# Lambda Calculus (PDCS 2)

## alpha-renaming, beta reduction, applicative and normal evaluation orders, Church-Rosser theorem, combinators

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# Mathematical Functions

Take the mathematical function:

$$f(x) = x^2$$

$f$ is a function that maps integers to integers:

Function

$$f: Z \to Z$$

Domain Range

We apply the function f to numbers in its domain to obtain a number in its range, e.g.:

$$f(-2) = 4$$

# Function Composition

Given the mathematical functions:

$$f(x) = x^2 \;,\;\; g(x) = x+1$$

$f \bullet g$ is the composition of $f$ and $g$:

$$f \bullet g\,(x) = f(g(x))$$

$$f \bullet g\,(x) = f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$$
$$g \bullet f\,(x) = g(f(x)) = g(x^2) = x^2 + 1$$

Function composition is therefore not commutative. Function composition can be regarded as a (*higher-order*) function with the following type:

$$\bullet : (Z \rightarrow Z) \times (Z \rightarrow Z) \rightarrow (Z \rightarrow Z)$$

# Lambda Calculus (Church and Kleene 1930's)

A unified language to manipulate and reason about functions.

Given

$$f(x) = x^2$$

$$\lambda x.\ x^2$$

represents the same ***f*** function, except it is *anonymous*.

To represent the function evaluation $f(2) = 4$,
we use the following λ-calculus syntax:

$$(\lambda x.\ x^2\ 2) \Rightarrow 2^2 \Rightarrow 4$$

# Lambda Calculus Syntax and Semantics

The syntax of a λ-calculus expression is as follows:

| | | | |
|---|---|---|---|
| **e** | **::=** | **v** | variable |
| | \| | ***λ*v.e** | functional abstraction |
| | \| | ***(*e e*)*** | function application |

The semantics of a λ-calculus expression is as follows:

$$(\lambda x.E\ M) \Rightarrow E\{M/x\}$$

where we alpha-rename the lambda abstraction **E** if necessary to avoid capturing free variables in **M**.

# Currying

The lambda calculus can only represent functions of *one* variable. It turns out that one-variable functions are sufficient to represent multiple-variable functions, using a strategy called *currying*.

E.g., given the mathematical function: $h(x,y) = x+y$

of type $h: Z \times Z \rightarrow Z$

We can represent $h$ as $h'$ of type: $h': Z \rightarrow Z \rightarrow Z$

Such that

$$h(x,y) = h'(x)(y) = x+y$$

For example,

$$h'(2) = g, \text{ where } g(y) = 2+y$$

We say that $h'$ is the *curried* version of $h$.

# Function Composition in Lambda Calculus

| | | |
|---|---|---|
| S: | $\lambda x.(s\ x)$ | (Square) |
| I: | $\lambda x.(i\ x)$ | (Increment) |
| | | |
| C: | $\lambda f.\lambda g.\lambda x.(f\ (g\ x))$ | (Function Composition) |

> ***Recall semantics rule:***
>
> $(\lambda x.E\ M) \Rightarrow E\{M/x\}$

((C S) I)

$$((\underline{\lambda f.\lambda g.\lambda x.(f\ (g\ x))\ \lambda x.(s\ x))}\ \lambda x.(i\ x))$$
$$\Rightarrow \underline{(\lambda g.\lambda x.(\lambda x.(s\ x)\ (g\ x))\ \lambda x.(i\ x))}$$
$$\Rightarrow \lambda x.(\lambda x.(s\ x)\ \underline{(\lambda x.(i\ x)\ x))}$$
$$\Rightarrow \lambda x.\underline{(\lambda x.(s\ x)\ (i\ x))}$$
$$\Rightarrow \lambda x.(s\ (i\ x))$$

# Free and Bound Variables

The lambda functional abstraction is the only syntactic construct that *binds* variables. That is, in an expression of the form:

$$\lambda \mathbf{v.e}$$

we say that free occurrences of variable **v** in expression **e** are *bound*. All other variable occurrences are said to be *free*.

E.g.,

$$(\lambda x.\lambda y.(x\ y)\ (y\ w))$$

Bound Variables

Free Variables

# $\alpha$-renaming

Alpha renaming is used to prevent capturing free occurrences of variables when reducing a lambda calculus expression, e.g.,

$$\underline{(\lambda x.\lambda y.(x\ y)\ (y\ w))}$$
$$\Rightarrow \lambda y.((y\ w)\ y)$$

This reduction **erroneously** captures the free occurrence of ***y***.

A correct reduction first renames ***y*** to ***z***, (or any other *fresh* variable) e.g.,

$$(\lambda x.\lambda y.(x\ y)\ (y\ w))$$
$$\Rightarrow \underline{(\lambda x.\lambda z.(x\ z)\ (y\ w))}$$
$$\Rightarrow \lambda z.((y\ w)\ z)$$

where ***y*** remains *free*.

# Order of Evaluation in the Lambda Calculus

Does the order of evaluation change the final result?
Consider:

$$\lambda x.(\lambda x.(s\ x)\ (\lambda x.(i\ x)\ x))$$

***Recall semantics rule:***

$$(\lambda x.E\ M) \Rightarrow E\{M/x\}$$

There are two possible evaluation orders:

$$\begin{aligned} &\lambda x.(\lambda x.(s\ x)\ \underline{(\lambda x.(i\ x)\ x)}) \\ &\Rightarrow \lambda x.\underline{(\lambda x.(s\ x)\ (i\ x))} \\ &\Rightarrow \lambda x.(s\ (i\ x)) \end{aligned}$$

Applicative Order

and:

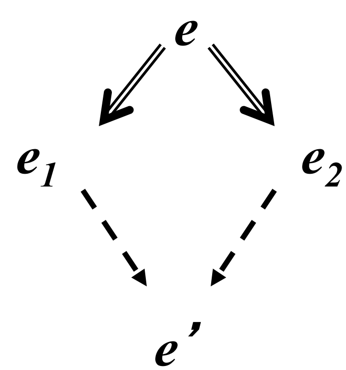
$$\begin{aligned} &\lambda x.\underline{(\lambda x.(s\ x)\ (\lambda x.(i\ x)\ x))} \\ &\Rightarrow \lambda x.(s\ \underline{(\lambda x.(i\ x)\ x)}) \\ &\Rightarrow \lambda x.(s\ (i\ x)) \end{aligned}$$

Normal Order

Is the final result always the same?

# Church-Rosser Theorem

If a lambda calculus expression can be evaluated in two different ways and both ways terminate, both ways will yield the same result.




Also called the *diamond* or *confluence* property.

Furthermore, if there is a way for an expression evaluation to terminate, using normal order will cause termination.

# Order of Evaluation and Termination

Consider:

$$(\lambda x.y\ (\lambda x.(x\ x)\ \lambda x.(x\ x)))$$

**Recall semantics rule:**

$$(\lambda x.E\ M) \Rightarrow E\{M/x\}$$

There are two possible evaluation orders:

$$(\lambda x.y\ \underline{(\lambda x.(x\ x)\ \lambda x.(x\ x))})$$
$$\Rightarrow (\lambda x.y\ (\lambda x.(x\ x)\ \lambda x.(x\ x)))$$

Applicative Order

and:

$$\underline{(\lambda x.y\ (\lambda x.(x\ x)\ \lambda x.(x\ x)))}$$
$$\Rightarrow y$$

Normal Order

In this example, normal order terminates whereas applicative order does not.

# Combinators

A lambda calculus expression with *no free variables* is called a *combinator*. For example:

| | | |
|---|---|---|
| I: | $\lambda x.x$ | (Identity) |
| App: | $\lambda f.\lambda x.(f\ x)$ | (Application) |
| C: | $\lambda f.\lambda g.\lambda x.(f\ (g\ x))$ | (Composition) |
| L: | $(\lambda x.(x\ x)\ \lambda x.(x\ x))$ | (Loop) |
| Cur: | $\lambda f.\lambda x.\lambda y.((f\ x)\ y)$ | (Currying) |
| Seq: | $\lambda x.\lambda y.(\lambda z.y\ x)$ | (Sequencing--normal order) |
| ASeq: | $\lambda x.\lambda y.(y\ x)$ | (Sequencing--applicative order) |

where ***y*** denotes a *thunk, i.e.*, a lambda abstraction wrapping the second expression to evaluate.

The meaning of a combinator is always the same independently of its context.

# Combinators in Functional Programming Languages

Most functional programming languages have a syntactic form for lambda abstractions.  For example the identity combinator:

$$\lambda x.x$$

can be written in Oz as follows:

```
fun {$ X} X end
```

and it can be written in Scheme as follows:

```
(lambda(x) x)
```

# Currying Combinator in Oz

The currying combinator can be written in Oz as follows:

```
fun {$ F}
    fun {$ X}
        fun {$ Y}
            {F X Y}
        end
    end
end
```

It takes a function of two arguments, F, and returns its curried version, e.g.,

{{{Curry Plus} 2} 3} $\Rightarrow$ 5

# Exercises

20. PDCS Exercise 2.11.1 (page 31).
21. PDCS Exercise 2.11.2 (page 31).
22. PDCS Exercise 2.11.5 (page 31).
23. PDCS Exercise 2.11.6 (page 31).