FoCS: Exam 1 Review Questions

1. Prove or disprove: the compound proposition $\neg s \wedge (t \rightarrow s) \rightarrow \neg t$ is a tautology.

2. Write a proposition using the variables p, q, and r that is true **only** when at least one of p, q, and r are true and $p \rightarrow \neg q$ is true.

3. Is the following compound proposition satisfiable? Prove your answer is correct.
   $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$

4. Let the domain be all integers. For each of the following statements, indicate **TRUE** or **FALSE**

   (a) $\exists! x\ (x \ > \ 1)$

   (b) $\exists! x\ (x^2 \ = \ 9)$

   (c) $\exists! x\ (x + 3 \ = \ 2x)$

   (d) $\exists! x\ P(x) \rightarrow \exists x\ P(x)$

   (e) $\forall x\ P(x) \rightarrow \exists! x\ P(x)$

   (f) $\exists! x\ \neg P(x) \rightarrow \neg \forall x\ P(x)$

5. Determine the truth value of the statement $\exists x \forall y (x \leq y^2)$ if the domain consists of

   (a) real numbers

   (b) integers

   (c) positive real numbers

6. Prove that the statement "x is a rational number" is equivalent to the statement "$3x - 1$ is a rational number".

Solution. First, we show "if x is rational, then $3x - 1$ is rational."
Assume x is rational. Then, $x = p/q$ for some integers p and q, with $q \neq 0$ (by definition of a rational number). So we have,

$$\begin{aligned} 3x - 1 &= 3\left(\frac{p}{q}\right) - 1 \\ &= \frac{3p}{q} - \frac{q}{q} \\ &= \frac{3p - q}{q}. \end{aligned}$$

Thus, $3x - 1 = r/s$ where $r = 3p - q$ is an integer and $s = q$ is an integer, with $s \neq 0$. Therefore, $3x - 1$ is an odd number.

We now show "if $3x - 1$ is rational, then x is rational".
If $3x - 1$ is rational, then $3x - 1 = p/q$ for some integers p and q with $q \neq 0$. So we have,

$$\begin{aligned} 3x - 1 &= \frac{p}{q} \\ x &= \frac{1}{3}\left(\frac{p}{q} + 1\right) \\ x &= \frac{p}{3q} + \frac{q}{3q} \\ x &= \frac{p + q}{3q}. \end{aligned}$$

Thus, $3x - 1 = r/s$ where $r = p + q$ and $s = 3q$ are integers, with $s \neq 0$ (since $q \neq 0$). Therefore, $3x - 1$ is rational. Q.E.D.

7. Use a direct proof to show that every odd integer is the difference of two squares.
This problem is a little more difficult than what might be on the exam. It is included because it is a nice proof.

Solution. We want to prove the statement "if x is odd, then $x = n^2 - m^2$ for some integers n and m. If x is odd, then $x = 2k + 1$ for some integer k (by definition of an odd number). It follows that $x = 2j - 1$ where $j = k + 1$ is an integer. Thus,

$$\begin{aligned} x &= 2j - 1 \\ &= j^2 - j^2 + 2j - 1 \\ &= j^2 - (j^2 - 2j + 1) \\ &= j^2 - (j - 1)^2. \end{aligned}$$

Therefore, $x = n^2 - m^2$ where $n = j$ and $m = j - 1$ are integers. Q.E.D.

8. Define the set A to be $A = \{\emptyset, \{c\}, \{\{\emptyset\}\}\}$.
   (a) What is $\mathcal{P}(A)$?

   (b) What is $|\mathcal{P}(\mathcal{A}) \cup \emptyset|$?

9. Determine whether the following are **TRUE** or **FALSE**.
   (a) $\{0\} \in \{0\}$

   (b) $\{0\} \subset \{0\}$

   (c) $\emptyset \subseteq \{x\}$

   (d) $\{x\} \subseteq \{\{w\}, \{x\}, y\}$

10. Let A and B be sets. Is $A \times B = B \times A$? Prove your answer.

11. For each of the following, determine if a set is a subset, proper subset, or equal to the other set, or state that none of these properties can be inferred.
    (a) What can we say for the sets A and B if we know that $A \cup B = A$?

    (b) What can we say for the sets A and B if we know that $A - B = A$?

12. Determine which of the following functions are bijections from **R** to **R**
    (a) $f(x) = 2x + 1$

    (b) $f(x) = -3x^2 + 17$

    (c) $f(x) = x^5 + 1$

    (d) $f(x) = \frac{x^2+1}{x^2+2}$