Loops
Weakest Precondition and Loops

• We would like to be able to find the weakest precondition \( \{P\} \):

\[
\begin{array}{l}
\{P\} \\
\text{while}(b) \{ \\
\quad S; \\
\} \\
\} \{Q\}
\end{array}
\]

• \( \{P\} \text{ while}(b) S; \ {Q} \) is a Hoare triple

• It turns out that computing the weakest precondition for loops is, in general, a hard problem

• Instead, we'll assume we can find an invariant for the loop
  • Something that gives us information about the loop and can be relied upon to be true before and after each the execution of the loop
Weakest Precondition for Loops

• If we knew how many iterations, we could unroll the loop.
  • Compiler optimization does this

• In general, finding the weakest precondition is complicated even for simple loops

• {??} while(x > 0) x = x-1; {x = 0}
• WP = ¬(x > 0) → x = 0 ^ (x > 0) →[(¬(x-1 > 0) → x-1 = 0 ^ (x-1 > 0) →[¬(x-2 > 0) → x-2 = 0 ^ (x-2 > 0) →[...]
  • When do we stop expanding the loop into a logical condition?
Reasoning about loops is a bit more complicated than reasoning about sequence or if ... else ...
-- Unknown number of iterations and unknown number of paths
-- Recursion adds an additional level of complexity

Instead we will use a loop invariant to reason about a loop

Two things to prove about loops:
-- It computes correct values (partial correctness)
    That is, the postcondition holds on loop exit
-- It terminates (it is not an infinite loop)

**Total correctness = Partial correctness + Loop termination**
Loop Invariant

A loop invariant is a property of a program loop
- That is true before 1st iteration of the loop
- That is true after each iteration.
  - Not necessarily between statements in the loop
- It is a logical assertion
  - Abstract specification of the loop
  - A statement about the loop

To show partial correctness
- Loop exit condition and the LI must imply the desired postcondition
- That is, if the loop exits, the correct result is calculated

Why do we care?
- If we have an LI that implies the postcondition at exit, we can be somewhat confident that the loop computes the correct result

How do we show partial correctness?
- Induction
Reasoning about Loops

PRECONDITION: \{ x \geq 0 \} // assume all variables are ints

```c
i = x;
z = 0;

{ LOOP INVARIANT (LI): i + z = x \ ^\ i \geq 0 } 
while (i > 0) {
   z = z + 1;
i = i - 1;
}
```

POSTCONDITION: x = z

Questions:
(A) Is LI true before 1st iteration?
(B) Is LI true after each iteration?
(C) If loop terminates, do loop exit condition and LI imply postcondition?
(D) Does the loop terminate?
Reasoning about Loops

Proof by Induction
(1) BASE CASE: Initially, \( i = x \) and \( z = 0 \) gives us \( i+z=x \), i.e.,
LI holds at iteration 0 (before the loop code executes)
from precondition \( x \geq 0 \land i = x \Rightarrow i \geq 0 \)

(2) INDUCTION: Assuming \( i+z=x \) holds after iteration \( k \), we show
that \( i+z=x \) holds after iteration \( k+1 \)

\[
\begin{align*}
\text{z\_new} &= z + 1 \quad \text{and} \quad \text{i\_new} = i - 1 \\
\text{therefore, i\_new} + \text{z\_new} &= i - 1 + z + 1 = i + z = x \\
\text{at iteration k, i} > 0 \text{ or we have exited loop, i\_new} = i-1 \land i > 0 \Rightarrow \\
i\_new + 1 > 0 = (i\_new \geq 0)
\end{align*}
\]

(3) If the loop terminates, we know \( i = 0 \).

\[
\begin{align*}
( \neg(i>0) \land (i \geq 0 \land i+z=x)) \Rightarrow (i = 0 \land i+z = x) \Rightarrow (z = x) \\
\text{we have z} = x \text{ (i.e., the POSTCONDITION)}
\end{align*}
\]

(4) How do we know if the loop terminates?
-- the PRECONDITION \( x \geq 0 \) guarantees that \( i \geq 0 \) before the loop.
At every iteration, \( i \) decreases by 1, thus it eventually reaches 0
We will get a bit more formal about this in a while.
Reasoning about Loops

Reasoning about Loops using Induction
-- \(i+z=x\) is a loop invariant, meaning that it holds true before the loop and also after each/every iteration of the loop

-- even though \(i\) and \(z\) change within the loop code,
  \(i+z=x\) stays true at the END of each iteration
  true at the closing "\}" of the loop

-- Above we made an inductive argument over the number of iterations of the given loop

-- Proof by Induction -- also called Computational Induction

-- Establish that the LI holds before iteration 0

-- Assuming LI holds after iteration \(k\), show that it holds after iteration \(k+1\)
Loop Invariant

{ P }  // Hoare triple
while (b) S;
{ Q }

Find an invariant, LI, such that
1. $P \implies LI$  // true initially
2. $\{ LI \& b \} \ S \{ LI \}$  // true if the loop executes
3. $\{ LI \& \neg b \} \implies Q$  // establishes the postcondition

Finding the invariant is the key to reasoning about loops.

Inductive assertions are a “complete method of proof”
Reasoning about Loops

Partial Correctness
-- Establish and prove the loop invariant (LI) using computational induction

-- Loop exit condition and the LI must imply the desired postcondition
  -- ¬ (i > 0) (loop exit condition) and (i ≥ 0 ^ i+z=x) (LI) imply z=x

Termination
-- Establish some decrementing function D such that
  D = minimum value implies loop exit condition
  \[ D = \text{minimum} \implies \neg b \]
  b is the loop condition
  D decreases at each loop iteration.
  Show that D reaches its minimum
  Ideally minimum D = 0
Example

**precondition:** arr != null ^ arr.length = len ^ len >= 0; assume ints

```java
int sum = 0;
int i = 0;
while ( i < len ) {
    sum = sum + arr[i];
    i = i + 1;
}
```

**postcondition:** (result is the sum of all elements in array arr)

```
sum = arr[0] + arr[1] + ... + arr[arr.length-1]
```
LI: \( i \leq \text{len} \land \text{sum} = \text{arr}[0] + \ldots + \text{arr}[i-1] \)

(1) BASE CASE: does the LI hold before the loop?

\[ i \leq \text{len} \land \text{sum} = \text{arr}[0] + \ldots + \text{arr}[i-1] \]

the LI holds, given that \( i = 0 \) and that no values from the array \( \text{arr} \) have been summed yet. sum is initially 0. 
\( (i \leq \text{len}) = (0 \leq \text{len}) \) by precondition

(2) INDUCTION: assume the LI holds at iteration \( k \), does it hold at iteration \( k+1 \)?

\[ \text{sum}_\text{new} = \text{sum} + \text{arr}[i] = \text{arr}[0] + \ldots + \text{arr}[i-1] + \text{arr}[i] \]
\[ i_{\text{new}} = i + 1 \]
\[ \text{sum}_\text{new} = \text{arr}[0] + \ldots + \text{arr}[i_{\text{new}}-1] \]

\( i_{\text{new}} \leq \text{len} \) also holds; \( i < \text{len} \) at iteration \( k \).

If \( i = \text{len} \) at iteration \( k \), there would be no iteration \( k+1 \)

(3) LI \( \land \neg b \Rightarrow \) postcondition

\[ i \leq \text{len} \land \text{sum} = \text{arr}[0] + \ldots + \text{arr}[i-1] \land \neg (i < \text{len}) \]
\[ \Rightarrow (i = \text{len}) \land \text{sum} = \text{arr}[0] + \ldots + \text{arr}[i-1] \]
\[ \Rightarrow \text{sum} = \text{arr}[0] + \ldots + \text{arr}[\text{len}-1] \]
\[ \Rightarrow \text{sum} = \text{arr}[0] + \ldots + \text{arr}[\text{arr}.\text{length}-1] \]  // by precondition
Does loop terminate?

Define $D = len - i$ // initially $i = 0$ and $len \geq 0$, $D \geq 0$

Loop can be rewritten:

```java
while((len-i) > 0) { // i.e. while(D > 0)
    sum = sum + arr[i];
    i = i + 1; // D_new = len - (i+1) = (len-i)-1 = D - 1
}
```

$D$ decreases by 1 with each step.

$D$ eventually reaches 0.

$D = 0$ => loop exit condition $i=len$

When $D = 0$, loop exits
What A Loop Invariant Is Not

A loop invariant is not just some statement that is true before, during, and after the loop. It must be effective.

LI ^ exit condition => postcondition

For example,

// precondition: x > 0
x = 10;
y = 0;
z = 42;  // LI: z = 42, D=x

while(x > 0) {
    x = x - 1;
y = y + 1;
}
// postcondition: y=x

z is always 42, but it has nothing to do with the loop. It is not a valid or useful loop invariant. Exit condition and LI do not imply postcondition
What is a LI?

**Precondition:** $x \geq 0 \land y = 0$

```java
while(x != y) {
    y = y + 1;
}
```

**Postcondition:** $x = y$

Assume ints.
Since initially $x \geq 0 \land y = 0$ we can rewrite the loop:

$D = x - y \geq 0$

```java
while((x - y) > 0) {
    y = y + 1;
}
```

At the end $D = 0 \Rightarrow x - y = 0 \Rightarrow x = y$

Initially, $x \geq 0 \land y = 0 \Rightarrow x \geq y$ (good guess?)

We want to show by induction:
Assume at iteration $k$: $x \geq y_k$.
But if $x = y_k$, we would exit so $x > y_k$

- $y_{new} = y_k + 1$
- $x > y_k \Rightarrow x \geq y_{new}$

**LI:** $x \geq y$

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Check the LI

Precondition: \( x \geq 0 \land y = 0 \)

while(\( x \neq y \)) {
    \( y = y + 1; \)
}

Postcondition \( x = y \)

LI: \( x \geq y \)

Base case:
\( x \geq 0 \land y = 0 \Rightarrow x \geq y \)

Assume: \( x \geq y \) holds at iteration \( k \)
If \( x = y \) at iteration \( k \), we would exit loop
\( x > y \) at iteration \( k \)
\( y_{\text{new}} = y + 1 \)
\( x \geq y_{\text{new}} \)

At exit: \( !(x \neq y) \land x \geq y \Rightarrow x = y \)

\( D = x - y \)
\( D_{\text{new}} = D - (y+1) = D - 1 \)  // \( D \) decreases at each iteration
\( D = 0 \Rightarrow x = y \)
What is a weaker LI?

Precondition: \( x \geq 0 \land y = 0 \)

```plaintext
while(x != y) {
    y = y + 1;
}
```

Postcondition: \( x = y \)

Notice that the negation of the loop condition immediately implies the postcondition.

A better (weaker) invariant: \( \text{true} \).

LI: \( \text{true} \)

Base case:
\[ x \geq 0 \land y = 0 \Rightarrow \text{true} \]

Assume: \( \text{true} \) holds at iteration \( k \),
\( \text{true} \) also holds after iteration \( k+1 \).

At exit:
\[ !(x=y) \land \text{true} \Rightarrow x = y \]

\[ D = x - y \]
\[ D_{\text{new}} = D - (y+1) = D - 1 \]  // \( D \) decreases at each iteration
\[ D = 0 \Rightarrow x = y \]
Example

Assume ints
PRECONDITION: n >= 0

i = 0;
r = 1;

while ( i < n ) {
  i = i + 1;
  r = r * i;
}

POSTCONDITION: r = n!
what is the LI here?
PRECONDITION: \( n \geq 0 \)

\[ i = 0; \]
\[ r = 1; \]

while \( i < n \) {
  \[ i = i + 1; \]
  \[ r = r * i; \]
}

POSTCONDITION: \( r = n! \)

\[ \text{POSTCONDITION: } r = n! \]
\[ D = n - i \]
\[ \text{Li: } r = i! \land i \leq n \]

show the above to be true in terms of Partial Correctness

BASE CASE: \( i = 0 \) and \( r = 1 \)
\[ (r = i!) = (r = 0!) = (1 = 0!) \]
\[ (i \leq n) = (0 \leq n) \quad // \text{precondition} \]
both parts of Li hold

INDUCTIVE CASE:
assume: \( r_{\text{old}} = i_{\text{old}}! \)
\[ i_{\text{new}} = i_{\text{old}} + 1 \]

\[ // \text{assume } r_{\text{old}} = i_{\text{old}}! \]
\[ r_{\text{new}} = r_{\text{old}} * i_{\text{new}} \]
\[ r_{\text{new}} = (i_{\text{new}}-1)! * i_{\text{new}} = i_{\text{new}}! \]
\[ r_{\text{new}} = i_{\text{new}}! \]

\[ i_{\text{old}} \leq n; \text{ if } i_{\text{old}} = n, \text{ we would have exited} \]
\[ i_{\text{old}} < n \]
\[ i_{\text{new}} = i_{\text{old}} + 1 \leq n \]

AT EXIT: 
\[ !(i < n) \land (i \leq n \land r = i!) \]
\[ \implies i = n \land r = i! \]
\[ \implies r = n! \]

Initially \( D \geq 0 \)
\[ D: n - i \]
\[ D_{\text{old}} = n - i_{\text{old}} \]
\[ i_{\text{new}} = i_{\text{old}} + 1 \]
\[ D_{\text{new}} = n - i_{\text{new}} \]
\[ = n - (i_{\text{old}} + 1) \]
\[ = D_{\text{old}} - 1 \]

\[ D = 0 \implies i = n \quad // \text{loop exit condition} \]

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Termination

Termination, a little more formally...

-- We need to find a decrementing function \( D \)

\( \{ P \} \) while ( \( b \) ) \( S \) \{ Q \}

We need \( D \) such that

1. \( \{ LI \wedge b \} S \{ D_{\text{after}} < D_{\text{before}} \} \) \hspace{1em} // One iteration of the loop reduces the value of \( D \)

2. \( D=\text{min} \implies \neg b \) \hspace{1em} // Exit condition

Note: In this case, if 0 is \( D \)'s minimal value and
must imply the loop exit condition.
You can replace \( b \) with \( D > 0 \)
Total correctness = Partial correctness + Loop termination

• Establish that the loop terminates
• Suppose the loop always reduces some variable’s value
  • Does the loop terminate if the variable is a
    • Natural number
    • Integer
    • Non-negative real
    • Boolean
    • List or Array
  • Loop terminates if the variable values are a subset of a well-ordered set and D decreases with each iteration
    • For an ordered set, every non-empty subset has a least element
Decrementing Function

• Decrementing function maps program variables to some well-ordered set

    // precondition: x ≥ 0 ^ y = 0
    // Loop invariant: true
    // D: (x-y)
    while (x != y) {
        y = y + 1;
    }
    // postcondition: x = y

• Is $x-y$ a good decrementing function?
Decrementing Function

• Does the loop reduce the decrementing function’s value?
  
  \[ D_k = x - y_k \]
  
  \[ y_{k+1} = y_k + 1 \]
  
  \[ D_{k+1} = x - y_{k+1} \]
  
  \[ = x - (y_k + 1) \]
  
  \[ = D_k - 1 \]

• If the function is at a minimum does the loop exit?
  
  \[ D == 0 \implies x - y = 0 \implies x = y \implies !(x = y) \]
Example

**PRECONDITION:** \( x \geq 0 \)

\[
i = x; \\
z = 0;
\]

\[
\{ \text{LOOP INVARIANT (LI): } i + z = x \} \\
\text{while } ( i > 0 ) \{ \\
\quad z = z + 1; \\
\quad i = i - 1; \\
\}\n\]

**POSTCONDITION:** \( x = z \)

a decrementing function \( D \) is \( D = i \)
Exercise

precondition: arr.length = len ^ len ≥ 0

int sum = 0;
int i = 0;
while (i < len) {
    sum = sum + arr[i];
    i = i + 1;
}

postcondition: (result is the sum of all elements on array arr)
    sum = arr[0] + arr[1] + ... + arr[arr.length-1]

D = len - i
Exercise

**PRECONDITION:** \( x > 0 \)

\[
\text{zeros} = 0; \\
y = x;
\]

while ( \( y \% 10 == 0 \) ) {
    \( y = y / 10; \)  // integer division
    \( \text{zeros} = \text{zeros} + 1; \)
}

\( \text{zeros} \)

**POSTCONDITION:** \( x = y \times 10 \quad ^\land (y \% 10 \neq 0) \)
Exercise

**PRECONDITION:** $x_1 > 0 \land x_2 > 0$

$y_1 = x_1$;
$y_2 = x_2$;

while ( $y_1 \neq y_2$ ) {
    if ( $y_1 > y_2$ ) {
        $y_1 = y_1 - y_2$;
    }
    else {
        $y_2 = y_2 - y_1$;
    }
}

**POSTCONDITION:** $y_1 = \text{gcd}(x_1, x_2)$
Loops - Summary

Total correctness = Partial correctness + Loop termination

(1) Partial correctness

-- "Guess" then prove the loop invariant (LI) by induction

-- Loop invariant and the loop exit condition must imply the given postcondition

-- This gives us:

"If the loop terminates, then the postcondition holds."

(2) Loop termination

-- "Guess" the decrementing function D.
   Each iteration of the loop decrements D, until D reaches a minimum.
   D at min must imply loop exit condition
Rules for Backward Reasoning: Method Call

// precondition: ??
x = foo();
// postcondition: Q

If method has no side-effects, just like assignment

// precondition: ??
x = Math.abs(y)
// postcondition: x = 1

Precondition is \( y = 1 \) \( \lor \) \( y = -1 \)
Recursion

- An effective recursive routine must
  - Have a base case
  - Assume algorithm is valid for step k
  - Show how to get from step k to step k+1
  - Show that algorithm terminates
    - Recurses towards base case

- Sounds like computational induction
Example

// precondition: x > 0

int factorial(int x) {
    if(x == 1) { // base case
        return 1;
    } else {
        return x * factorial(x-1);
    }
}
// postcondition: returns x!

Invariant:
factorial(x) = x! ^ x>= 1

Base case:
1! = 1

Induction:
Assume factorial(y) = y! For y < x
factorial(x) = x * factorial(x-1) = x * (x-1)! = x!

Termination:
D = x-1; x decreases at each iteration
D = 0 ^ factorial(x) = x! ^ x>= 1 => x=1

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Summary So Far

• Intro to reasoning about code. Concepts
  • Specifications, preconditions and postconditions, forward and backward reasoning

• Hoare triples

• Rules for backward reasoning
  • Rule for assignment
  • Rule for sequence of statements
  • Rule for if-then-else
In Practice

- Write loop invariants when unsure about a loop
- When you have evidence that a loop is not working
  - Add invariant and decrementing function
  - Write code to check them
  - Understand why the code doesn't work
  - Fix
  - Reason to ensure that no similar bugs remain
In Practice

• Use the loop invariant to guide writing the loop
  • Determine the set of variables for the loop
  • Express the required condition at the end of the loop
    • Postcondition for the loop
  • Determine what holds before the loop executes
    • Precondition
  • Determine a decrementing function
    • What decreases with each iteration
    • Try to find a decrementing function with 0 as a minimum
• Construct a loop invariant
  • What has to be true after each iteration
• Use the loop invariant to construct the loop body
Why Do We Care?

• Correctness is important
  • Bugs are frustrating, expensive, and in some case dangerous
• Pre and postconditions for functions are specifications
• Optimizing compilers
  • Transform loops
  • Is the transformed loop the same as the original?
• Thinking about code in a formal way leads to better code
  • Helps us solve problems
  • Helps us create code from specifications