CSCI-1200 Data Structures — Spring 2014
Lecture 23 – Operators

Review from Lecture 22

- Basic exception mechanisms: `try/throw/catch`
- Functions & exceptions, constructors & exceptions

Today's Lecture

- Finish Leftist Heaps
- Operators as non-member functions, as member functions, and as friend functions.
- Some tree practice problems

23.1 Leftist Heaps — Overview

- Our goal is to be able to merge two heaps in $O(\log n)$ time, where n is the number of values stored in the larger of the two heaps.
  - Merging two binary heaps (where every row but possibly the last is full) requires $O(n)$ time
- Leftist heaps are binary trees where we deliberately attempt to eliminate any balance.
  - Why? Well, consider the most *unbalanced* tree structure possible. If the data also maintains the heap property, we essentially have a sorted linked list.
- Leftists heaps are implemented explicitly as trees (rather than vectors).

23.2 Leftist Heaps — Mathematical Background

- **Definition:** The *null path length* (NPL) of a tree node is the length of the shortest path to a node with 0 children or 1 child. The NPL of a leaf is 0. The NPL of a NULL pointer is -1.
- **Definition:** A *leftist tree* is a binary tree where at each node the null path length of the left child is greater than or equal to the null path length of the right child.
- **Definition:** The *right path* of a node (e.g. the root) is obtained by following right children until a NULL child is reached.
  - In a leftist tree, the right path of a node is at least as short as any other path to a NULL child.
- **Theorem:** A leftist tree with $r > 0$ nodes on its right path has at least $2^r - 1$ nodes.
  - This can be proven by induction on r.
- **Corollary:** A leftist tree with n nodes has a right path length of at most $\lfloor \log(n+1) \rfloor = O(\log n)$ nodes.
- **Definition:** A *leftist heap* is a leftist tree where the value stored at any node is less than or equal to the value stored at either of its children.

23.3 Leftist Heap Operations

- The `insert` and `delete_min` operations will depend on the `merge` operation.
- Here is the fundamental idea behind the merge operation. Given two leftist heaps, with `h1` and `h2` pointers to their root nodes, and suppose `h1->value <= h2->value`. Recursively merge `h1->right` with `h2`, making the resulting heap `h1->right`.
- When the leftist property is violated at a tree node involved in the merge, the left and right children of this node are swapped. This is enough to guarantee the leftist property of the resulting tree.
- `Merge` requires $O(\log n + \log m)$ time, where m and n are the numbers of nodes stored in the two heaps, because it works on the right path at all times.

23.4 Merge Code

```
template <class T>
class LeftNode {
public:
  LeftNode() : npl(0), left(0), right(0) {}
  LeftNode(const T& init) : value(init), npl(0), left(0), right(0) {}
  T value;
  int npl;         //  the null-path length
  LeftNode* left;
  LeftNode* right;
};
```

Here are the two functions used to implement leftist heap merge operations. Function merge is the driver. Function merge1 does most of the work. These functions call each other recursively.

```
template <class Etype>
LeftNode<Etype>* merge(LeftNode<Etype> *H1,LeftNode<Etype> *H2) {
  if (!h1)
    return h2;
  else if (!h2)
    return h1;
  else if (h2->value > h1->value)
    return merge1(h1, h2);
  else
    return merge1(h2, h1);
}
```

```
template <class Etype>
LeftNode<Etype>* merge1(LeftNode<Etype> *h1, LeftNode<Etype> *h2) {
  if (h1->left == NULL)
    h1->left = h2;
  else {
    h1->right = merge(h1->right, h2);
    if(h1->left->npl < h1->right->npl)
      swap(h1->left, h1->right);
    h1->npl = h1->right->npl + 1;
  }
  return h1;
}
```

23.5 Exercises

1. Explain how `merge` can be used to implement `insert` and `delete_min`, and then write code to do so.

2. Show the state of a leftist heap at the end of:

```
insert 1, 2, 3, 4, 5, 6
delete_min
insert 7, 8
delete_min
delete_min
```

23.6 Complex Numbers — A Brief Review

- Complex numbers take the form $z = a + bi$, where $i = \sqrt{-1}$ and a and b are real. a is called the real part, b is called the imaginary part.

- If $w = c + di$, then
  - $w + z = (a + c) + (b + d)i$,
  - $w - z = (a - c) + (b - d)i$, and
  - $w \times z = (ac - bd) + (ad + bc)i$
- The magnitude of a complex number is $\sqrt{a^2 + b^2}$.

23.7 `Complex` Class declaration (`complex.h`)

```
class Complex {
public:
  Complex(double x=0, double y=0) : real_(x), imag_(y) {}  // default constructor
  Complex(Complex const& old) : real_(old.real_), imag_(old.imag_) {}  // copy constructor
  Complex& operator= (Complex const& rhs); // Assignment operator
  double Real() const { return real_; }
  void SetReal(double x) { real_ = x; }
  double Imaginary() const { return imag_; }
  void SetImaginary(double y) { imag_ = y; }
  double Magnitude() const { return sqrt(real_*real_ + imag_*imag_); }
  Complex operator+ (Complex const& rhs) const;
  Complex operator- () const; // unary operator- negates a complex number
  friend istream& operator>> (istream& istr, Complex& c);
private:
  double real_, imag_;
};

Complex operator- (Complex const& left, Complex const& right); // non-member function
ostream& operator<< (ostream& ostr, Complex const& c);  // non-member function
```

23.8 Implementation of `Complex` Class (`complex.cpp`)

```
// Assignment operator
Complex& Complex::operator= (Complex const& rhs) {
  real_ = rhs.real_;
  imag_ = rhs.imag_;
  return *this;
}

// Addition operator as a member function.
Complex Complex::operator+ (Complex const& rhs) const {
  double re = real_ + rhs.real_;
  double im = imag_ + rhs.imag_;
  return Complex(re, im);
}
// Subtraction operator as a non-member function.
Complex operator- (Complex const& lhs, Complex const& rhs) {
  return Complex(lhs.Real()-rhs.Real(), lhs.Imaginary()-rhs.Imaginary());
}
// Unary negation operator.  Note that there are no arguments.
Complex Complex::operator- () const {
  return Complex(-real_, -imag_);
}

// Input stream operator as a friend function
istream& operator>> (istream & istr, Complex & c) {
  istr >> c.real_ >> c.imag_;
  return istr;
}
// Output stream operator as an ordinary non-member function
ostream& operator<< (ostream & ostr, Complex const& c) {
  if (c.Imaginary() < 0)  ostr << c.Real() << " - " << -c.Imaginary() << " i ";
  else                    ostr << c.Real() << " + " <<  c.Imaginary() << " i ";
  return ostr;
}
```

23.9 Operators as Non-Member Functions and as Member Functions

- We have already written our own operators, especially `operator<`, to sort objects stored in STL containers and to create our own keys for maps.
- We can write them as non-member functions (e.g., `operator-`). When implemented as a non-member function, the expression: `z - w` is translated by the compiler into the function call: `operator- (z, w)`
- We can also write them as member functions (e.g., `operator+`). When implemented as a member function, the expression: `z + w` is translated into: `z.operator+ (w)`

  This shows that `operator+` is a member function of `z`, since `z` appears on the left-hand side of the operator. Observe that the function has **only one** argument!

  There are several important properties of the implementation of an operator as a member function:

  - It is within the scope of class `Complex`, so private member variables can be accessed directly.
  - The member variables of `z`, whose member function is actually called, are referenced by directly by name.
  - The member variables of `w` are accessed through the parameter `rhs`.
  - The member function is `const`, which means that `z` will not (and can not) be changed by the function. Also, since `w` will not be changed since the argument is also marked `const`.
- Both `operator+` and `operator-` return `Complex` objects, so both must call `Complex` constructors to create these objects. Calling constructors for `Complex` objects inside functions, especially member functions that work on `Complex` objects, seems somewhat counter-intuitive at first, but it is common practice!

23.10 Assignment Operators

- The assignment operator: `z1 = z2;` becomes a function call: `z1.operator=(z2);`

  And cascaded assignments like: `z1 = z2 = z3;` are really: `z1 = (z2 = z3);`
  which becomes: `z1.operator= (z2.operator= (z3));`

  Studying these helps to explain how to write the assignment operator, which is usually a member function.
- The argument (the right side of the operator) is passed by constant reference. Its values are used to change the contents of the left side of the operator, which is the object whose member function is called. A reference to this object is returned, allowing a subsequent call to `operator=` (`z1`'s `operator=` in the example above).

  The identifier `this` is reserved as a pointer inside class scope to the object whose member function is called. Therefore, `*this` is a a reference to this object.
- The fact that `operator=` returns a reference allows us to write code of the form: `(z1 = z2).real();`

23.11 Exercise

Write an `operator+=` as a member function of the `Complex` class. To do so, you must combine what you learned about `operator=` and `operator+`. In particular, the new operator must return a reference, `*this`.

23.12 Returning Objects vs. Returning References to Objects

- In the `operator+` and `operator-` functions we create new `Complex` objects and simply return the new object. The return types of these operators are both `Complex`.

  Technically, we don't return the new object (which is stored only locally and will disappear once the scope of the function is exited). Instead we create a copy of the object and return the copy. This automatic copying happens outside of the scope of the function, so it is *safe* to access outside of the function. *Note: It's important that the copy constructor is correctly implemented!* Good compilers can minimize the amount of redundant copying without introducing semantic errors.
- When you change an existing object inside an operator and need to return that object, you must return a **reference** to that object. This is why the return types of `operator=` and `operator+=` are both `Complex&`. This avoids creation of a new object.
- A common error made by beginners (and some non-beginners!) is attempting to return a reference to a locally created object! This results in someone having a pointer to stale memory. The pointer may behave correctly for a short while... until the memory under the pointer is allocated and used by someone else.

23.13 Friend Classes vs. Friend Functions

- In the example below, the `Foo` class has designated the `Bar` to be a **friend**. This must be done in the `public` area of the declaration of `Foo`.

  ```
  class Foo {
  public:
    friend class Bar;
    ...
  };
  ```

  This allows member functions in class `Bar` to access *all of* the private member functions and variables of a `Foo` object as though they were public (but not vice versa). Note that `Foo` is giving friendship (access to its private contents) rather than `Bar` claiming it. What could go wrong if we allowed friendships to be claimed?

- Alternatively, within the definition of the class, we can designate specific functions to be "`friend`"s, which grants these functions access similar to that of a member function. The most common example of this is operators, and especially stream operators.

23.14 Stream Operators as Friend Functions

- The operators `>>` and `<<` are defined for the `Complex` class. These are binary operators.
  The compiler translates: `cout << z3` into: `operator<< (cout, z3)`
  Consecutive calls to the `<<` operator, such as: `cout << "z3 = " << z3 << endl;`
  are translated into: `((cout << "z3 = ") << z3) << endl;`
  Each application of the operator returns an `ostream` object so that the next application can occur.

- If we wanted to make one of these stream operators a regular member function, it would have to be a member function of the `ostream` class because this is the first argument (left operand). *We cannot make it a member function of the* `Complex` *class.* This is why stream operators are never member functions.

- Stream operators are either ordinary non-member functions (if the operators can do their work through the public class interface) or friend functions (if they need non public access).

23.15 Summary of Operator Overloading in C++

- Unary operators that can be overloaded: `+ - * & ~ ! ++ -- -> ->*`

- Binary operators that can be overloaded: `+ - * / % ^ & | << >> += -= *= /= %= ^= &= |= <<= >>= < <= > >= == != && || , [] () new new[] delete delete[]`

- There are only a few operators that can not be overloaded: `. .* ?: ::`

- We can't create new operators and we can't change the number of arguments (except for the function call operator, which has a variable number of arguments).

- There are three different ways to overload an operator. When there is a choice, we recommend trying to write operators in this order:
  - Non-member function
  - Member function
  - Friend function

- The most important rule for clean class design involving operators is to **NEVER change the intuitive meaning of an operator**. The whole point of operators is lost if you do. One (bad) example would be defining the increment operator on a `Complex` number.

23.16 Extra Practice

- Implement the following operators for the `Complex` class (or explain why they cannot or should not be implemented). Think about whether they should be non-member, member, or friend.

  `operator*   operator==   operator!=   operator<`

23.17 A Tree Practice Problem

- Draw a *balanced binary tree* that contains the values: 6, 13, 9, 17, 32, 23, and 20.

- What is the height of a *balanced binary tree* storing n elements?

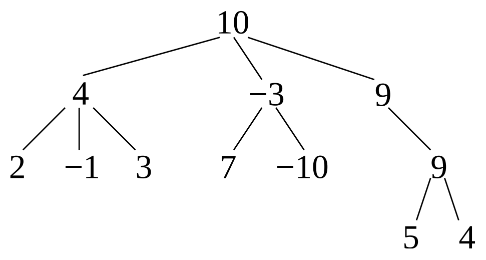
- Draw a *binary search tree* that has *post-order traversal*: 6 13 9 17 32 23 20.

- How many other correct answers are possible for the previous problem?

23.18 Another Tree Practice Problem

A *trinary tree* is similar to a binary tree except that each node has at most 3 children. Write a *recursive* function named `EqualsChildrenSum` that takes one argument, a pointer to the root of a trinary tree, and returns true if the value at each non-leaf node is the sum of the values of all of its children and false otherwise. In the examples below, the tree on the left will return true and the tree on the right will return false.

```
class Node {
public:
  int value;
  Node* left;
  Node* middle;
  Node* right;
};
```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