

CSCI-1200 Data Structures — Spring 2021

Lecture 7 — Order Notation & Basic Recursion

Today's Lecture

- Algorithm Analysis, Formal Definition of Order Notation
- Simple recursion, Visualization of recursion, Iteration vs. Recursion
- “Rules” for writing recursive functions, Lots of examples!

7.1 Algorithm Analysis

Why should we bother?

- We want to do better than just implementing and testing every idea we have.
- We want to know why one algorithm is better than another.
- We want to know the best we can do. (This is often quite hard.)

How do we do it? There are several options, including:

- Don't do any analysis; just use the first algorithm you can think of that works.
- Implement and time algorithms to choose the best.
- Analyze algorithms by counting operations while assigning different weights to different types of operations based on how long each takes.
- Analyze algorithms by assuming each operation requires the same amount of time. Count the total number of operations, and then multiply this count by the average cost of an operation.

7.2 Exercise: Counting Example

- Suppose `arr` is an array of `n` doubles. Here is a simple fragment of code to sum of the values in the array:

```
double sum = 0;
for (int i=0; i<n; ++i)
    sum += arr[i];
```

- What is the total number of operations performed in executing this fragment? Come up with a function describing the number of operations *in terms of* n .

7.3 Exercise: Which Algorithm is Best?

A venture capitalist is trying to decide which of 3 startup companies to invest in and has asked for your help. Here's the timing data for their prototype software on some different size test cases:

n	foo-a	foo-b	foo-c
10	10 u-sec	5 u-sec	1 u-sec
20	13 u-sec	10 u-sec	8 u-sec
30	15 u-sec	15 u-sec	27 u-sec
100	20 u-sec	50 u-sec	1000 u-sec
1000	?	?	?

Which company has the “best” algorithm?

7.4 Order Notation Definition

In this course we will focus on the intuition of order notation. This topic will be covered again, in more depth, in later computer science courses.

- Definition: Algorithm A is order $f(n)$ — denoted $O(f(n))$ — if constants k and n_0 exist such that A requires no more than $k * f(n)$ time units (operations) to solve a problem of size $n \geq n_0$.
- For example, algorithms requiring $3n + 2$, $5n - 3$, and $14 + 17n$ operations are all $O(n)$.

This is because we can select values for k and n_0 such that the definition above holds. (What values?)

Likewise, algorithms requiring $n^2/10 + 15n - 3$ and $10000 + 35n^2$ are all $O(n^2)$.

- Intuitively, we determine the order by finding the *asymptotically dominant term (function of n)* and throwing out the leading constant. This term could involve logarithmic or exponential functions of n . Implications for analysis:
 - We don't need to quibble about small differences in the numbers of operations.
 - We also do not need to worry about the different costs of different types of operations.
 - We don't produce an actual time. We just obtain a rough count of the number of operations. This count is used for comparison purposes.
- In practice, this makes analysis relatively simple, quick and (sometimes unfortunately) rough.

7.5 Common Orders of Magnitude

- $O(1)$, *a.k.a. CONSTANT*: The number of operations is independent of the size of the problem. e.g., compute quadratic root.
- $O(\log n)$, *a.k.a. LOGARITHMIC*. e.g., dictionary lookup, binary search.
- $O(n)$, *a.k.a. LINEAR*. e.g., sum up a list.
- $O(n \log n)$, e.g., sorting.
- $O(n^2)$, $O(n^3)$, $O(n^k)$, *a.k.a. POLYNOMIAL*. e.g., find closest pair of points.
- $O(2^n)$, $O(k^n)$, *a.k.a. EXPONENTIAL*. e.g., Fibonacci, playing chess.

7.6 Exercise: A Slightly Harder Example

- Here's an algorithm to determine if the value stored in variable `x` is also in an array called `foo`. Can you analyze it? What did you do about the `if` statement? What did you assume about where the value stored in `x` occurs in the array (if at all)?

```
int loc=0;
bool found = false;
while (!found && loc < n) {
    if (x == foo[loc]) found = true;
    else loc++;
}
if (found) cout << "It is there!\n";
```

7.7 Best-Case, Average-Case and Worst-Case Analysis

- For a given fixed size array, we might want to know:
 - The fewest number of operations (best case) that might occur.
 - The average number of operations (average case) that will occur.
 - The maximum number of operations (worst case) that can occur.
- The last is the most common. The first is rarely used.
- On the previous algorithm, the best case is $O(1)$, but the average case and worst case are both $O(n)$.

7.8 Approaching An Analysis Problem

- Decide the important variable (or variables) that determine the “size” of the problem. For arrays and other “container classes” this will generally be the number of values stored.
- Decide what to count. The order notation helps us here.
 - If each loop iteration does a fixed (or bounded) amount of work, then we only need to count the number of loop iterations.
 - We might also count specific operations. For example, in the previous exercise, we could count the number of comparisons.
- Do the count and use order notation to describe the result.

7.9 Exercise: Order Notation

For each version below, give an order notation estimate of the number of operations as a function of n :

- | | | |
|---|--|---|
| 1. <pre>int count=0; for (int i=0; i<n; ++i) for (int j=0; j<n; ++j) ++count;</pre> | 2. <pre>int count=0; for (int i=0; i<n; ++i) ++count; for (int j=0; j<n; ++j) ++count;</pre> | 3. <pre>int count=0; for (int i=0; i<n; ++i) for (int j=i; j<n; ++j) ++count;</pre> |
|---|--|---|

7.10 Recursive Definitions of Factorials and Integer Exponentiation

- Factorial is defined for non-negative integers as:
- Computing integer powers is defined as:

$$n! = \begin{cases} n \cdot (n-1)! & n > 0 \\ 1 & n == 0 \end{cases}$$

$$n^p = \begin{cases} n \cdot n^{p-1} & p > 0 \\ 1 & p == 0 \end{cases}$$

- These are both examples of *recursive definitions*.

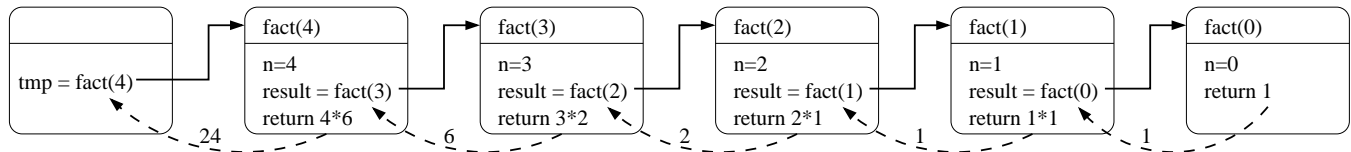
7.11 Recursive C++ Functions

C++, like other modern programming languages, allows functions to call themselves. This gives a direct method of implementing recursive functions. Here are the recursive implementations of factorial and integer power:

<pre>int fact(int n) { if (n == 0) { return 1; } else { int result = fact(n-1); return n * result; } }</pre>	<pre>int intpow(int n, int p) { if (p == 0) { return 1; } else { return n * intpow(n, p-1); } }</pre>
--	---

7.12 The Mechanism of Recursive Function Calls

- For each recursive call (or any function call), a program creates an *activation record* to keep track of:
 - **Completely separate instances** of the parameters and local variables for the newly-called function.
 - The location in the calling function code to return to when the newly-called function is complete. (Who asked for this function to be called? Who wants the answer?)
 - Which activation record to return to when the function is done. For recursive functions this can be confusing since there are multiple activation records waiting for an answer from the same function.
- This is illustrated in the following diagram of the call `fact(4)`. Each box is an activation record, the solid lines indicate the function calls, and the dashed lines indicate the returns. Inside of each box we list the parameters and local variables and make notes about the computation.



- This chain of activation records is stored in a special part of program memory called *the stack*.

7.13 Iteration vs. Recursion

- Each of the above functions could also have been written using a `for` or `while` loop, i.e. *iteratively*. For example, here is an iterative version of factorial:

```
int ifact(int n) {  
    int result = 1;  
    for (int i=1; i<=n; ++i)  
        result = result * i;  
    return result;  
}
```

- Often writing recursive functions is more natural than writing iterative functions, especially for a first draft of a problem implementation.
- You should learn how to recognize whether an implementation is recursive or iterative, and practice rewriting one version as the other. Note: We'll see that not all recursive functions can be *easily* rewritten in iterative form!
- Note: The order notation for the number of operations for the recursive and iterative versions of an algorithm is usually the same. However in C, C++, Java, and some other languages, *iterative functions are generally faster than their corresponding recursive functions*. This is due to the overhead of the function call mechanism. Compiler optimizations will sometimes (but not always!) reduce the performance hit by automatically eliminating the recursive function calls. This is called *tail call optimization*.

7.14 Exercises

1. Draw a picture to illustrate the activation records for the function call

```
cout << intpow(4, 4) << endl;
```

2. Write an iterative version of `intpow`.
3. What is the order notation for the two versions of `intpow`?

7.15 Rules for Writing Recursive Functions

Here is an outline of five steps that are useful in writing and debugging recursive functions. Note: You don't have to do them in exactly this order...

1. Handle the base case(s).
2. Define the problem solution in terms of smaller instances of the problem. Use *wishful thinking*, i.e., if someone else solves the problem of `fact(4)` I can extend that solution to solve `fact(5)`. This defines the necessary recursive calls. It is also the hardest part!
3. Figure out what work needs to be done before making the recursive call(s).
4. Figure out what work needs to be done after the recursive call(s) complete(s) to finish the computation. (What are you going to do with the result of the recursive call?)
5. Assume the recursive calls work correctly, but make sure they are progressing toward the base case(s)!

7.16 Location of the Recursive Call — Example: Printing the Contents of a Vector

- Here is a function to print the contents of a vector. Actually, it's two functions: a *driver function*, and a true recursive function. It is common to have a driver function that just initializes the first recursive function call.

```
void print_vec(std::vector<int>& v, unsigned int i) {
    if (i < v.size()) {
        cout << i << ": " << v[i] << endl;
        print_vec(v, i+1);
    }
}

void print_vec(std::vector<int>& v) {
    print_vec(v, 0);
}
```

- What will this print when called in the following code?

```
int main() {
    std::vector<int> a;
    a.push_back(3); a.push_back(5); a.push_back(11); a.push_back(17);
    print_vec(a);
}
```

- How can you change the second `print_vec` function as little as possible so that this code prints the contents of the vector in reverse order?

7.17 Fibonacci Optimization: Order Notation of Time vs. Space

The Fibonacci sequence is defined:

$$\text{Fib}_n = \begin{cases} 1 & n == 0 \\ 1 & n == 1 \\ \text{Fib}_{n-1} + \text{Fib}_{n-2} & n > 1 \end{cases}$$

1. Write a *simple* recursive version of Fibonacci that is a direct translation of the mathematical definition of the Fibonacci sequence.
2. Write an *iterative* version of Fibonacci that uses a vector to improve the running time of the function.
3. What is the order notation of the *running time* for each version?
4. What is the order notation of the *memory usage* for each version?

7.18 Binary Search

- Suppose you have a `std::vector<T> v` (for a placeholder type T), sorted so that:

`v[0] <= v[1] <= v[2] <= ...`

- Now suppose that you want to find if a particular value x is in the vector somewhere. How can you do this without looking at every value in the vector?
- The solution is a recursive algorithm called *binary search*, based on the idea of checking the middle item of the search interval within the vector and then looking either in the lower half or the upper half of the vector, depending on the result of the comparison.

```
template <class T>
bool binsearch(const std::vector<T> &v, int low, int high, const T &x) {
    if (high == low) return x == v[low];
    int mid = (low+high) / 2;
    if (x <= v[mid])
        return binsearch(v, low, mid, x);
    else
        return binsearch(v, mid+1, high, x);
}
template <class T>
bool binsearch(const std::vector<T> &v, const T &x) {
    return binsearch(v, 0, v.size()-1, x);
}
```

7.19 Exercises

1. Write a non-recursive version of binary search.
2. If we replaced the if-else structure inside the recursive binsearch function (above) with

```
if ( x < v[mid] )
    return binsearch( v, low, mid-1, x );
else
    return binsearch( v, mid, high, x );
```

would the function still work correctly?