FoCS: Exam 2 Review Questions

1. (Rosen 2.4: prob. 15) Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

   (a) $a_n = 5 \cdot (-1)^n - n + 2$

   (b) $a_n = 7 \cdot 2^n - n + 2$

2. For each of the recurrence relations below, write the solution without recursion.

   (a) $a_n = a_{n-1} + 2$ with $a_0 = 4$

   (b) $a_n = a_{n-1} + n$ with $a_0 = 5$

3. (Rosen 2.4: prob. 19) Suppose that the number of bacteria in a colony triples every hour.

   (a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.

   (b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

4. Find a closed formula for $\sum_{k=0}^{n} k(k+1)^2$.

5. Use mathematical induction to prove that $n^3 + 3n^2 + 2n$ is divisible by 3 for all integers $n \geq 1$.

6. Use mathematical induction to prove that $\sum_{j=1}^{n} \frac{1}{j^2} < 2 - \frac{1}{n}$ for all integers $n > 1$.

7. Let $\{d_n\}$ be the sequence defined by
   $d_1 = 9/10$
   $d_2 = 10/11$
   $d_n = d_{n-1} \cdot d_{n-2}$ for $n \geq 3$
   Use strong induction to show that $d_n \leq 1$ for all $n \geq 1$.

8. What is wrong with the following proof by strong induction?

   Theorem: For every non-negative integer n, 5n = 0.

   Basis Step: $5 \cdot 0 = 0$.

   Inductive Step: Assume that $5 \cdot j = 0$ for all non-negative integers $j = 0, 1, ..., k$.
   We will show that $5 \cdot (k+1) = 0$.
   We can write $k + 1 = i + j$, where i and j are natural numbers strictly less than $k + 1$. By the inductive hypothesis, $5 \cdot i = 5 \cdot j = 0$, and thus

   $$5 \cdot (k+1) = 5 \cdot (i+j) = 5i + 5j = 0.$$

9. (Rosen 5.3: prob. 9) Let F be the function such that $F(n)$ is the sum of the first n positive integers. Give a recursive definition of $F(n)$ for $n \geq 0$.

10. (Rosen 5.3: probs. 24 and 25) Give a recursive definition of

    (a) the set of positive integer powers of 3.

    (b) the set of positive integers not divisible by 5.

11. List five elements of the set S of strings with the following recursive definition.

    $10 \in S$

    if $w \in S$ then $w10 \in S$

12. (Rosen 5.2: prob. 39) When does a string belong to the set A of bit strings defined recursively by

    $\lambda \in A$

    if $x \in A$ then $0x1 \in A$

13. (Rosen 5.4: probs. 7 and 21) Give a recursive algorithm for computing nx where n is a positive integer and x is an integer, just using addition. Then, using induction, prove your algorithm is correct.

14. State the answer for each of the following counting problems.

    (a) How many bit strings (strings of zeros and ones) of length 14 contain exactly three ones?

    (b) How many bit strings (strings of zeros and ones) of length 14 contain at most three ones?

    (c) In how many ways can a photographer at a wedding arrange six people in a row, including the bride and the groom, if the bride must be next to the groom?

15. There are 800,000 pine trees in a forest. Each pine tree has no more than 600,000 needles. Show that at least two trees have the same number of needles.

16. (Rosen 6.4: prob. 9) What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x - 3y)^{200}$?

17. Let f be a function that maps from $\{1, 2, \ldots n\}$ to $\{0, 1, 2, \ldots n\}$.

    (a) How many such functions are there?

    (b) How many of these functions are injective?

    (c) How many of these functions are surjective?

18. How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$?