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Natural Number Orderings

Goal: to reason about properties of natural number ordering relations, and its application to ordered lists and binary search trees.

- properties of natural number ordering functions
  - trichotomy properties
  - transitive and asymmetric properties
  - less-equal properties
  - combining ordering and arithmetic

- natural number subtraction
- ordered lists
- binary search trees
Natural number ordering functions

We begin with properties of a strict ordering operator <. To define strict inequality on natural numbers, we introduce the following symbol declaration and module nested within module N. Within module N.Less we define three axioms:

```plaintext
extend-module N {
  open Plus
  declare <: [N N] -> Boolean [[int->nat int->nat]

  module Less {
    assert* def := [(zero < S n)
      (~ _ < zero)
      (S m < S n <===> m < n)]
    define [zero<S not-zero injective] := def
  } # close module Less
```
Natural number ordering functions

Within module N.Less we also define a number of sentences we intend to prove as theorems:

```plaintext
extend-module N {
    extend-module Less {
        define irreflexive := (forall n . ~ n < n)
        define <S := (forall n . n < S n)
        define =zero := (forall n . ~ zero < n ==> n = zero)
        define zero< := (forall n . n =/= zero <==> zero < n)
        define S1 := (forall x y . S x < y ==> x < y)
        define S2 := (forall x y . x < y ==> x < S y)
        define S4 := (forall m n . S m < n ==> exists n' . n = S n')
        define S-step := (forall x y . x < S y & x =/= y ==> x < y)
        define discrete := (forall n . ~ exists x . n < x & x < S n)
    }
}
```
Natural number ordering functions

Each of these defined properties can be proved by induction. The first two are quite easy (their proofs and those that follow are expressed within the scope of the module `N` extension, but outside the scope of module `N.Less`):

```plaintext
by-induction Less.irreflexive {
  zero => (!uspec Less.not-zero zero)
| (S n) => (!chain-> [(~ n < n) ==> (~ S n < S n) [Less.injective]])
}

by-induction Less.<S {
  zero => (!uspec Less.zero<S zero)
| (S n) => (!chain-> [(n < S n) ==> (S n < S S n) [Less.injective]])
}
```
Natural number ordering functions

More typically, proofs about $<$ require case splitting in one or both of the basis case and inductive step, and proof by contradiction is also frequently useful, as in the following proof of $\text{Less}\cdot S1$.

```
by-induction Less.S1 {
  zero =>
  conclude (forall y . S zero < y ==> zero < y)
  <D1>
  | (S n) =>
  let {ind-hyp := (forall y . S n < y ==> n < y)}
  conclude (forall y . S S n < y ==> S n < y)
  <D2>
}
```

where D1 and D2 deductions prove the basis and inductive cases respectively.
Natural number ordering functions

The basis case D1 proving \((\forall y . \text{S zero } < y \implies \text{zero } < y)\) follows:

```
pick-any \text{y}
  
  assume \text{Szero<y := (S zero < y)}

  (!two-cases
    assume \text{y=zero := (y = zero)}
    \(<\text{D1a}>\)
    assume \text{y!=zero := (y /= zero)}
    \(<\text{D1b}>\)
  )
```

where D1a and D1b deductions prove the \(y = 0\) and \(y \neq 0\) cases respectively.
Natural number ordering functions

The case D1a proving \((y = \text{zero} \implies \text{zero} < y)\) follows:

\[
\begin{align*}
& \text{(!by-contradiction (zero < y)} \\
& \text{assume (~ zero < y)} \\
& \text{let \{-Szero<y :=} \\
& \text{conclude (~ S zero < y)} \\
& \text{(!chain->)} \\
& \text{[true ==> (~ S zero < zero) [Less.not-zero] } \\
& \implies (~ S zero < y) [y=zero]}) \\
& \text{(!absurd Szero<y -Szero<y))}
\end{align*}
\]

and D1b proving \((y \neq \text{zero} \implies \text{zero} < y)\) follows:

\[
\begin{align*}
& \text{let \{has-predecessor :=} \\
& \text{(!chain-> [y!=zero} \\
& \implies (exists m . y = S m) [nonzero-S]])} \\
& \text{pick-witness m for has-predecessor} \\
& \text{(!chain-> [true ==> (zero < S m) [Less.zerom]]}} \\
& \implies (zero < y) [(y = S m)]])
\end{align*}
\]
Natural number ordering functions

The inductive case D2 proving

\((\forall y \cdot S \ S \ n \lt y \implies S \ n \lt y)\) follows:

<table>
<thead>
<tr>
<th>pick-any (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>assume (\text{Less} := (S \ S \ n \lt y))</td>
</tr>
<tr>
<td>(!two-cases)</td>
</tr>
<tr>
<td>assume ((y = \text{zero}))</td>
</tr>
<tr>
<td>&lt;D2a&gt;</td>
</tr>
<tr>
<td>assume (\text{nonzero} := (y \neq \text{zero}))</td>
</tr>
<tr>
<td>&lt;D2b&gt;</td>
</tr>
</tbody>
</table>

where D2a and D2b deductions prove the \(y = 0\) and \(y \neq 0\) cases respectively.
### Natural number ordering functions

The case D2a proving \((y = \text{zero} \implies S \ n < y)\) follows:

<table>
<thead>
<tr>
<th>by-contradiction ((S \ n &lt; y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>assume (\sim S \ n &lt; y)</td>
</tr>
<tr>
<td>let {not-Less :=</td>
</tr>
<tr>
<td>(\sim S S \ n &lt; \text{zero}) [Less.not-zero]</td>
</tr>
<tr>
<td>(\sim S S \ n &lt; y) [(y = \text{zero})]]]))</td>
</tr>
<tr>
<td>(!absurd Less not-Less)</td>
</tr>
</tbody>
</table>
Natural number ordering functions

The case D2b proving \((y \neq \text{zero} \implies S \ n < y)\) follows:

```
let {has-predecessor :=
    (!chain->
        [nonzero
            ==> (exists m . y = S m) [nonzero-S]])}

pick-witness m for has-predecessor

# we now have \((y = S m)\)

(!chain->
    [(S S n < y) ==> (S S n < S m) [(y = S m)]
        ==> (S n < m) [Less.injective]
        ==> (n < m) [ind-hyp]
        ==> (S n < S m) [Less.injective]
        ==> (S n < y) [(y = S m)]]))
```

Note that in both cases D1b and D2b, we prove the existentially quantified sentence \((\exists m . y = S m)\) and pick a witness for it. We will see a simpler way of writing these parts of the proof.
Natural number ordering functions

Consider Less.S2:

\[(\forall x \ y \ . \ x < y \implies x < S \ y)\]

The proof is much simpler:

by-induction Less.S2 {
  zero =>
    conclude (\forall y \ . \ zero < y \implies zero < S \ y)
    pick-any y
    assume (zero < y)
    (!uspec Less.zero<S y)
  | (S m) =>
    conclude (\forall y \ . \ S \ m < y \implies S \ m < S \ y)
    pick-any y
    (!chain [(S m < y)
      ==> (m < y) \ [Less.S1]
      ==> (S m < S y) \ [Less.injective]]))
}

Natural number ordering functions

Notice the inductive hypothesis was not used. In such case, we can use Athena’s datatype-cases form instead of by-induction.

```haskell
datatype-cases Less.S2 {
  zero =>
    conclude (forall y . zero < y ==> zero < S y)
    pick-any y
    assume (zero < y)
    (!uspec Less.zero<S y)
  | (S m) =>
    conclude (forall y . S m < y ==> S m < S y)
    pick-any y
    (!chain [(S m < y)
        ===> (m < y) [Less.S1]
      ]
      ===> (S m < S y) [Less.injective]])
}
```

Other than the initial keyword, everything is the same as before.
Natural number ordering functions

Reconsidering Less . S1, we can use datatype-cases for the case splitting within both the basis case and the inductive step, thus significantly simplifying the proof:

```
by-induction Less.S1 {
  zero =>
  <D1'>
| (S n) =>
  let {ind-hyp := (forall y . S n < y ==> n < y)}
  <D2'>
} # by-induction Less.S1
```

where D1' and D2' deductions prove the basis and inductive cases respectively.
Natural number ordering functions

The basis case D1' proving

\[(\forall y \cdot S\ zero < y \implies zero < y)\]

follows:

```
datatype-cases (\forall y \cdot S\ zero < y \implies zero < y) \{ 
  zero => assume less := (S\ zero < zero) 
  let {-less := (!uspec Less.not-zero (S\ zero))} 
    (!from-complements (zero < zero) 
      less 
      -less) 
  | (S m) => assume (S\ zero < S\ m) 
    (!uspec Less.zero<S\ m)
}
```
Natural number ordering functions

The inductive case D2' proof follows:

```plaintext
datatype-cases (forall y . S S n < y ==> S n < y) {
  zero =>
    assume less := (S S n < zero)
    let {-less := (!uspec Less.not-zero (S S n))}
    (!from-complements (S n < zero)
      less
      -less)
  | (S m) =>
    (!chain [(S S n < S m) ==> (S n < m) [Less.injective]
      ==> (n < m) [ind-hyp]
      ==> (S n < S m) [Less.injective]])
}
```
Trichotomy properties

- Strict inequality defines a *total ordering* on the natural numbers.
- The property expressing this is called *trichotomy*, reflecting the fact that one of three possibilities always holds.
- While it could be expressed as a disjunction of the three possibilities, other forms that are often more convenient in proofs are stated here:

```plaintext
extend-module Less {
  define trichotomy := (forall m n . ~ m < n & m /= n ==> n < m)
  define trichotomy1 := (forall m n . ~ m < n & ~ n < m ==> m = n)
  define trichotomy2 := (forall m n . m = n <=> ~ m < n & ~ n < m)
}
```
Trichotomy properties

The basis case of the proof of trichotomy by induction follows:

by-induction Less.trichotomy {
  zero =>
  pick-any n
  assume (~ zero < n & zero /= n)
  conclude (n < zero)
  let {has-pred := (!chain-> [(zero /= n) => (n /= zero) [sym]
  ==> (exists k . n = S k) [nonzero-S]])}
  pick-witness k for has-pred
  let {less := (!chain-> [true ==> (zero < S k) [Less.zero<S]
  ==> (zero < n) [(n = S k)]]); -less := (~ zero < n)}
  (!from-complements (n < zero) less -less)
Trichotomy properties

The inductive step follows:

| (S m) =>

let {ind-hyp := (forall n . ~ m < n & m =/= n ==> n < m)}

datatype-cases (forall n . ~ S m < n & S m =/= n ==> n < S m) {

zero => assume (~ S m < zero & S m =/= zero)

(!uspec Less.zero<S m)

| (S k) => assume A := (~ S m < S k & S m =/= S k)

(!chain->

[A ==> (~ m < k & m =/= k) [Less.injective

S-injective]]

==> (k < m) [ind-hyp]

==> (S k < S m) [Less.injective]]})
Transitive and asymmetric properties

The next collection of properties of $<$ concerns transitivity:

```plaintext
extend-module Less {
  define transitive := (forall x y z . x < y & y < z ==> x < z)
  define transitive1 := (forall x y z . x < y & ~ z < y ==> x < z)
  define transitive2 := (forall x y z . x < y & ~ x < z ==> z < y)
  define transitive3 := (forall x y z . ~ y < x & y < z ==> x < z)
}
```
Transitive and asymmetric properties

Proving transitivity by induction requires case splitting on zero and nonzero values for both $y$ and $z$.
Instead, we follow the approach of commuting universally quantified variables, and then use the chain method to deduce the original property as follows:

```
conclude Less.transitive
let {transitive0 :=

    # A version with the easiest-to-induct-on variable first:

    (forall z x y . x < y & y < z ==> x < z);

    <D>

    }

pick-any x y z

(!chain [(x < y & y < z) ==> (x < z) [transitive0]])
```

where $D$ corresponds to the deduction of the alternative property by induction.
Transitive and asymmetric properties

The deduction $D$ by induction on $z$ for

$$(\forall z \ x \ y . \ x < y & y < z \implies x < z)$$

follows (basis case):

```
_ := by-induction transitive0 {  
    zero =>                        
    pick-any x y                  
    assume (x < y & y < zero)     
    let {-y<0 := (!uspec Less.not-zero y)}  
    (!from-complements (x < zero) (y < zero) -y<0)  
  } | (S n) =>  
    <D'>  
}  
```

# close by-induction
Transitive and asymmetric properties

The deduction $D'$ for the inductive case $z = S\ n$, i.e.,

$$(\forall x\ y.\ x < y \land y < S\ n \Rightarrow x < S\ n)$$

follows:

```
let \{ind-hyp := (\forall x\ y.\ x < y \land y < n \Rightarrow x < n)\}

pick-any x y

assume (x < y \land y < S\ n)

conclude (x < S\ n)

let \{_ := conclude (x < n)\)

(!two-cases

assume (y = n)

(!chain-> [(x < y) \Rightarrow (x < n) [(y = n)]])

assume (y /= n)

(!chain-> [(y /= n) \

===> (y < S\ n \land y /= n) [\text{augment}]

===> (y < n) [\text{Less.S-step}]

===> (x < y \land y < n) [\text{augment}]

===> (x < n) [\text{ind-hyp}])])

(!chain-> [(x < n) \Rightarrow (x < S\ n) [\text{Less.S2}]])
```
Transitive and asymmetric properties

Another key property of $<$ is *asymmetry*:

```
extend-module Less {
    define asymmetric := (forall m n . m < n ==> ~ n < m)
}
```

With the transitive and irreflexive properties at hand to use as lemmas, Less.asymmetric doesn’t even require induction:

```
conclude Less.asymmetric
    pick-any x y
    assume (x < y)
    (!by-contradiction (~ y < x)
    assume (y < x)
    let {x<x := (!chain-> [(x < y & y < x)
    ==> (x < x)] [Less.transitive])};
    -x<x := (!uspec Less.irreflexive x})
    (!absurd x<x -x<x))
```
Transitive and asymmetric properties

The following property is an easy consequence of Less.S1 and irreflexivity:

```plaintext
extend-module Less {
  define S-not-< := (forall n . ~ S n < n)
}

conclude Less.S-not-<
  pick-any n
  (!by-contradiction (~ S n < n)
    assume (S n < n)
    (!absurd
      (!chain-> [(S n < n) ==> (n < n) [Less.S1]])
      (!uspec Less.irreflexive n))
```
Less-equal properties

Other ordering operators—≤, >, and ≥—can be defined in terms of < and =. We restrict our attention here to ≤:

declare <=: [N N] -> Boolean [[int->nat int->nat]]

module Less= {

  assert definition := (forall x y . x <= y <=> x < y | x = y)

  define Implied-by-< := (forall m n . m < n ==> m <= n)
  define Implied-by-equal := (forall m n . m = n ==> m <= n)
  define reflexive := (forall n . n <= n)
  define zero<= := (forall n . zero <= n)
  define S-zero-S-n := (forall n . S zero <= S n)
  define injective := (forall n m . S n <= S m <=> n <= m)
  define not-S := (forall n . ~ S n <= n)
  define S-not-equal := (forall k n . S k <= n ==> k /= n)
  define discrete := (forall m n . m < n ==> S m <= n)
}

Less-equal properties

More properties defined for \( \leq \):

```
extend-module Less= {
    define transitive := (forall x y z . x \leq y & y \leq z ==> x \leq z)
    define transitive1 := (forall x y z . x < y & y \leq z ==> x < z)
    define transitive2 := (forall x y z . x \leq y & y < z ==> x < z)
    define S1 := (forall n m . n \leq m ==> n < S m)
    define S2 := (forall n m . n \leq m ==> n \leq S m)
    define S3 := (forall n . n \leq S n)
    define trichotomy1 := (forall m n . ~n \leq m ==> m < n)
    define trichotomy2 := (forall m n . ~n < m ==> m \leq n)
    define trichotomy3 := (forall m n . n < m ==> ~m \leq n)
    define trichotomy4 := (forall m n . n \leq m ==> ~m < n)
    define trichotomy5 := (forall m n . m \leq n & n \leq m ==> m = n)
}
```
Less-equal properties

And even more properties defined for ≤:

```plaintext
extend-module Less= {
  define Plus-cancellation := (forall k m n . m + k <= n + k ==> m <= n)
  define Plus-k := (forall k m n . m <= n ==> m + k <= n + k)
  define Plus-k1 := (forall k m n . m <= n ==> m <= n + k)
  define k-Less= := (forall k m n . n = m + k ==> m <= n)
  define zero2 := (forall n . n <= zero ==> n = zero)
  define not-S-zero := (forall n . ~ S n <= zero)
  define S4 := (forall m n . S m <= n ==> exists n' . n = S n')
  define S5 := (forall n m . n <= S m & n /= S m ==> n <= m)
  define =zero := (forall m . m < one ==> m = zero)
  define zero<=one := (forall m . m = zero ==> m <= one)
}
```

Only Less=.definition needs to be asserted as an axiom; the rest are theorems provable from Less=.definition and properties of < and equality. (Induction is not necessary.)
Less-equal properties

The first two properties follow simply from the definition, but offer examples of using alternate in implication chains.

conclude Less=.Implied-by-<
    pick-any m n
    (!chain [(m < n) ==> (m < n | m = n) [alternate]
             ==> (m <= n) [Less=.definition]])

conclude Less=.Implied-by-equal
    pick-any m:N n:N
    (!chain [(m = n) ==> (m < n | m = n) [alternate]
             ==> (m <= n) [Less=.definition]])

conclude Less=.reflexive
    pick-any n
    (!chain-> [(n = n) ==> (n <= n) [Less=.Implied-by-equal]])
**Less-equal properties**

To prove $\mathtt{Less}=.\mathtt{zero}<=$, we split into zero and nonzero cases, for which the best tool at hand is `datatype-cases`:

<table>
<thead>
<tr>
<th></th>
<th><code>datatype-cases Less=.zero&lt;=</code> {</th>
</tr>
</thead>
<tbody>
<tr>
<td>zero =&gt;</td>
<td>(!uspec Less=.reflexive zero) # zero &lt;= zero</td>
</tr>
<tr>
<td>(S n) =&gt;</td>
<td>(!chain-&gt; [true ==&gt; (zero &lt; S n)] [Less.zero&lt;S]</td>
</tr>
<tr>
<td></td>
<td>==&gt; (zero &lt;= S n) [Less=.Implied-by-&lt;]]}</td>
</tr>
</tbody>
</table>
Less-equal properties

The proof of \texttt{Less=\.injective} provides a nice illustration of the power of equivalence chaining:

<table>
<thead>
<tr>
<th>conclude \texttt{Less=.injective}</th>
</tr>
</thead>
<tbody>
<tr>
<td>pick-any (n\ m)</td>
</tr>
<tr>
<td>(!chain [(S (n\ &lt;=\ S\ m)])</td>
</tr>
<tr>
<td>(\iff (S\ n\ &lt;\ S\ m\</td>
</tr>
<tr>
<td>(\iff (n\ &lt;\ m\</td>
</tr>
<tr>
<td>(\iff (n\ &lt;=\ m)) [\texttt{Less=.definition}]</td>
</tr>
</tbody>
</table>
Less-equal properties

Since Less=.not-S is a (quantified) negation, we use proof by contradiction:

\[
\begin{align*}
\text{conclude } \text{Less=.not-S} \\
pick\text{-any } n \\
\quad \text{(by-contradiction } (\sim S n \leq n) \\
\quad \text{assume } Sn\leq n := (S n \leq n) \\
\quad \text{let } \{\text{disjunction } := \\
\quad \quad \quad (\text{by-} \text{chain-> } [Sn\leq n \implies (S n < n \mid S n = n) [\text{Less=.definition}]])
\}
\quad \text{(cases disjunction}
\quad \quad \text{assume } Sn<n := (S n < n) \\
\quad \quad \quad \text{let } \{-Sn<n := (\text{by-} \text{chain-> } [\text{true } \implies (\sim Sn<n) [\text{Less.S-not-<}])})
\quad \quad \quad (\text{absurd } Sn<n \sim Sn<n)
\quad \quad \text{assume } Sn=n := (S n = n) \\
\quad \quad \text{let } \{-Sn=n := (\text{by-} \text{chain-> } [\text{true } \implies (\sim Sn=n) [\text{S-not-same}]})
\quad \quad (\text{absurd } Sn=n \sim Sn=n))
\end{align*}
\]
Less-equal properties

The following proof of Less=.discrete provides an interesting example of generating an existential property in order to contradict the negated one in Less.discrete:

```plaintext
conclude Less=.discrete
 .pick-any m n
  .assume (m < n)
  (by-contradiction (S m <= n)
    .assume -Sm<=n := (~ S m <= n)
    .let {in-between := (exists k . m < k & k < S m)}
      .absurd
        .chain-> [-Sm<=n
          ==> (~ (S m < n | S m = n)) [Less=.definition]
          ==> (~ S m < n & S m =/= n) [dm]
          ==> (n < S m) [Less.trichotomy]
          ==> (m < n & n < S m) [augment]
          ==> in-between [existence]]
        )
  )
) # ~ in-between
```
Less-equal properties

A proof of Less=.zero2 is possible using dsyl (disjunctive syllogism) and the proof of Less=.not-S-zero is a proof by contradiction:

```
conclude Less=.zero2
  pick-any n
  assume hyp := (n <= zero)
  (!dsyl
    (!chain-> [hyp ==> (n < zero | n = zero) [Less=.definition]])
    (!uspec Less.not-zero n))  # (~ n < zero)

conclude Less=.not-S-zero
  pick-any n
  (!by-contradiction (~ S n <= zero)
    assume hyp := (S n <= zero)
    (!absurd
      (!chain-> [hyp ==> (S n = zero) [Less=.zero2]])
      (!uspec S-not-zero n)))  # (S n /= zero)
```
Combining ordering and arithmetic

Consider interaction between ordering operators and addition:

```plaintext
eextend-module Less {
    define Plus-cancellation := (forall k m n . m + k < n + k ==> m < n)
    define Plus-k := (forall k m n . m < n ==> m + k < n + k)
}
by-induction Less.Plus-cancellation {
    zero =>
        pick-any m n
        (!chain [(m + zero < n + zero)
            ==> (m < n) ] [Plus.right-zero]])
    | (S k) =>
        let {induction-hypothesis := (forall m n . m + k < n + k ==> m < n)}
        pick-any m n
        (!chain [(m + S k < n + S k)
            ==> (S (m + k) < S (n + k)) [Plus.right-nonzero]
            ==> (m + k < n + k) [Less.injective]
            ==> (m < n) [induction-hypothesis]])
}
```
Combining ordering and arithmetic

For the proof of \texttt{Less.Plus-k} we use \texttt{Less=.Plus-cancellation} as well as a couple of the trichotomy properties.

```
conclude Less.Plus-k
pick-any \, k \, m \, n
assume hyp1 := (m < n)
let \{goal := (m + k < n + k)\}
(!by-contradiction goal
(!chain [(~ goal)
  ==> (n + k <= m + k) [Less=.trichotomy2]
  ==> (n <= m) [Less=.Plus-cancellation]
  ==> (~ m < n) [Less=.trichotomy4]
  ==> (hyp1 & ~ m < n) [augment]
  ==> false [prop-taut]]))
```

Similar properties hold between ordering relations and natural number multiplication. See the Athena library file lib/main/nat-less.ath.