CSCI 2200 - Spring 2014

Exam 1 - Solutions

Name: ______________________________________

1. (15 = 7+8 points) Using propositional logic, write a statement that contains the propositions p, q, and r that is true when both $p \to q$ and $q \leftrightarrow \neg r$ are true and is false otherwise. Your statement must be written as specified below.

   Graded by Yuriy (both (a) and (b))

   (a) Write the statement in disjunctive normal form.

   $$(p \land q \land \neg r) \lor (\neg p \land q \land \neg r) \lor (\neg p \land \neg q \land r)$$

   (b) Write the statement using only the $\lor$ and $\neg$ connectives.

   $$\neg(\neg p \lor \neg q \lor r) \lor \neg(p \lor \neg q \lor r) \lor \neg(p \lor q \lor \neg r)$$

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2. ($15 = 5 \times 3$ points) Consider the following propositional functions:

   $D(x)$: "x is a dog."
   $H(x)$: "x is happy."
   $F(x)$: "x is fluffy."
   $L(x)$: "x likes cheese."

   Let the universe of discourse be all mammals.
   Translate the following statements into predicate logic.

   Graded by Scott ((a) - (e))

   (a) Not all dogs are fluffy.

   $\exists x(D(x) \land \neg F(x))$

   (b) All fluffy dogs like cheese.

   $\forall x((D(x) \land F(x)) \to L(x))$

   (c) All dogs are happy and some of them are fluffy.

   $\forall x(D(x) \to H(x)) \land \exists x(D(x) \land F(x))$

   (d) If Taz is a dog, then Taz likes cheese.

   $D(\text{Taz}) \to L(\text{Taz})$

   (e) There is only one dog who is fluffy, happy, and likes cheese.

   $\exists! x(D(x) \land F(x) \land H(x) \land L(x))$

3. (10 points) Determine whether the following argument is valid and give a formal proof of your answer.

$$\begin{array}{ll} & p \rightarrow (\neg r \rightarrow \neg q) \\ \hline \therefore & (p \wedge q) \rightarrow r \end{array}$$

Graded by Dean

The argument is valid.

1. $p \rightarrow (\neg r \rightarrow \neg q)$ premise
2. $\neg p \vee (\neg r \rightarrow \neg q)$ Table 7, equiv. 1 applied to (1)
3. $\neg p \vee (\neg\neg r \vee \neg q)$ Table 7, equiv. 1 applied to (2)
4. $\neg p \vee (r \vee \neg q)$ double negation equiv. applied to (3)
5. $\neg p \vee (\neg q \vee r)$ commutative equiv. applied to (4)
6. $(\neg p \vee \neg q) \vee r$ associative equiv. applied to (5)
7. $\neg(p \wedge q) \vee r$ DeMorgan's Law applied to (6)
8. $(p \wedge q) \rightarrow r$ Table 7, equiv. 1 applied to (7)

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4. (10 points) A number n is a multiple of 3 if $n = 3k$ for some integer k.

   Prove that if n^2 is a multiple of 3, then n is a multiple of 3.

   Graded by Stacy
   Note: This question should have specified that n is an integer. To help compensate for this omission, the lowest score you can receive on this question is 5/10. Contact Stacy if you have any questions.

   This is a proof by contraposition. Assume that n is not a multiple of 3. Then, either $n = 3k + 1$ for some integer k or $n = 3k + 2$ for some integer k. We consider each case separately.

   Case 1: $n = 3k + 1$
   If $n = 3k + 1$, then $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. So, $n^2 = 3j + 1$ where $j = 3k^2 + 2k$ is an integer. Therefore, n^2 is not a multiple of 3.

   Case 2: $n = 3k + 2$
   If $n = 3k + 2$, then $n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. So, $n^2 = 3j + 1$ where $j = 3k^2 + 4k + 1$ is an integer. Therefore, n^2 is not a multiple of 3.

   Q.E.D.

5. (10 points) Prove that if x is an irrational number, then $\frac{1}{x}$ is irrational.
   Note that if x is irrational, then $x \neq 0$.

   Graded by Dean

   This is a proof by contraposition. Assume that $\frac{1}{x}$ is rational. By definition of a rational number, $\frac{1}{x} = \frac{p}{q}$ for some integers p and q, with $q \neq 0$. We also know that $\frac{1}{x}$ cannot equal 0, since there is no way to divide 1 by anything and get 0. Thus, $p \neq 0$.

   It follows that $x = \frac{q}{p}$, which means that x can be written as $\frac{r}{s}$, where $r = q$ and $s = p$ are integers, with $s \neq 0$. Therefore, x is rational. Q.E.D.

6. ($6 = 3 + 3$ points) Let $A_i = \{0, 1, 2, \ldots i\}$. What are the contents of the set A, where A is as defined below? You can explain your answer using English or set builder notation. You do not need to use logical expressions.

Graded by Scott

(a) $A = \bigcap_{i=1}^{n} A_i$.

Answer: $A = A_1 = \{0, 1\}$

(b) $A = \bigcup_{i=1}^{n} A_i$.

Answer: $A = A_n = \{0, 1, 2, \ldots, n\}$.

7. ($10 = 5 \times 2$ points) Circle **True** or **False** for each of the following statements.

Graded by Dean

(a) There is a set A such that $\mathcal{P}(A) = 12$ **True** <u>**False**</u>

(b) $|\{\emptyset\} \times \{1, 2\}| = 0$ **True** <u>**False**</u>

(c) For all sets, A, B, and C,
$(A \cup B) \cup (A \cap C) - B = A$ **True** <u>**False**</u>

Counterexample:
$A = \{1, 2\},\ B = \{1\},\ C = \emptyset$
$(A \cup B) \cup (A \cap C) - B = \{2\} \neq A$

(d) $\sqrt{2} \in \mathbf{Q}$ **True** <u>**False**</u>

(e) For all sets A and B, $A \cup B = \overline{\overline{A} \cup \overline{B}}$ **True** <u>**False**</u>

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8. (**10 points**) Using the definitions from set theory and the set identities, prove that $A - (B \cap C) = (A \cap \overline{B}) \cup (A \cap \overline{C})$.

Graded by Scott

$A - (B \cap C)$	$= \{x \mid x \in (A - (B \cap C))\}$	by definition of set builder notation
	$= \{x \mid (x \in A) \wedge (x \notin (B \cap C))\}$	by definition of difference
	$= \{x \mid (x \in A) \wedge \neg(x \in (B \cap C))\}$	by definition of $\notin$
	$= \{x \mid (x \in A) \wedge ((\neg(x \in B) \vee \neg(x \in C))\}$	by DeMorgan's Law for propositional logic
	$= \{x \mid (x \in A) \wedge ((x \notin B) \vee (x \notin C))\}$	by definition of $\notin$
	$= \{x \mid (x \in A) \wedge ((x \in \overline{B}) \vee (x \in \overline{C}))\}$	by definition of complement
	$= \{x \mid (x \in A) \wedge (x \in \overline{B} \cup \overline{C})\}$	by definition of union
	$= \{x \mid x \in A \cap (\overline{B} \cup \overline{C})\}$	by definition of intersection
	$= \{x \mid x \in (A \cap \overline{B}) \cup (A \cap \overline{C})\}$	by distributive identity for sets
	$= (A \cap \overline{B}) \cup (A \cap \overline{C})$	by definition of set builder notation

9. ($4 = 4 \times 1$ points) Circle **True** or **False** for each of the following statements.

Graded by Stacy (both (a) and (b)))

(a) Let $f : \mathbf{N} \to \mathbf{N}$ be $f(x) = 4x + 1$

f is injective. **True** **False**

f is surjective. **True** **False** There is no $x \in \mathbf{N}$ for which $f(x) = 2$.

(b) Let $f : \mathbf{Z} \to \mathbf{Z}$ be $f(x) = 3x^4 - 3$

f is injective. **True** **False**

f is surjective. **True** **False**