CSCI.6962/4962 Software Verification—Fundamental Proof Methods in Computer Science (Arkoudas and Musser)—Chapter 3.1-3.8

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Spring 2018

Proving Equalities

Goal: to become familiar with *equational proofs* as a type of inference.

- numeric equations
- equality chaining
- terms and sentences as trees
- logic behind equality chaining
- sample proofs
- mathematical induction
- *list equations*
- *polymorphic datatypes*
- *ground terms*
- *top-down proof development*

Datatype for natural numbers

```
datatype N := zero | (S N)
```

This defines a datatype `N` with two constructors: `zero` and `S`. The definition says that the values of sort `N` are:

```
zero
(S zero)
(S (S zero))
(S (S (S zero)))
...
```

- 0 is represented by `zero`,
- $n + 1$ is obtained by applying the "successor function" `S` to the natural number n,
- these are *the only values* that are natural numbers.

Functions over natural numbers

```
declare Plus: [N N] -> N [+]
```

- `Plus` takes two natural numbers as inputs and produces a natural number as output.
- The expression `[+]` at the end of the declaration overloads the built-in symbol + so that it can be used as an alias for `Plus` whenever the context allows it.

We can now write the following equivalent terms:

```
(Plus (S zero) zero)
```

```
((S zero) Plus zero)
```

```
(S zero Plus zero)
```

```
(S zero + zero)
```

after setting the precedence of `S` to be higher than `Plus`.

Reasoning about functions

We can express commutativity of addition as follows:

```
> (forall n m . n Plus m = m Plus n)

Sentence: (forall ?n:N
            (forall ?m:N
              (= (Plus ?n:N ?m:N)
                 (Plus ?m:N ?n:N))))
```

Axioms for + can be added to the global assumption base using universally quantified equations:

```
assert right-zero := (forall n . n + zero = n)

assert right-nonzero := (forall n m . n + S m = S (n + m))
```

Instantiating axioms

The meaning of (S zero + zero) is determined by the equation that is just the special case of right-zero with the ground term (S zero) substituted for n.

```
(!instance right-zero [(S zero)])
```

to which Athena responds:

```
Theorem: (= (Plus (S zero)
                  zero)
            (S zero))
```

That is, $1 + 0 = 1$. Likewise:

```
> (!instance right-nonzero [zero (S zero)])

Theorem: (= (Plus zero
                  (S (S zero)))
            (S (Plus zero
                     (S zero))))
```

The `instance` method

- The first argument to `instance` is a universally quantified sentence p in the assumption base.
- The second is a list L of terms.

If $p =$ `(forall` $v_1 \cdots v_n$ `.` q`)` and $L =$ `[`$t_1 \cdots t_k$`]`, where $k \leq n$, then `instance` produces the sentence

$$\texttt{(forall } v_{k+1} \cdots v_n \texttt{ . } q'\texttt{)}$$

where q' results from substituting t_i for v_i in q, for $i = 1, \ldots, k$.

```
> (!instance right-nonzero [zero])

Theorem: (forall ?v303:N
           (= (Plus zero
                    (S ?v303:N))
              (S (Plus zero ?v303:N))))
```

Equality chaining

What about the meaning of `Plus` for larger ground term inputs, like

`(S S zero + S S zero)`?

In other words, can we now deduce that $2 + 2 = 4$?

Yes, and here is one way to do it:

```
(!chain [(S S zero + S S zero)
       = (S (S S zero + S zero))    [right-nonzero]
       = (S S (S S zero + zero))    [right-nonzero]
       = (S S S S zero)             [right-zero]
        ])
```

Here we have used `chain`, an Athena method for proving equations by chaining together a sequence of terms connected by equalities.

The `chain method`

In general,

$$\texttt{(!chain [}t_0 \texttt{ = } t_1 \texttt{ [}p_1\texttt{] = } t_2 \texttt{ [}p_2\texttt{] = } \cdots \texttt{ = } t_n \texttt{ [}p_n\texttt{]])}$$

- attempts to derive the identity $\texttt{(}t_0 \texttt{ = } t_n\texttt{)}$,
- each p_i must be in the assumption base and
- each equation $\texttt{(}t_{i-1} \texttt{ = } t_i\texttt{)}$ must follow from p_i, typically by one of five fundamental axioms of equality, for $i = 1, \ldots, n$:
  - reflexivity
  - symmetry
  - transitivity
  - functional substitution, or
  - relational substitution

Equality chaining

```
(!chain [(S S zero + S S zero)
       = (S (S S zero + S zero))    [right-nonzero]
       = (S S (S S zero + zero))    [right-nonzero]
       = (S S S S zero)             [right-zero]
        ])
```

- $n = 3$,
- $p_1 = p_2 =$ `right-nonzero` and $p_3 =$ `right-zero`,
- Athena responds with the theorem proved:

```
Theorem: (= (Plus (S (S zero))
                  (S (S zero)))
            (S (S (S (S zero)))))
```

For each step i (from t_{i-1} to t_i), `[`p_i`]` is its *justification list* and each p_i is its *justifier*.

The logic behind equality chaining

A firm foundation for reasoning about equalities is provided by the basic *equality axioms*:

1. **Reflexivity:** $\forall\, x \,.\, x = x$.
2. **Symmetry:** $\forall\, x\, y \,.\, x = y \Rightarrow y = x$.
3. **Transitivity:** $\forall\, x\, y\, z \,.\, x = y \wedge y = z \Rightarrow x = z$.
4. **Functional Substitution:** For any n-arity function symbol f,

$$\forall\, x_1 \cdots x_n\, y_1 \cdots y_n \,.\, x_1 = y_1 \wedge \ldots \wedge x_n = y_n \Rightarrow$$

$$f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n).$$

5. **Relational Substitution:** For any n-arity relation symbol R,

$$\forall\, x_1 \cdots x_n\, y_1 \cdots y_n \,.\, x_1 = y_1 \wedge \ldots \wedge x_n = y_n \wedge R(x_1, \ldots, x_n) \Rightarrow$$

$$R(y_1, \ldots, y_n).$$

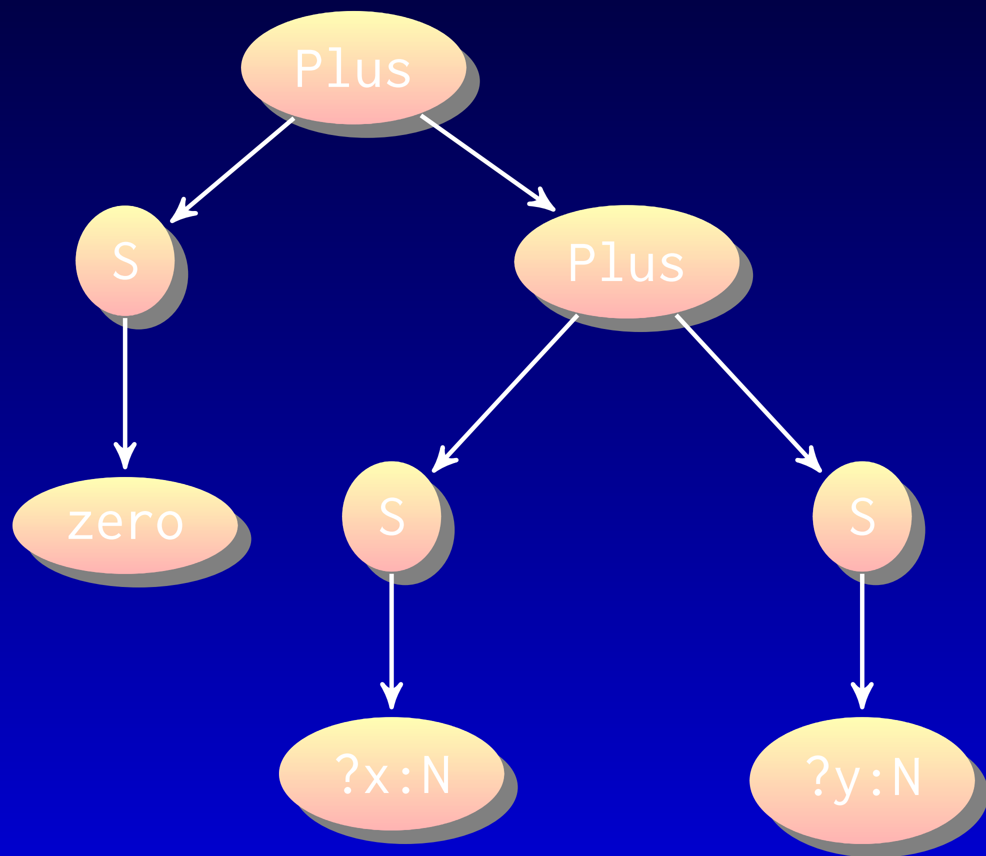
Terms and sentences as trees

Terms and sentences are tree-structured objects.

- A variable or a constant symbol can be viewed as a simple one-node tree (a *leaf* node)
- An application of the form $(f\ t_1\ \cdots\ t_n)$ for $n > 0$ can be viewed as a tree with
  - the symbol f at the root and
  - with the trees corresponding to $t_1, \ldots, t_n$ as its immediate subtrees, arranged from left to right in that order.

Terms and sentences as trees

The term `(Plus (S zero) (Plus (S x) (S y)))` can be depicted as follows:



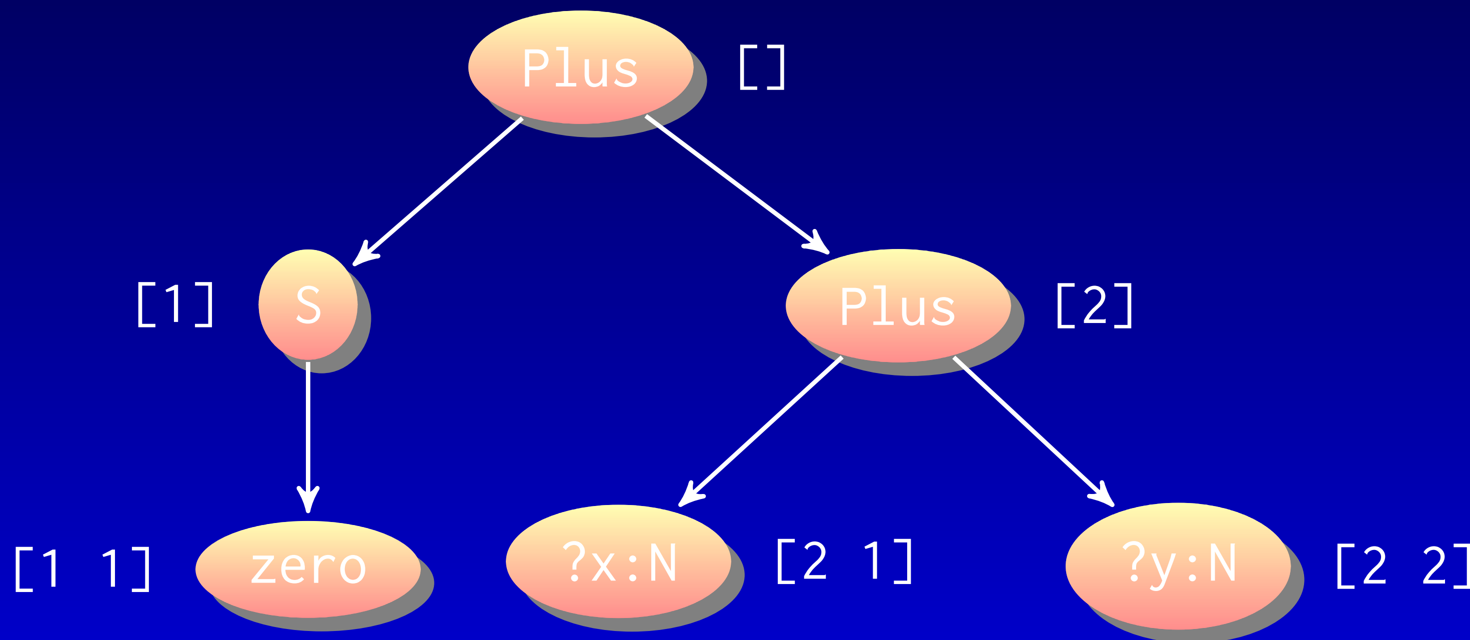
Abstract syntax

Tree representations of terms depict their *abstract syntax*:

- not specified whether prefix, infix, or postfix notation was used
- no argument separation syntax (e.g., commas, periods, spaces, indentation, etc.)
- these belong to a *concrete syntax*, e.g., Lisp prefix notation is used by Athena to output terms.

Dewey paths

Every node can be assigned a unique list of positive integers $[i_1 \ \cdots \ i_m]$ indicating the path that must be traversed in order to get from the root of the tree to the node in question, e.g., for `(Plus (S zero) (Plus x y))`:



These integer sequences are called *Dewey paths* (or Dewey positions).

Computing with Dewey paths

The Athena procedure `positions-and-subterms`

- takes a term and
- produces a list of all positions in the term, each paired in a sublist with the subterm at that position.

For example:

```
> (positions-and-subterms (Plus (S zero) (Plus x y)))

List: [[[] (Plus (S zero) (Plus ?x ?y))]
       [[1] (S zero)]
       [[1 1] zero]
       [[2] (Plus ?x ?y)]
       [[2 1] ?x]
       [[2 2] ?y]]
```

Computing with Dewey paths

`subterm` is a useful procedure that

- takes a term t and
- a position I (as a list of positive integers) and
- returns the subterm of t that is located at position I in the tree representation of t.

```
define (subterm t I) :=
  match I {
    [] => t
  | (list-of i rest) => (subterm (ith (children t) i)
                                 rest)
  }
```

The primitive binary procedure `ith` takes a list of $n > 0$ values `[`$V_1 \cdots V_n$`]` and an integer $i \in \{1, \ldots, n\}$ and returns V_i.

Computing with Dewey paths

If we only want the node at a given position, we can use the following procedure: `subterm-node`

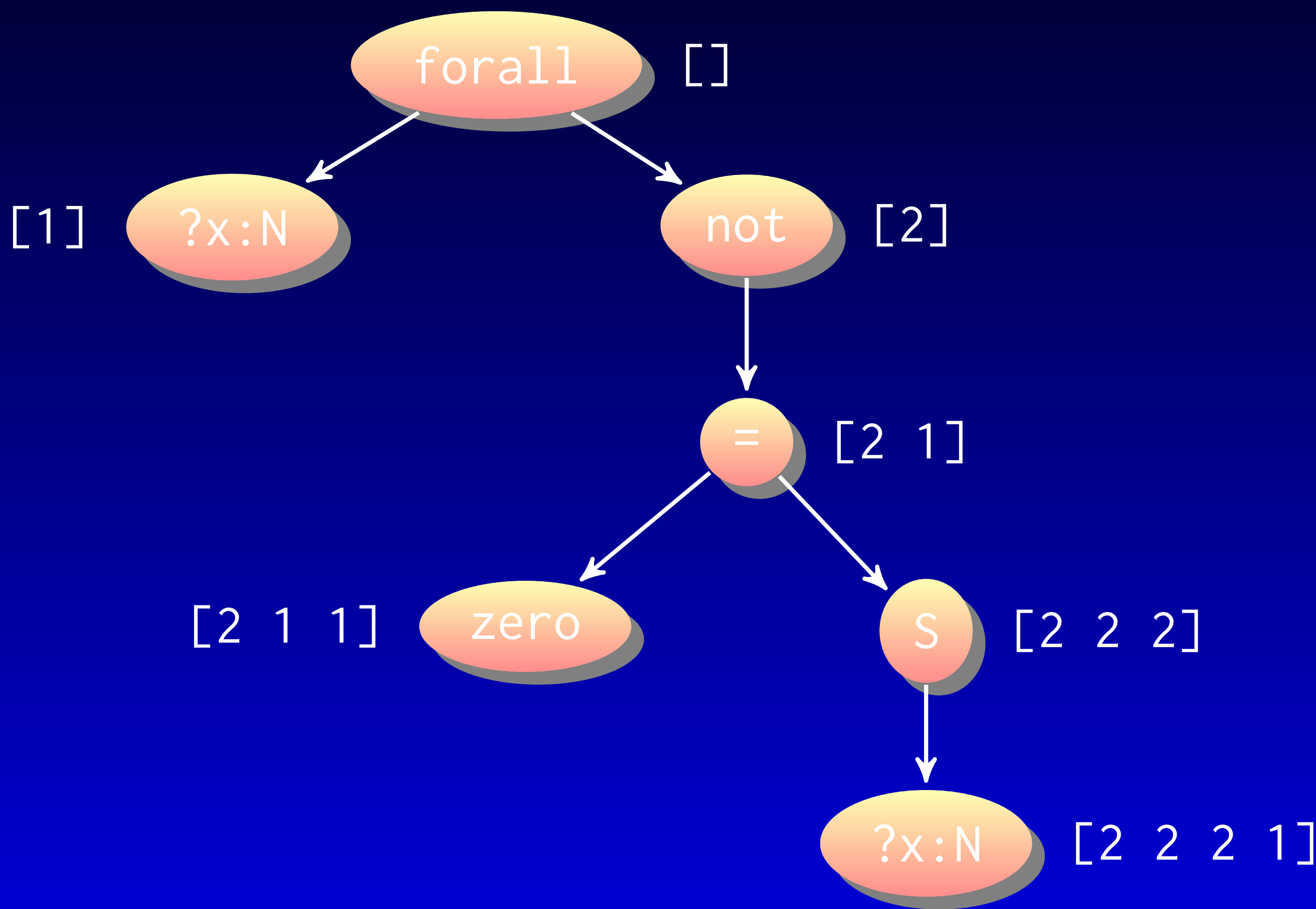
```
define (subterm-node t I) := (root (subterm t I))

> (subterm-node (x + S S zero) [2 1])

Symbol: S
```

Another useful procedure is `replace-subterm`, where (`replace-subterm` t I t')

- returns the term obtained from t by replacing the subterm at position I by t',
- provided that the result is well sorted.
- **Exercise:** Define the `replace-subterm` procedure.

Sentences as trees

Dewey paths also apply to sentences depicted as trees, e.g.:




```
(forall ?x (not (= zero (S ?x))))
```

More examples of equality chaining

We begin with the following property:

```
define left-zero := (forall n . zero + n = n)
```

which differs from `right-zero` in that `zero` appears as the first input to `Plus` rather than the second.

Recall we called it `Plus-S-property` in Chapter 1 (see induction.ath).

Similarly:

```
assert left-nonzero := (forall m n . (S n) + m = S (n + m))
```

We are treating `left-nonzero` as an axiom (we used `assert`) though it can be proven from `right-zero` and `right-nonzero`.

More examples of equality chaining

A multiplication function, `Times`, takes two natural numbers as inputs and returns their product:

```
declare Times: [N N] -> N [*]
```

We overloaded `*` to mean `Times` when applied to `N` arguments.

Next, the semantics:

```
assert Times-zero    := (forall x . x * zero = zero)
assert Times-nonzero := (forall x y . x * S y = x * y + x)
```

If we read (`S` n) as $n + 1$, the second axiom just says

$$x \cdot (y + 1) = x \cdot y + x.$$

Note that `*` has a built-in precedence greater than that of `+`, so that, for example, (x `*` y `+` z) is parsed as ((x `*` y) `+` z).

More examples of equality chaining

Let's also introduce a name one and give its meaning with an equation:

```
declare one: N
assert one-definition := (one = S zero)
```

The proof of the following property:

```
define Times-right-one := (forall x . x * one = x)
```

provides another simple illustration of equality chaining:

```
conclude Times-right-one
  pick-any x:N
    (!chain [(x * one)
           = (x * S zero)      [one-definition]
           = (x * zero + x)    [Times-nonzero]
           = (zero + x)        [Times-zero]
           = x                 [left-zero]])
```

More examples of equality chaining

Associativity of `Times`, which for the moment we will treat as an axiom:

```
assert Times-associative := (forall x y z . (x * y) * z = x * (y * z))
```

An exponentiation function, `**`, follows:

```
declare **: [N N] -> N [310]
```

We set the precedence of `**` higher than that of `*` (predefined as 300). For semantics, we write:

```
assert Power-right-zero    := (forall x . x ** zero = one)

assert Power-right-nonzero := (forall x n . x ** S n = x * x ** n)
```

or

$$x^{n+1} = x \cdot x^n$$

Power square theorem

Recall the following result from elementary algebra: $(x^2)^n = x^{2n}$

```
define power-square-theorem := (forall n x . (x * x) ** n = x ** (n + n))
```

If we define the following procedure:

```
define (power-square-property n) :=
  (forall x . (x * x) ** n = x ** (n + n))
```

we can express `power-square-theorem` as the proposition that every natural number has the `power-square-property`:

`(forall n . power-square-property n).`

Athena can verify that the two formulations are identical:

```
> (power-square-theorem equals? (forall n . power-square-property n))

Term: true
```

We call `power-square-property` a *property procedure*.

Power square theorem

So how do we go about proving `power-square-theorem`?

For this theorem, instantiating only the variable `n` for a few small values yields:

```
(forall x . (x * x) ** zero = x ** (zero + zero))

(forall x . (x * x) ** S zero = x ** (S zero + S zero))

(forall x . (x * x) ** S S zero = x ** (S S zero + S S zero))

(forall x . (x * x) ** S S S zero =
                x ** (S S S zero + S S S zero))

⋮
```

Power square theorem

The proof of `(power-square-property zero)` is simple:

```
conclude power-zero-case := (power-square-property zero)
  pick-any x:N
    (!chain [((x * x) ** zero)
           = one                          [Power-right-zero]
           = (x ** zero)                  [Power-right-zero]
           = (x ** (zero + zero))         [right-zero]])
```

Power square theorem

For `n` $\equiv$ `(S zero)`, first consider the following proof:

```
conclude power-one-case := (power-square-property (S zero))
  pick-any x:N
    (!combine-equations
      (!chain [((x * x) ** S zero)
              = ((x * x) * (x * x) ** zero)  [Power-right-nonzero]
              = ((x * x) * one)              [Power-right-zero]
              = (x * x)                      [Times-right-one]])
      (!chain [(x ** (S zero + S zero))
              = (x ** (S (S zero + zero)))   [right-nonzero]
              = (x ** (S S zero))            [right-zero]
              = (x * (x ** S zero))          [Power-right-nonzero]
              = (x * (x * (x ** zero)))      [Power-right-nonzero]
              = (x * x * one)                [Power-right-zero]
              = (x * x)                      [Times-right-one]]))
```

Power square theorem

The structure of this proof, combines *two* applications of `chain`:

$$\begin{tikzcd}[column sep=small]
t_0 \arrow[dr] & & & & & & u_0 \arrow[dl] \\
& t_1 \arrow[dr] & & & & u_1 \arrow[dl] & \\
& & \ddots \arrow[dr] & & \iddots \arrow[dl] & & \\
& & & t_n = u_m & & &
\end{tikzcd}$$

If we can rewrite each side of the equation to the very same term, then we can combine the two chain conclusions to obtain the proof.

The `combine-equations` method

The method `combine-equations` does just that:

`(!combine-equations (`s_0` = `s_n`) (`t_0` = `t_m`))`

proves `(`s_0` = `t_0`)` when

- both `(`s_0` = `s_n`)` and `(`t_0` = `t_m`)` are in the assumption base and
- s_n and t_m are identical.

`chain` allows the direction of rewriting to be indicated on each step:

- If `-->` is used, `chain` only attempts to rewrite left-to-right: $t_i \longrightarrow t_{i+1}$.
- If `<--` is used, it only attempts to rewrite right-to-left: $(t_{i+1} \longrightarrow t_i)$.
- If = is specified, `chain` first tries left-to-right, and if that fails, it tries right-to-left.

Power square theorem

The proof given for `power-one-case` can be shortened by taking advantage of the `power-zero-case` theorem, as follows:

```
conclude (forall x . (x * x) ** S zero = x ** (S zero + S zero))
  pick-any x:N
    (!combine-equations
      (!chain [((x * x) ** S zero)
            --> ((x * x) * ((x * x) ** zero))     [Power-right-nonzero]
            --> ((x * x) * (x ** (zero + zero)))  [power-zero-case]
            --> (x * x * x ** (zero + zero))      [Times-associative]])
      (!chain [(x ** (S zero + S zero))
            --> (x ** (S (S zero + zero)))        [right-nonzero]
            --> (x ** (S (S (zero + zero))))      [left-nonzero]
            --> (x * (x ** (S (zero + zero))))    [Power-right-nonzero]
            --> (x * x * x ** (zero + zero))
[Power-right-nonzero]]))
```

This proof is not just shorter, it can be generalized for *any* value of `n`.

Power square theorem

```
define power-square-step :=
  method (n)
    let {previous-result := (power-square-property n)}
      conclude (power-square-property (S n))
        pick-any x:N
          (!combine-equations
            (!chain [((x * x) ** S n)
                 --> ((x * x) * ((x * x) ** n))   [Power-right-nonzero]
                 --> ((x * x) * (x ** (n + n)))   [previous-result]
                 --> (x * x * (x ** (n + n)))     [Times-associative]])
            (!chain [(x ** (S n + S n))
                 --> (x ** (S (S n + n)))         [right-nonzero]
                 --> (x ** (S S (n + n)))         [left-nonzero]
                 --> (x * (x ** (S (n + n))))     [Power-right-nonzero]
                 --> (x * x * (x ** (n + n)))   [Power-right-nonzero]]))
```

if we apply `power-square-step` to n, we obtain theorem

`(power-square-property (S` n`))`

Power square theorem

Encapsulating the n $\equiv$ zero case in a separate method, power-square-base:

```
define power-square-base :=
  method ()
    conclude (power-square-property zero)
      pick-any x:N
        (!chain [((x * x) ** zero)
                   = one                        [Power-right-zero]
                   = (x ** zero)                [Power-right-zero]
                   = (x ** (zero + zero))       [right-zero]])
```

The following sequence of calls could be extended to obtain the proof of (power-square-property n) for *any* natural number n:

```
(!power-square-base)
(!power-square-step zero)
(!power-square-step (S zero))
```

$\vdots$

The principle of mathematical induction

To prove $\forall\, n \,.\, P(n)$ where n ranges over the natural numbers, it suffices to prove:

1. *Basis case*: $P(0)$.
2. *Induction step*: $\forall\, n \,.\, P(n) \Rightarrow P(n+1)$.

In the induction step, the antecedent assumption $P(n)$ is called the *induction hypothesis*.

The `by-induction` proof construct

This principle is embodied in Athena's `by-induction` proof construct. We can use it to prove `power-square-theorem` as follows:

```
by-induction power-square-theorem {
  zero  => (!power-square-base)
| (S n) => (!power-square-step n)
}
```

The keyword `by-induction` is followed by the sentence to be derived, which is a goal of the form

$$\forall\, n : N \;.\; P(n),$$

followed by a number of *clauses*, enclosed in curly braces and separated by `|`, expressing the cases that together are sufficient to complete the proof.

The `by-induction` proof construct

There are usually two clauses (there can be more):

- one that expresses the basis case, corresponding to $P(0)$, and
- the other expressing the induction step (or "inductive step"), corresponding to

$$\forall\, n \,.\, P(n) \Rightarrow P(n+1).$$

Each clause is essentially a pair consisting of

- a constructor pattern π_i that represents one of the cases of the inductive argument, and
- a corresponding subproof D_i.

The arrow keyword => separates π_i from D_i.

The subproof D_i will be evaluated in the original assumption base *augmented with all appropriate inductive hypotheses*.

Power square theorem

The induction-step sentence for our example can be written in Athena as follows:

```
(forall n . power-square-property n ==> power-square-property (S n))
```

If we were trying to prove this sentence from scratch, without the benefit of `by-induction`, we could do it with a proof along the following lines:

```
pick-any n:N
  assume induction-hypothesis := (power-square-property n)
    conclude (power-square-property (S n))
      (!power-square-step n)
```

But with `by-induction` it is not necessary to write this much detail: `pick-any`, `assume`, and `conclude` are implicit.

A schema for inductive proofs

```
# Start by defining a unary ``property procedure'':

define (P t) := ...

# Then use it to define a goal which says that
# every object has this property:

define goal := (forall n:N . P n)

# Finally, prove the goal by induction:

by-induction goal {
  zero  => conclude (P zero)
             (!basis-case ...)
| (n as (S m)) =>
    conclude (P n)           # Here the assumption base contains
      (!induction-step ...)  # the inductive hypothesis (P m).
}
```

A simpler proof by induction

Recall the `left-zero` property that we defined earlier:

```
define left-zero := (forall n . zero + n = n)
```

Here is a proof using `by-induction`:

```
by-induction left-zero {
  zero => conclude (zero + zero = zero)
            (!chain [(zero + zero) --> zero [right-zero]])
| (n as (S m)) =>
    conclude (zero + n = n)
      let {induction-hypothesis := (zero + m = m)}
        (!chain [(zero + S m)
             --> (S (zero + m))     [right-nonzero]
             --> (S m)              [induction-hypothesis]])
}
```