CSCI 2200 - Spring 2014

Exam 2

Name: ____

Instructions:

- You have **105 minutes** to complete this exam. The exam is worth a total of 90 points.
- Put away laptop computers and other electronic devices. Calculators are NOT allowed. Cheating on an exam will result in an **immediate F** for the whole course.
- A one page, two-sided crib sheet (legit sheet) is allowed. Rulers are also allowed.
- **Write your answers clearly and completely.**
- Logical equivalences, rules of inference, set identities, and summation formulae are given on the last full page (front and back).
- Please read each question carefully several times before beginning to work and especially before asking questions. We generally will not answer questions except when there is a glaring mistake or ambiguity in the statement of a question.

1. (10=5+5 points) The Super Ball is an extremely elastic, small rubber ball made by Wham-O. According to the Richard Knerr, a co-founder of Wham-O, "each Super Ball bounce is 92% as high as the last". Suppose I drop a Super Ball from a height of 2 meters.

   (a) Write a recurrence relation for the maximum height (in meters) reached by the Super Ball after the n^{th} bounce (assuming Mr. Knerr is correct).

   (b) Write a closed formula for the maximum height (in meters) reached by the Super Ball after the n^{th} bounce (assuming Mr. Knerr is correct).

   Answer (graded by Scott):

   (a) Each bounce is 92% of the last, so, if h_n is the maximum height after the n^{th} bounce, then

   $$\begin{aligned} h_0 &= 2 \\ h_n &= 0.92h_{n-1} \quad \text{(for } n \geq 1\text{)} \end{aligned}$$

   (b) To find a closed form formula, we notice that:
   $h_n = 0.92h_{n-1}$ and $h_{n-1} = 0.92h_{n-2}$
   so $h_n = 0.92 \cdot 0.92 \cdot h_{n-2}$
   and similarly, $h_n = 0.92^3 h_{n-3}$, etc. Thus, the closed formula is

   $$h_n = 0.92^n h_0 = 2 \cdot 0.92^n$$

2. (15=5+10 points) Our friend Zeno is running a 10 km footrace on Saturday. Zeno has a very peculiar way of running a race. In the first minute, Zeno runs half the distance of the race, arriving at the 5 km mark after exactly 1 minute. In the second minute of the race, Zeno runs another 2.5 km; in the third minute, he runs 1.25 km, and so on. In other words, in each minute of the race, Zeno runs half as far as he ran in the previous minute.

   (a) Write a recurrence relation for the distance that Zeno runs during the n^{th} minute of the race (not the total distance).

   (b) Give a closed formula for the total distance that Zeno has run after exactly n minutes of the race. You may want to use summation in your solution. If your formula contains a summation, you should simplify the summation. Do not give the summation expression with Σ in your final answer.

   Answer: (graded by Stacy)

   (a) The distance that Zeno runs during any given minute of the race is half of the distance he ran in the last minute of the race, so, letting d_n be the distance Zeno runs in the n^{th} minute, we have

$$\begin{aligned} d_1 &= 5 \\ d_n &= d_{n-1}/2 \end{aligned}$$

   (b) The total distance that Zeno has run after n minutes, denoted t_n is the sum of all the distances run in the minutes up to the n_{th} minute,

$$t_n = \sum_{i=1}^{n} 5\left(\frac{1}{2}\right)^{i-1} = 5\sum_{i=1}^{n}\left(\frac{1}{2}\right)^{i-1}$$

   Note that this sum runs from 1 to n. To apply the formula from Table 2, we want the index to start at 0. To apply the formula, we first rewrite the above sum as

$$t_n = 5\sum_{i=0}^{n-1}\left(\frac{1}{2}\right)^{i}$$

   We can then the formula to simplify the sum (here $a = 1$ and $r = (1/2)$.

$$\begin{aligned} t_n &= 5\left(\frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1}\right) \\ &= 10\left(1 - \left(\frac{1}{2}\right)^n\right) \end{aligned}$$

3. (10 points) Using mathematical induction, prove that $2n + 3 \leq 2^n$ for all integers $n \geq 4$. You must write the complete proof.

Answer (graded by Dean):
This is a proof by induction.

Basis Step: $(n = 4)$
$2 \cdot 4 + 3 = 11 \leq 2^4 = 16$

Inductive Step:
Assume that $2k + 3 \leq 2^k$ for an arbitrary integer $k \geq 4$. We will show that this implies $2(k + 1) + 3 \leq 2^{k+1}$.

$$2(k + 1) + 3 = 2k + 3 + 2 \leq 2^k + 2,$$

where the last inequality follows from the inductive hypothesis. Note that for $k \geq 4$, $2 < 2^k$. Therefore,

$$2(k + 1) + 3 \leq 2^k + 2^k = 2^{k+1}.$$

Q.E.D.

4. (15 points) Consider the sequence $\{b_n\}$ defined by

$$\begin{aligned} b_1 &= 1 \\ b_2 &= 2 \\ b_3 &= 3 \\ b_n &= b_{n-1} + b_{n-2} + b_{n-3} \quad \text{for } n > 3 \end{aligned}$$

Using strong induction, prove that $b_n < 2^n$ for all $n \geq 1$. You must write the complete proof.

Answer (graded by Scott):
This is a proof by strong induction.

Basis Step: $1 < 2$, $2 < 4$ and $3 < 8$, so $b_n < 2^n$ for $n = 1, 2, 3$

Inductive Step: Assume for some fixed $k \geq 3$ that $b_j < 2^j$ for all $j \leq k$.
We want to show this implies $b_{k+1} < 2^{k+1}$.

$$b_{k+1} = b_k + b_{k-1} + b_{k-2} < 2^k + 2^{k-1} + 2^{k-2} \text{ (per our induction hypothesis)}$$

We want the right hand side to be less than 2^{k+1}, so we factor it out 2^{k+1},

$$2^k + 2^{k-1} + 2^{k-2} = 2^{k+1}(1/2 + 1/4 + 1/8) = 2^{k+1}(7/8) < 2^{k+1}$$

So, $b_{k+1} < 2^{k+1}$. Thus, if the theorem holds for all j up to some fixed $k \geq 1$, then it holds for $k + 1$.
Q.E.D.

5. (14=5+9 points) Consider the following recursive algorithm.

```
procedure mystery(n:  nonnegative integer)
  if (n == 0) then return 1
  else return mystery(n − 1) + 2
```

(a) Give a closed formula for the output of `mystery(n)` where n is a nonnegative integer.

(b) Give a complete proof by induction that the algorithm is correct with respect to your closed formula, i.e., prove that for every input n, where n is a nonnegative integer, the algorithm outputs the expression for your closed formula.

Answer (graded by Dean):

(a) Testing the first few values returned by our algorithm, we obtain:
`mystery(0)` $= 1$
`mystery(1)` $= 3$
`mystery(2)` $= 5$
We assert that `mystery(n)` $= 2n + 1$

(b) This is a proof by induction.
Basis step: `mystery(0)` $= 1 = 2(0) + 1$, so the algorithm is correct for $n = 0$
Inductive step: Assume that `mystery(k)` outputs $2k + 1$ for some fixed $k \geq 0$.
`mystery(k+1)` will output `mystery(k)` + 2, by the definition of the procedure.
`mystery(k)` + 2 $= 2k + 1 + 2$ (by our inductive hypothesis)
$2k + 1 + 2 = 2(k + 1) + 1$
So `mystery(k+1)` will output $2(k + 1) + 1$
Thus, if the `mystery` algorithm works for some $k \geq 0$, it works for $k + 1$.

Q.E.D.

6. (3 points) Consider a set S of strings made from the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}\}$. Let S have the following recursive definition.

   Basis Step: $\mathbf{a} \in S$

   Recursive Step: if $w \in S$ then $\mathbf{b}w \in S$

   List 3 elements of S.

   Answer (graded by Dean):

   $\mathbf{a}, \mathbf{ba}, \mathbf{bba}$

7. (3 points) Give a recursive definition of the set of integers that are multiples of 5.

   Answer (graded by Dean):

   $0 \in S$

   if $x \in S$ then $(x + 5) \in S$ and $(x - 5) \in S$

8. (9=3+3+3 points) Let f be a function that maps from the set $X = \{1, 2\}$ to the set $Y = \{0, 1, 2, \ldots, n\}$, where $n \geq 2$.

   (a) How many possible functions are there in total?

   (b) How many of these functions are injective?

   (c) How many of these functions are surjective?

   Answer (graded by Yuriy):

   (a) For each $x \in X$, we must pick a $y \in Y$, but we can pick any $y \in Y$, so there are $(n + 1)(n + 1) = (n + 1)^2$ ways to construct a function from X to Y

   (b) We can't pick the same $y \in Y$ for both of the $x \in X$, so there are $(n + 1)(n) = n^2 + n$ ways to construct an injective function from X to Y

   (c) There are too many values in Y to cover with just the values in X, so there are no surjective functions from X to Y

9. (6=3+3 points) Suppose Jack and Jill go to the movies with 6 of their friends.

   (a) How many different ways can the 8 people sit if they all want to sit together and Jack and Jill must sit next to each other?

   (b) How many different ways can they sit if they all want to sit together and Jack and Jill refuse to sit next to each other?

   Answer (graded by Yuriy):

   (a) Suppose Jack and Jill are a couple (whom their friends refer to, collectively, as "the Jillck"), so we can count them, temporarily, as one person. Now there are 7 people to arrange. This gives 7! ways. But, for every way that we've arranged everyone, we can arrange "the Jillck"... So each of these 7! different arrangements is actually 2 arrangements: one with Jill to the left of Jack and one with Jack to the left of Jill. So in total there are $2 * 7!$ arrangements.

   (b) Suppose Jack and Jill are brother and sister (and thus naturally the bitterest of enemies). Counting the different ways to arrange everyone with them apart directly is rather difficult, but we notice that this problem is related to the problem of having them sit together. If we want to count the ways to seat them apart, we can just count all the ways to seat them and subtract the ways in which they sit together. The total number of ways to seat all 8 friends is 8!. So the number of ways to have Jack and Jill sit apart is: $8! - 2 \cdot 7!$

10. (5 points) Carnegie Hall has 2,804 seats in its main auditorium. Prove that, in a packed performance, there will be at least two people in the audience who have the same first and last initials (as each other).

    Answer (graded by Stacy):
    There are $26 \cdot 26 = 676$ different first/last initial combinations. Thus, if more than 676 people show up for a performance, since there are more people than first/last initial possibilities, the pigeonhole principle guarantees that there will be at least two people at the performance with the same first and last initials.