Converting NPDAs to Context-Free Grammars

For any NPDA M

we will construct

a context-free grammar G with

$$L(M) = L(G)$$

Intuition: The grammar simulates the machine

A derivation in Grammar G :

$$S \Rightarrow \cdots \Rightarrow abc\ldots ABC\ldots \Rightarrow \ldots \Rightarrow abc\ldots$$

Current configuration in NPDA M

A derivation in Grammar G :

terminals variables

$$S \Rightarrow \cdots \Rightarrow abc\ldots ABC\ldots \Rightarrow \ldots \Rightarrow abc\ldots$$

Input processed

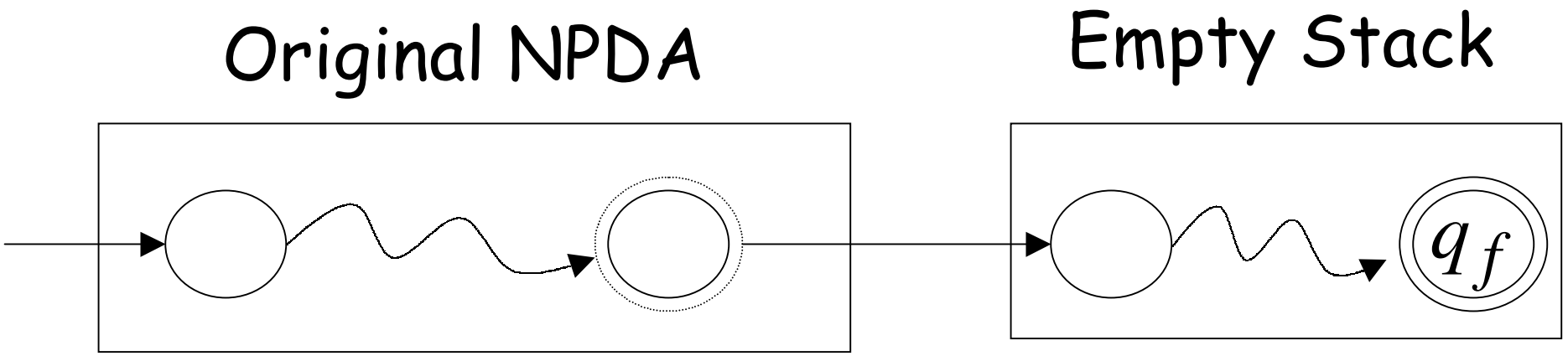
Stack contents

in NPDA M

Some Necessary Modifications

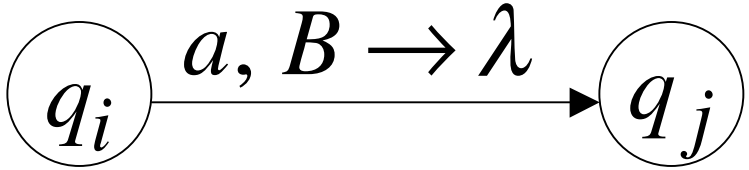
First, we modify the NPDA:

- It has a single final state q_f
- It empties the stack
  when it accepts the input

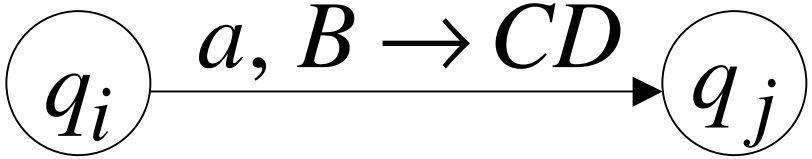


Second, we modify the NPDA transitions:

all transitions will have form



or



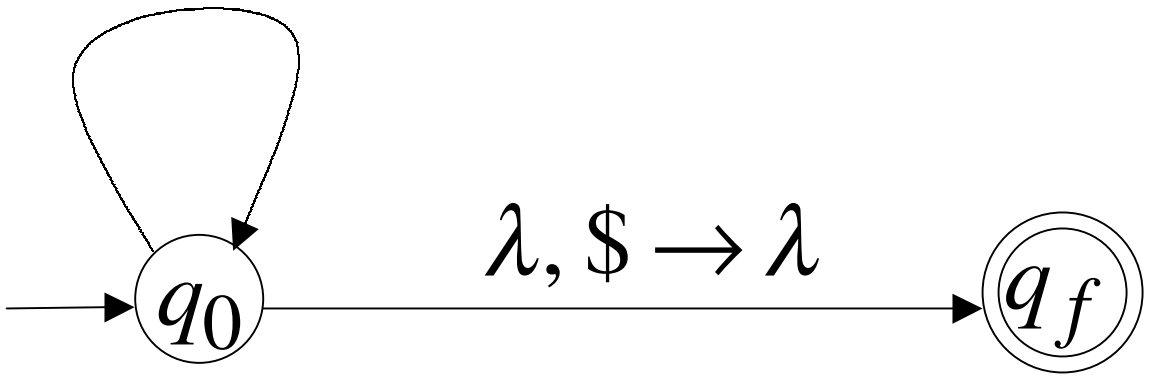
B, C, D : stack symbols

Example of a NPDA in correct form:

$$L(M) = \{w: \quad n_a = n_b\}$$

$\$$: initial stack symbol

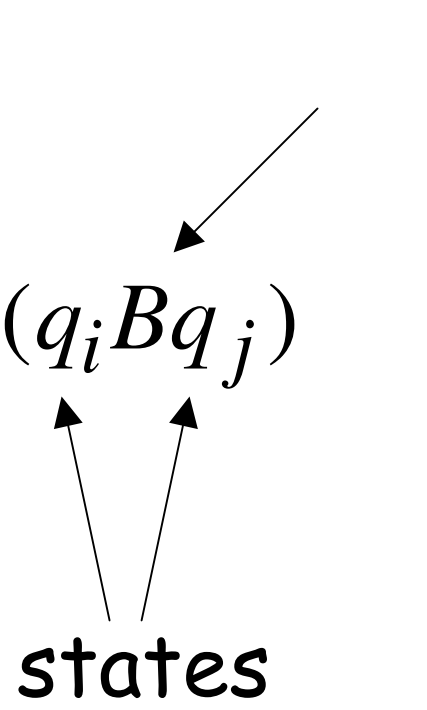
$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$
$$a, 0 \to 00 \qquad b, 1 \to 11$$
$$a, 1 \to \lambda \qquad b, 0 \to \lambda$$



The Grammar Construction

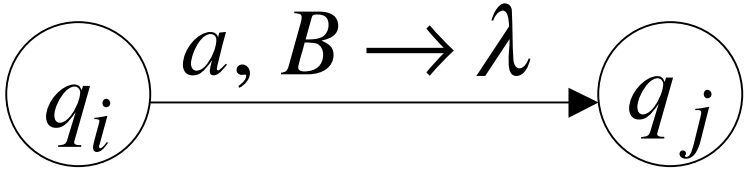
In grammar G

variables have form $(q_i B q_j)$



Terminals are input symbols

For each transition

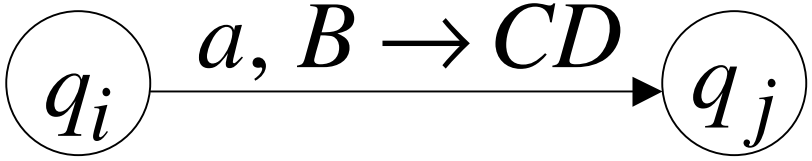


We add production

$$(q_i B q_j) \to a$$

For all states q_k

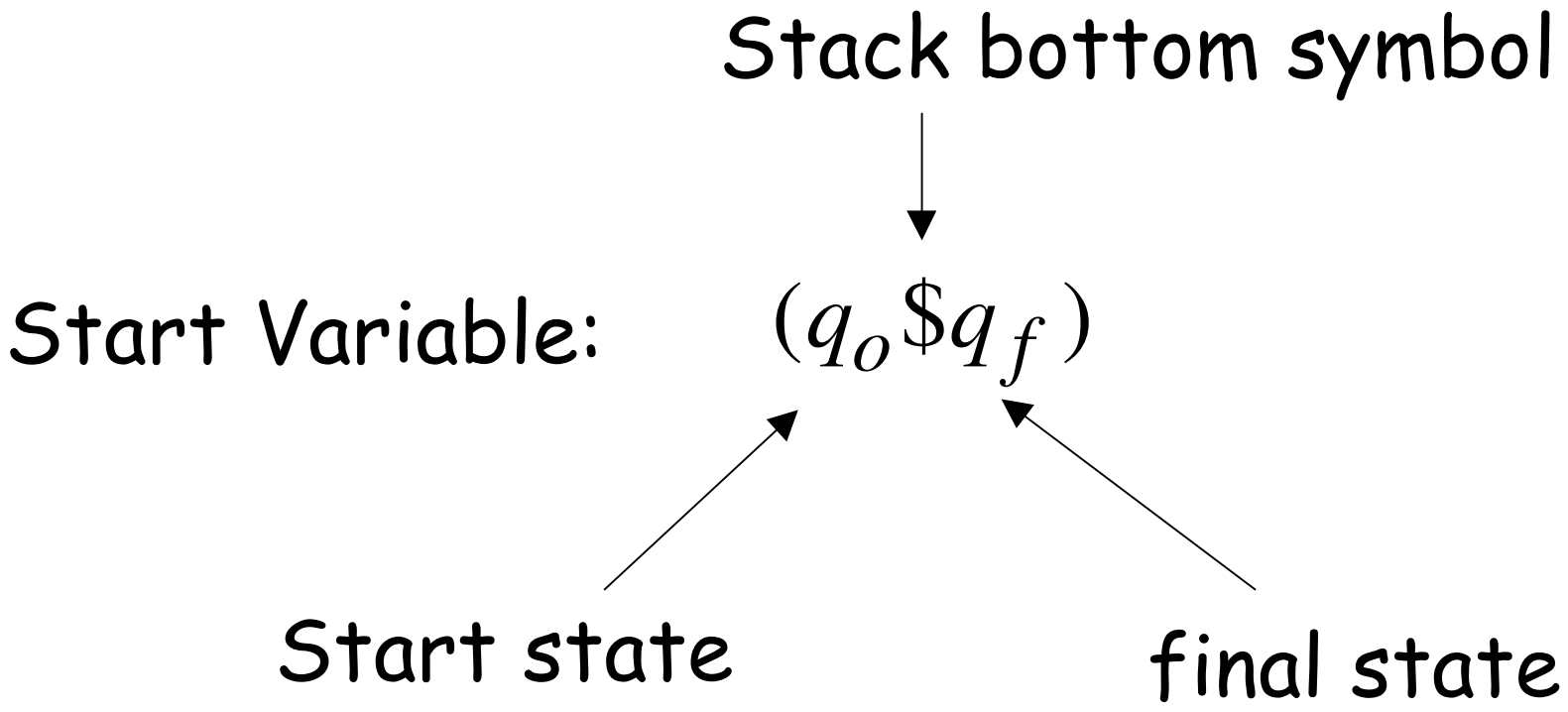
For each transition



We add production

$$(q_i B q_k) \to a(q_j C q_l)(q_l D q_k)$$

For all states q_k, q_l in the NPDA

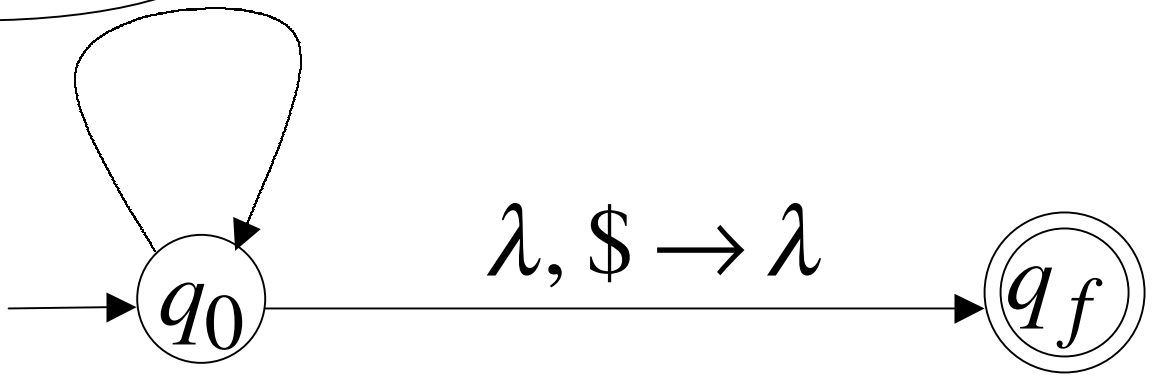

Stack bottom symbol
Start Variable:
$(q_o \$ q_f)$
Start state
final state

Example:

$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad b, 0 \rightarrow \lambda$



Grammar production:

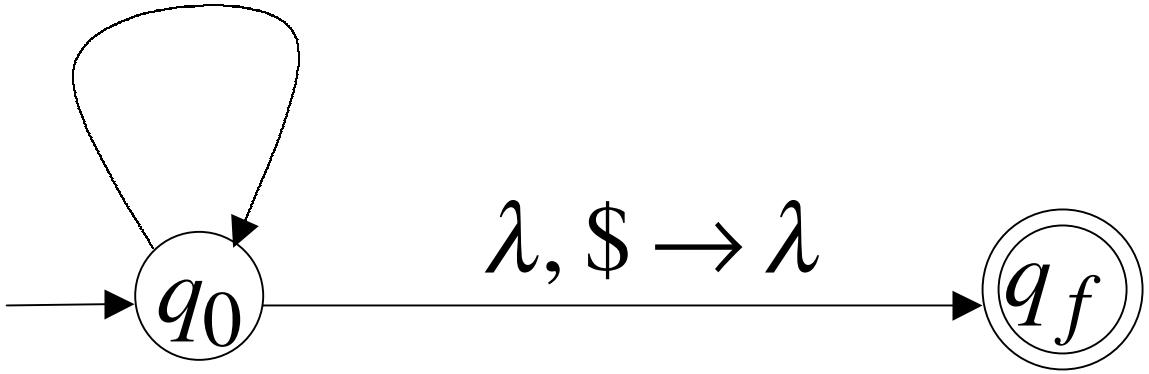
$$(q_0 1 q_0) \rightarrow a$$

Example:

$a, \$ \to 0\$$ $b, \$ \to 1\$$

$a, 0 \to 00$ $b, 1 \to 11$

$a, 1 \to \lambda$ $b, 0 \to \lambda$



Grammar productions:

$$(q_0\$q_0) \to b(q_0 1 q_0)(q_0\$q_0) \mid b(q_0 1 q_f)(q_f\$q_0)$$

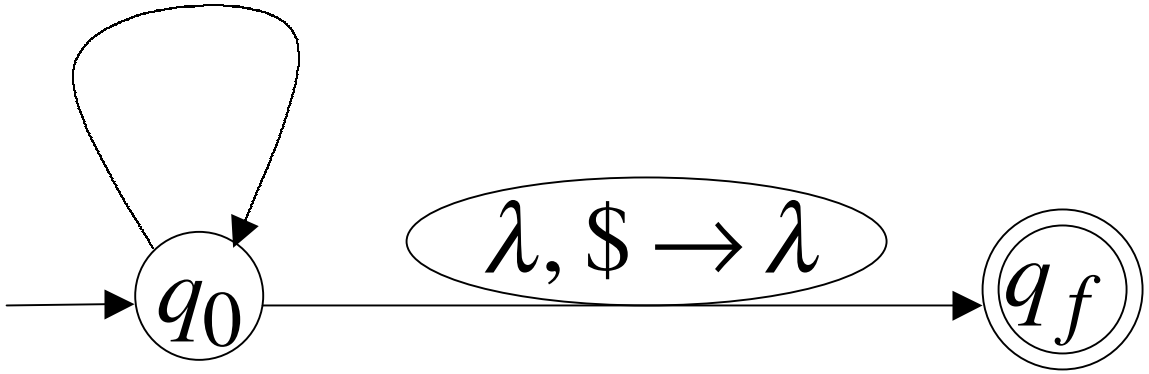
$$(q_0\$q_f) \to b(q_0 1 q_0)(q_0\$q_f) \mid b(q_0 1 q_f)(q_f\$q_f)$$

Example:

$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$

$$a, 0 \to 00 \qquad b, 1 \to 11$$

$$a, 1 \to \lambda \qquad b, 0 \to \lambda$$



Grammar production:

$$(q_0 \$ q_f) \to \lambda$$

Resulting Grammar: $(q_0\$q_f)$: start variable

$$(q_0\$q_0) \to b(q_0 1 q_0)(q_0\$q_0) \mid b(q_0 1 q_f)(q_f\$q_0)$$

$$(q_0\$q_f) \to b(q_0 1 q_0)(q_0\$q_f) \mid b(q_0 1 q_f)(q_f\$q_f)$$

$$(q_0 1 q_0) \to b(q_0 1 q_0)(q_0 1 q_0) \mid b(q_0 1 q_f)(q_f 1 q_0)$$

$$(q_0 1 q_f) \to b(q_0 1 q_0)(q_0 1 q_f) \mid b(q_0 1 q_f)(q_f 1 q_f)$$

$$(q_0\$q_0) \to a(q_0 0 q_0)(q_0\$q_0) \mid a(q_0 0 q_f)(q_f\$q_0)$$

$$(q_0\$q_f) \to a(q_0 0 q_0)(q_0\$q_f) \mid a(q_0 0 q_f)(q_f\$q_f)$$

$$(q_0 0 q_0) \rightarrow a(q_0 0 q_0)(q_0 0 q_0) \,|\, a(q_0 0 q_f)(q_f 0 q_0)$$

$$(q_0 0 q_f) \rightarrow a(q_0 0 q_0)(q_0 0 q_f) \,|\, a(q_0 0 q_f)(q_f 0 q_f)$$

$$(q_0 1 q_0) \rightarrow a$$

$$(q_0 0 q_0) \rightarrow b$$

$$(q_0 \$ q_f) \rightarrow \lambda$$

Derivation of string $abba$

$$(q_0\$q_f) \Rightarrow \quad a(q_0 0 q_0)(q_0\$q_f) \Rightarrow$$

$$ab(q_0\$q_f) \Rightarrow$$

$$abb(q_0 1 q_0)(q_0\$q_f) \Rightarrow$$

$$abba(q_0\$q_f) \Rightarrow \quad abba$$

In general, in grammar

$$(q_0 \$ q_f) \overset{*}{\Rightarrow} w$$

if and only if

w is accepted by the NPDA

Explanation:

By construction of grammar:

$$(q_i A q_j) \overset{*}{\Rightarrow} w$$

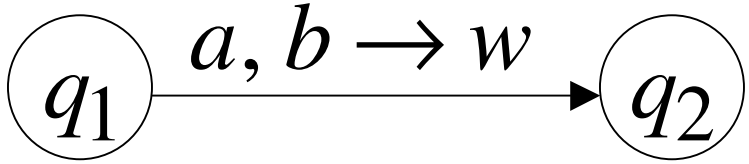
If and only if

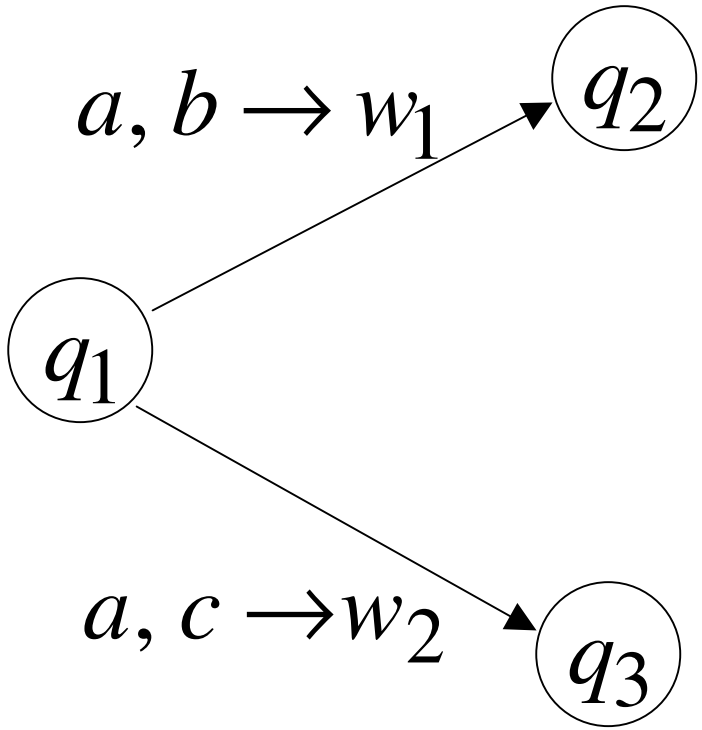
in the NPDA going from q_i to q_j
the stack doesn't change below A
and A is removed from stack

Deterministic PDAs (DPDAs)

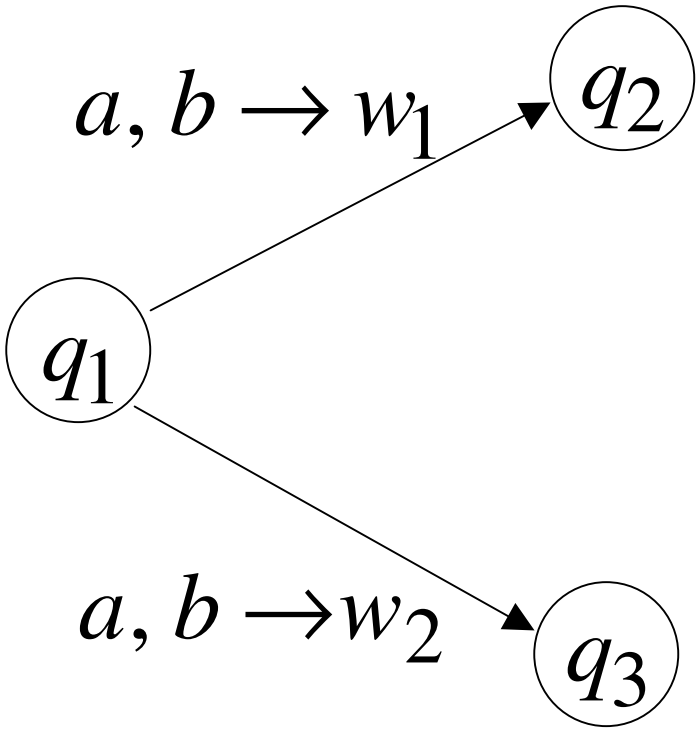
DPDAs

Allowed Transitions:



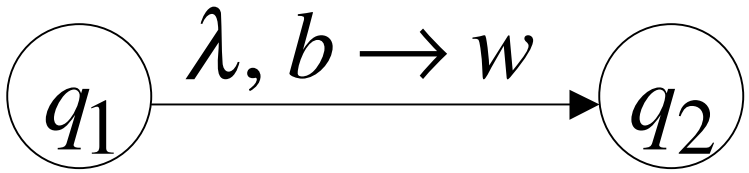


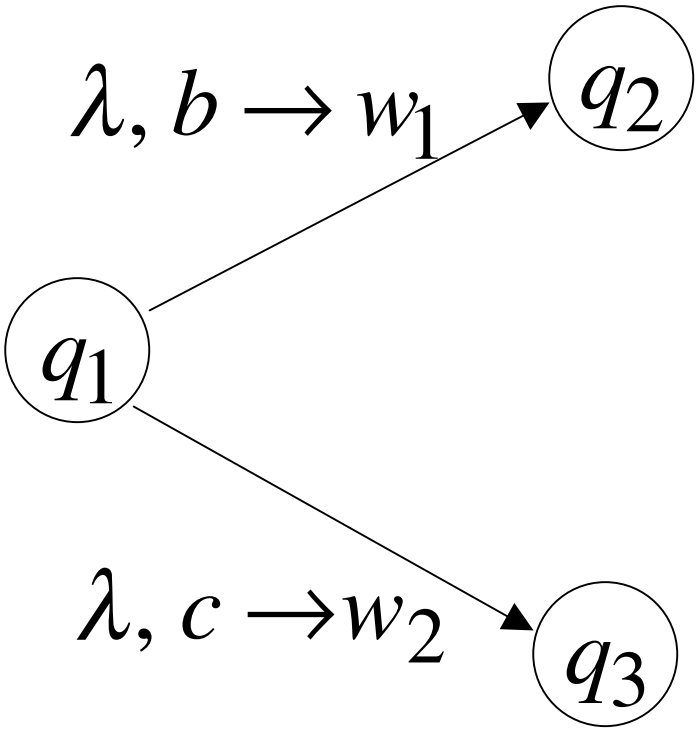
Not allowed:



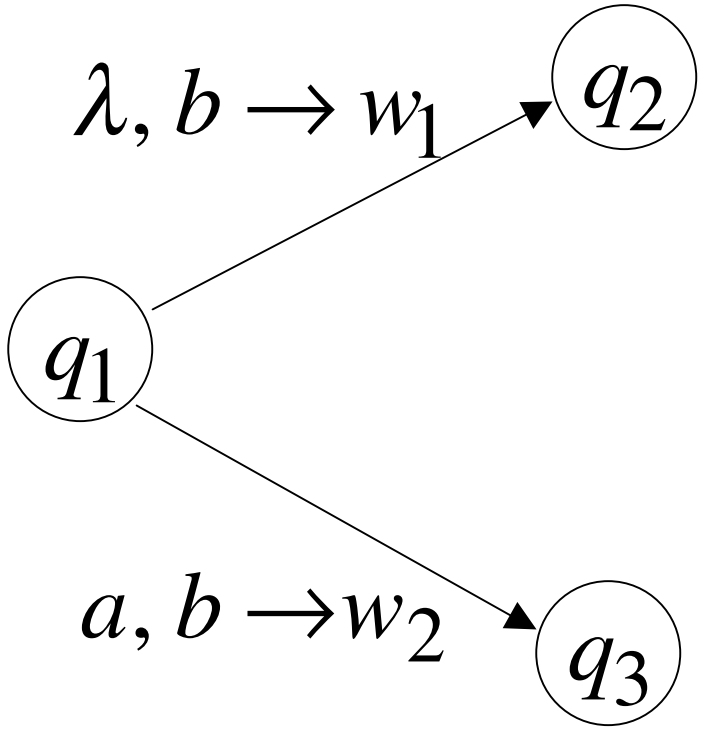
Not allowed even when b is λ

Allowed:





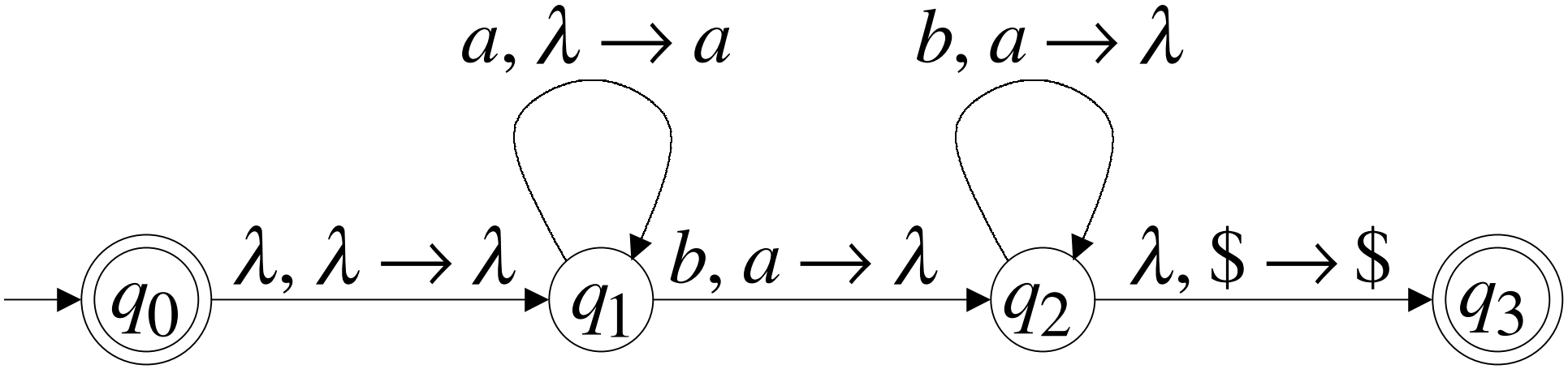
Not allowed:



Not allowed even when b is λ

DPDA example

$$L(M) = \{a^n b^n : n \geq 0\}$$



The language $L(M) = \{a^n b^n : n \geq 0\}$

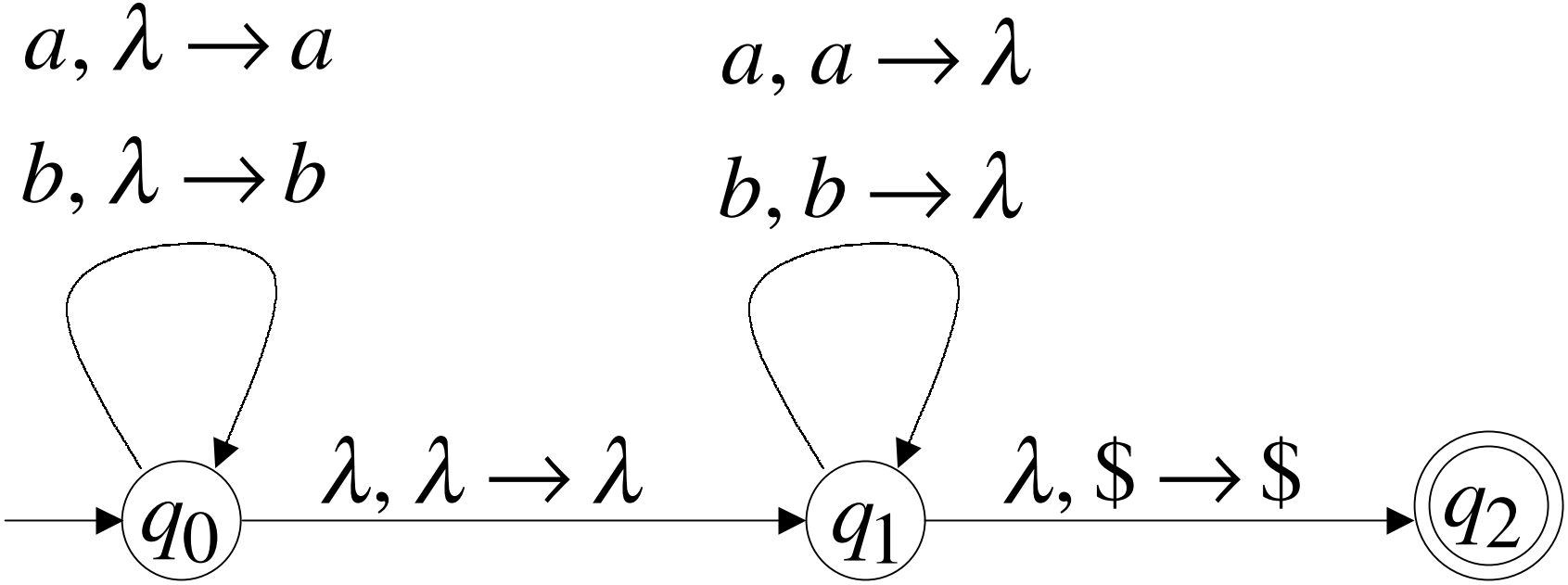
is **deterministic context-free**

In general:

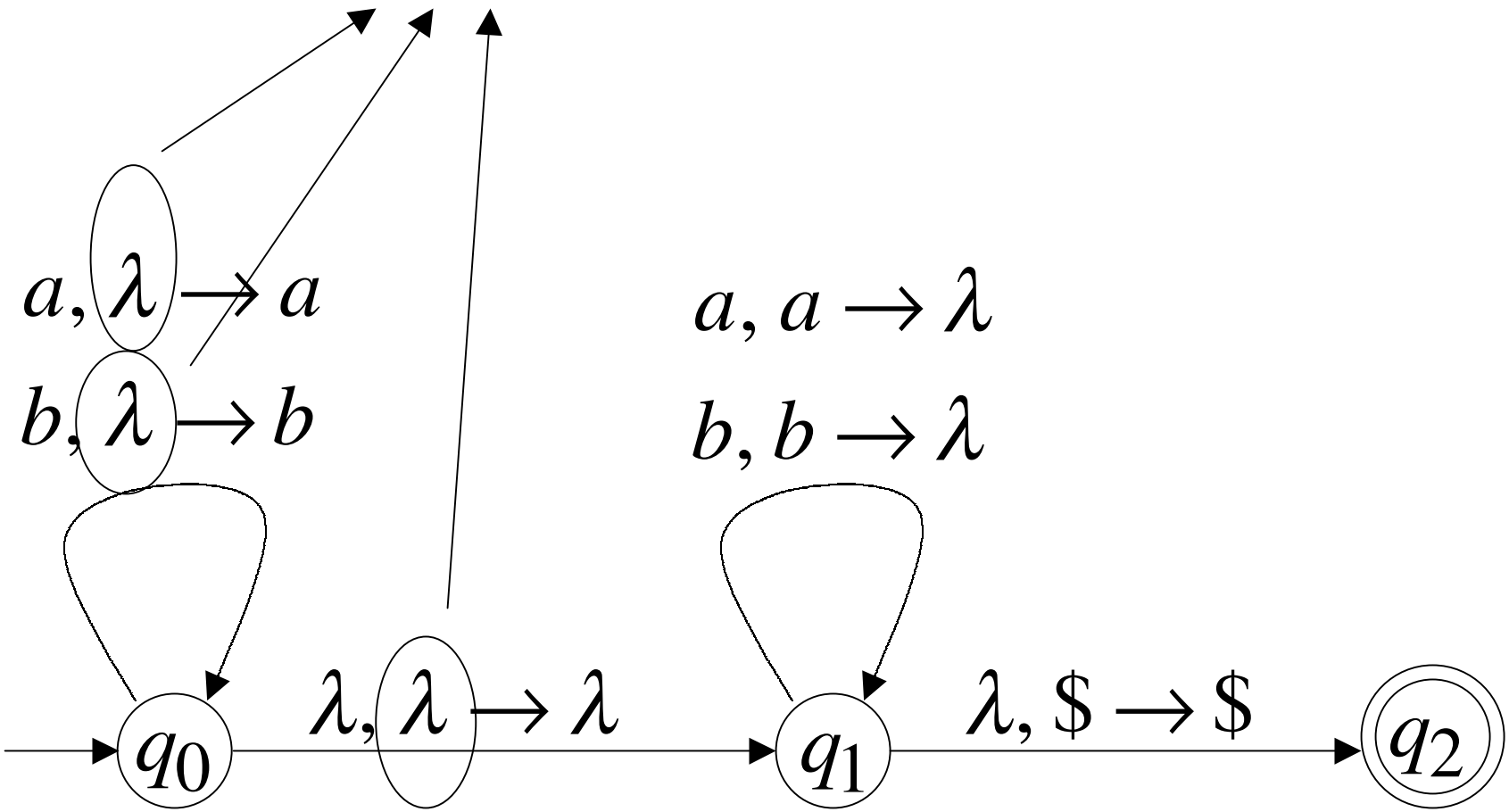
A language is **deterministic context-free** if there is some NPDA that accepts it

Example of Non-DPDA

$$L(M) = \{ww^R\}$$



Not allowed transitions for DPDAs



NPDAs
have more power than
DPDAs

We will show:

There is

a context-free language L (accepted by a NPDA)

which is **not**

deterministic context-free (**not** accepted by a DPDA)

The language is:

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

$$n \geq 0$$

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

The language L is context-free

Context-free grammar for L :

$$S \to S_1 \,|\, S_2$$

$$S_1 \to aS_1b \,|\, \lambda$$

$$S_2 \to aS_2bb \,|\, \lambda$$

(there is an NPDA that accepts L)

Theorem:

The language

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

is **not** deterministic context-free

(there is **no** DPDA that accepts L)

Proof:

Assume for contradiction that

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

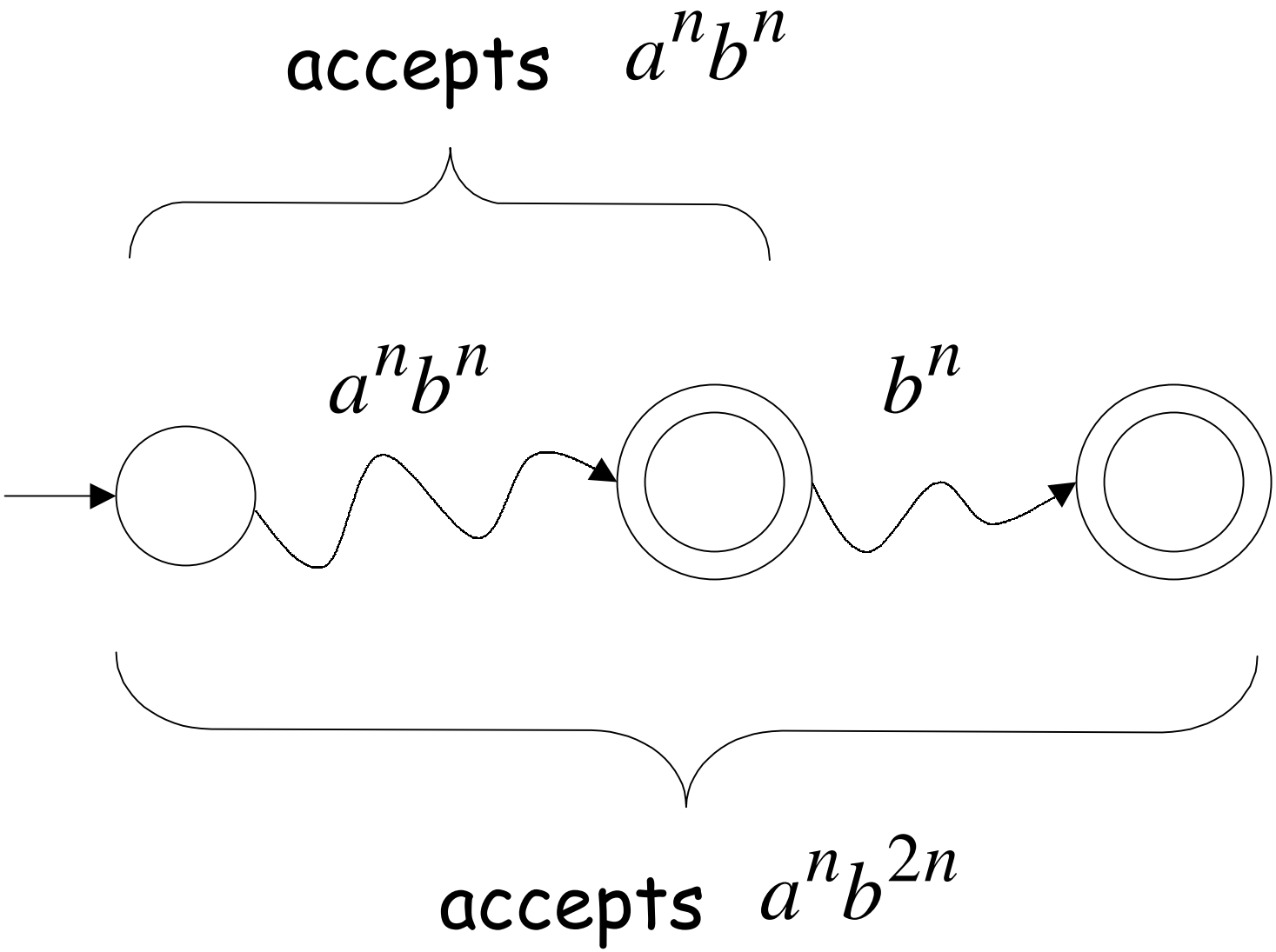
is deterministic context free

Therefore:

There exists a DPDA M

that accepts L

The DPDA M with $L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$



A fact:

The language $\{a^n b^n c^n\}$

is **not** context-free

(we will prove it at the next class)

Another fact:

The language $L' = L \cup \{a^n b^n c^n\}$

is **not** context-free

$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$

We will construct a NPDA that accepts

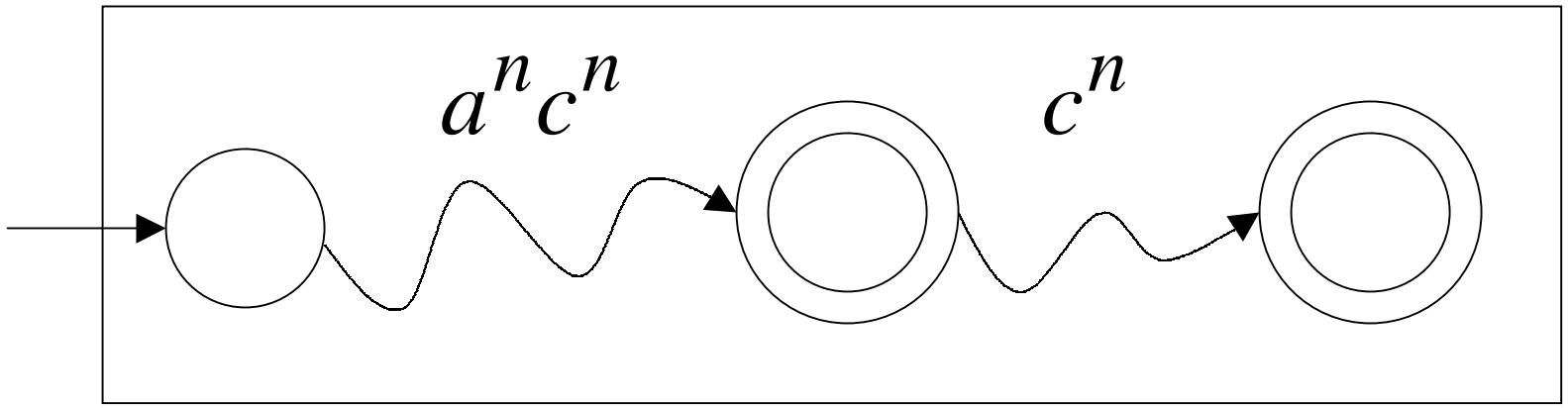
$$L' = L \cup \{a^n b^n c^n\}$$

$$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$$

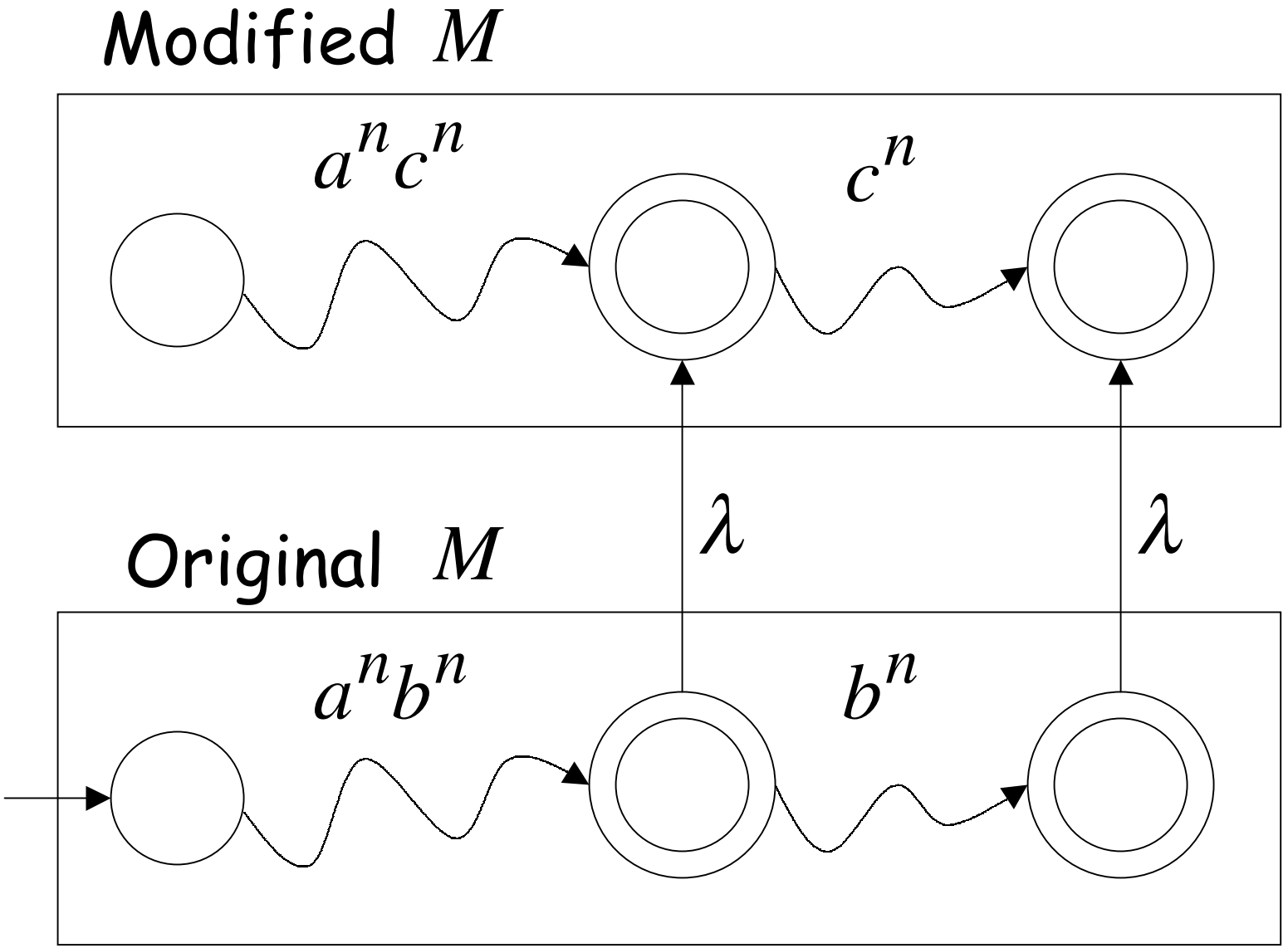
First, we modify M:

Replace b with c

Modified M



The NPDA that accepts $L' = L \cup \{a^n b^n c^n\}$



$L' = L \cup \{a^n b^n c^n\}$ is accepted by a NPDA

Therefore: L' is context-free

Contradiction!

(L' is not context-free)

Therefore:

There is **no** DPDA that accepts

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

End of Proof