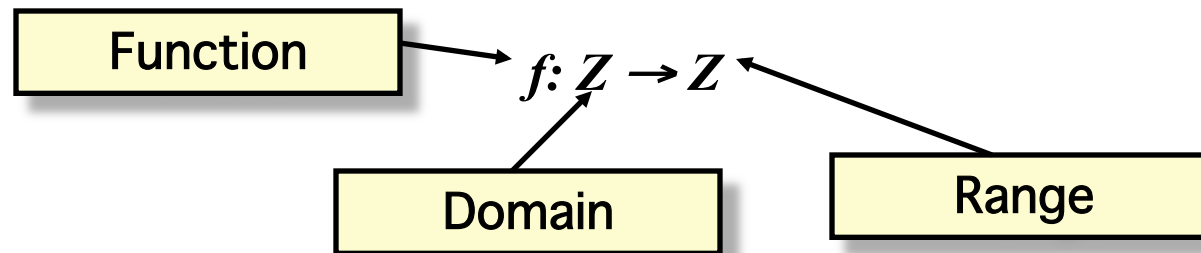
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Mathematical Functions

Take the mathematical function:

$$f(x) = x^2$$

f is a function that maps integers to integers:



We apply the function f to numbers in its domain to obtain a number in its range, e.g.:

$$f(-2) = 4$$

Function Composition

Given the mathematical functions:

$$f(x) = x^2, \quad g(x) = x+1$$

$f \circ g$ is the composition of f and g :

$$f \circ g (x) = f(g(x))$$

$$f \circ g (x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$$

$$g \circ f (x) = g(f(x)) = g(x^2) = x^2 + 1$$

Function composition is therefore not commutative. Function composition can be regarded as a (*higher-order*) function with the following type:

$$\bullet : (\mathbb{Z} \rightarrow \mathbb{Z}) \times (\mathbb{Z} \rightarrow \mathbb{Z}) \rightarrow (\mathbb{Z} \rightarrow \mathbb{Z})$$

Lambda Calculus (Church and Kleene 1930's)

A unified language to manipulate and reason about functions.

$$\text{Given}$$
$$f(x) = x^2$$

$$\lambda x. x^2$$

represents the same f function, except it is *anonymous*.

To represent the function evaluation $f(2) = 4$,
we use the following λ -calculus syntax:

$$(\lambda x. x^2 \ 2) \Rightarrow 2^2 \Rightarrow 4$$

Lambda Calculus Syntax and Semantics

The syntax of a λ -calculus expression is as follows:

e	$::=$	v	variable
		$\lambda v.e$	functional abstraction
		$(e e)$	function application

The semantics of a λ -calculus expression is as follows:

$$(\lambda x.E M) \Rightarrow E\{M/x\}$$

where we choose a *fresh* x , alpha-renaming the lambda abstraction if necessary to avoid capturing free variables in M .

Currying

The lambda calculus can only represent functions of *one* variable. It turns out that one-variable functions are sufficient to represent multiple-variable functions, using a strategy called *currying*.

E.g., given the mathematical function:
of type

$$h(x,y) = x+y$$
$$h: Z \times Z \rightarrow Z$$

We can represent h as h' of type:
Such that

$$h': Z \rightarrow Z \rightarrow Z$$

$$h(x,y) = h'(x)(y) = x+y$$

For example,

$$h'(2) = g, \text{ where } g(y) = 2+y$$

We say that h' is the *curried* version of h .

Function Composition in Lambda Calculus

S: $\lambda x.x^2$ (Square)

I: $\lambda x.x+1$ (Increment)

C: $\lambda f.\lambda g.\lambda x.(f (g x))$ (Function Composition)

((C S) I)

Recall semantics rule:

$(\lambda x.E M) \Rightarrow E\{M/x\}$

$$\begin{aligned} & ((\lambda f.\lambda g.\lambda x.(f (g x)) \lambda x.x^2) \lambda x.x+1) \\ & \Rightarrow (\lambda g.\lambda x.(\lambda x.x^2 (g x)) \lambda x.x+1) \\ & \Rightarrow \lambda x.(\lambda x.x^2 (\lambda x.x+1 x)) \\ & \Rightarrow \lambda x.(\lambda x.x^2 x+1) \\ & \Rightarrow \lambda x.x+1^2 \end{aligned}$$

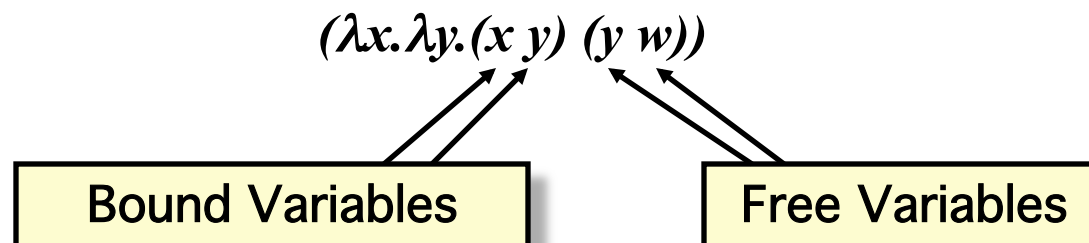
Free and Bound Variables

The lambda functional abstraction is the only syntactic construct that *binds* variables. That is, in an expression of the form:

$$\lambda v.e$$

we say that free occurrences of variable v in expression e are *bound*. All other variable occurrences are said to be *free*.

E.g.,



α -renaming

Alpha renaming is used to prevent capturing free occurrences of variables when reducing a lambda calculus expression, e.g.,

$$\begin{aligned} & (\lambda x. \lambda y. (x y) (y w)) \\ & \Rightarrow \lambda y. ((y w) y) \end{aligned}$$

This reduction **erroneously** captures the free occurrence of y .

A correct reduction first renames y to z , (or any other *fresh* variable) e.g.,

$$\begin{aligned} & (\lambda x. \lambda y. (x y) (y w)) \\ & \Rightarrow (\lambda x. \lambda z. (x z) (y w)) \\ & \Rightarrow \lambda z. ((y w) z) \end{aligned}$$

where y remains *free*.

Order of Evaluation in the Lambda Calculus

Does the order of evaluation change the final result?

Consider:

$$\lambda x. (\lambda x. x^2 (\lambda x. x+1 x))$$

Recall semantics rule:

$$(\lambda x. E M) \Rightarrow E\{M/x\}$$

There are two possible evaluation orders:

$$\begin{aligned} &\lambda x. (\lambda x. x^2 (\lambda x. x+1 x)) \\ &\Rightarrow \lambda x. (\lambda x. x^2 x+1) \\ &\Rightarrow \lambda x. x+1^2 \end{aligned}$$

Applicative
Order

and:

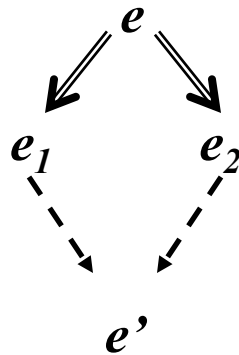
$$\begin{aligned} &\lambda x. (\lambda x. x^2 (\lambda x. x+1 x)) \\ &\Rightarrow \lambda x. (\lambda x. x+1 x)^2 \\ &\Rightarrow \lambda x. x+1^2 \end{aligned}$$

Normal Order

Is the final result always the same?

Church-Rosser Theorem

If a lambda calculus expression can be evaluated in two different ways and both ways terminate, both ways will yield the same result.



Also called the *diamond* or *confluence* property.

Furthermore, if there is a way for an expression evaluation to terminate, using normal order will cause termination.

Order of Evaluation and Termination

Consider:

$$(\lambda x. y (\lambda x. (x x) \lambda x. (x x)))$$

There are two possible evaluation orders:

$$\begin{aligned} &(\lambda x. y (\lambda x. (x x) \lambda x. (x x))) \\ \Rightarrow &(\lambda x. y (\lambda x. (x x) \lambda x. (x x))) \end{aligned}$$

and:

$$\begin{aligned} &\underline{(\lambda x. y (\lambda x. (x x) \lambda x. (x x)))} \\ \Rightarrow &y \end{aligned}$$

Recall semantics rule:

$$(\lambda x. E M) \Rightarrow E\{M/x\}$$

Applicative
Order

Normal Order

In this example, normal order terminates whereas applicative order does not.

Combinators

A lambda calculus expression with *no free variables* is called a *combinator*. For example:

I:	$\lambda x.x$	(Identity)
App:	$\lambda f.\lambda x.(f\ x)$	(Application)
C:	$\lambda f.\lambda g.\lambda x.(f\ (g\ x))$	(Composition)
L:	$(\lambda x.(x\ x)\ \lambda x.(x\ x))$	(Loop)
Cur:	$\lambda f.\lambda x.\lambda y.((f\ x)\ y)$	(Currying)
Seq:	$\lambda x.\lambda y.(\lambda z.y\ x)$	(Sequencing--normal order)
ASeq:	$\lambda x.\lambda y.(y\ x)$	(Sequencing--applicative order)

where y denotes a *thunk*, *i.e.*, a lambda abstraction wrapping the second expression to evaluate.

The meaning of a combinator is always the same independently of its context.

Combinators in Functional Programming Languages

Most functional programming languages have a syntactic form for lambda abstractions. For example the identity combinator:

$$\lambda x.x$$

can be written in Oz as follows:

```
fun {$ X} X end
```

and it can be written in Scheme as follows:

```
(lambda(x) x)
```

Currying Combinator in Oz

The currying combinator can be written in Oz as follows:

```
fun {$ F}
  fun {$ X}
    fun {$ Y}
      {F X Y}
    end
  end
end
end
```

It takes a function of two arguments, F, and returns its curried version, e.g.,

$$\{\{\{\text{Curry Plus}\} 2\} 3\} \Rightarrow 5$$

Exercises

- 20. Lambda Calculus Handout Exercise 1.
- 21. Lambda Calculus Handout Exercise 2.
- 22. Lambda Calculus Handout Exercise 5.
- 23. Lambda Calculus Handout Exercise 6.