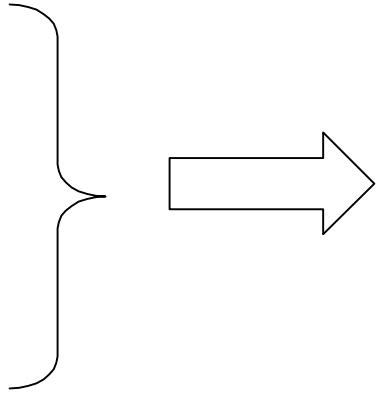
# Positive Properties of Context-Free languages

Context-free languages
are closed under: **Union**

$L_1$ is context free

$L_2$ is context free



$L_1 \cup L_2$

is context-free

Union Example:

$$S_1 \to aS_1b \mid \lambda \qquad L_1 = \{a^n b^n\}$$

$$S_2 \to aS_2a \mid bS_2b \mid \lambda \qquad L_2 = \{ww^R\}$$

Union grammar:

$$S \to S_1 \mid S_2 \qquad L = \{a^n b^n\} \cup \{ww^R\}$$

In general:

Take any two context-free grammars $G_1, G_2$ with start variables $S_1,\ S_2$

The **union** grammar $G$ has start variable $S \to S_1 \mid S_2$

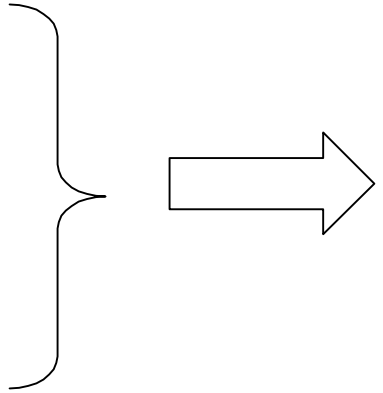
and $L(G) = L(G_1) \cup L(G_2)$

is context-free

Context-free languages
are closed under: **Concatenation**

$L_1$ is context free

$L_2$ is context free



$L_1 L_2$

is context-free

Concatenation Example:

$$S_1 \rightarrow aS_1b \mid \lambda \qquad L_1 = \{a^n b^n\}$$

$$S_2 \rightarrow aS_2a \mid bS_2b \mid \lambda \qquad L_2 = \{ww^R\}$$

Concatenation grammar:

$$S \rightarrow S_1 S_2 \qquad L = \{a^n b^n\}\{ww^R\}$$

In general:

Take any two context-free grammars $G_1, G_2$ with start variables $S_1,\ S_2$

The **concatenation** grammar $G$ has start variable $S \to S_1 \mid S_2$

and $L(G) = L(G_1)L(G_2)$

is context-free

Context-free languages
are closed under: **Star-operation**

$L$ is context free $\Longrightarrow$ $L^*$ is context-free

Star operation Example:

$$S \to aSb \mid \lambda \qquad\qquad L = \{a^n b^n\}$$

Star grammar:

$$S_1 \to SS_1 \mid \lambda \qquad\qquad L = \{a^n b^n\}*$$

In general:

Take any context-free grammars $G$
with start variable $S$

The **concatenation** grammar $G_1$
has start variable $S_1 \to SS_1 \mid \lambda$

and $L(G_1) = L(G)^*$

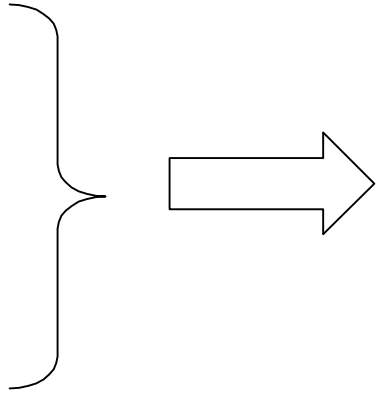
is context-free

# Negative Properties of Context-Free Languages

Context-free languages
are **not** closed under: **intersection**

$L_1$ is context free

$L_2$ is context free



$L_1 \cap L_2$

**not** necessarily
context-free

Counter Example:

$L_1 = \{a^n b^n c^m\}$

Context-free:

$S \to AC$

$A \to aAb \mid \lambda$

$C \to cC \mid \lambda$

$L_2 = \{a^n b^m c^m\}$

Context-free:

$S \to AB$

$A \to aA \mid \lambda$

$B \to bBb \mid \lambda$

$L_1 \cap L_2 = \{a^n b^n c^n\}$ **NOT** context-free

Context-free languages
are **not** closed under: **complement**

$L$ is context free $\Rightarrow$ $\overline{L}$ **not** necessarily context-free

Counter Example:

$$L_1 = \{a^n b^n c^m\}$$

Context-free:

$$S \to AC$$

$$A \to aAb \mid \lambda$$

$$C \to cC \mid \lambda$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \to AB$$

$$A \to aA \mid \lambda$$

$$B \to bBb \mid \lambda$$

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

**NOT** context-free

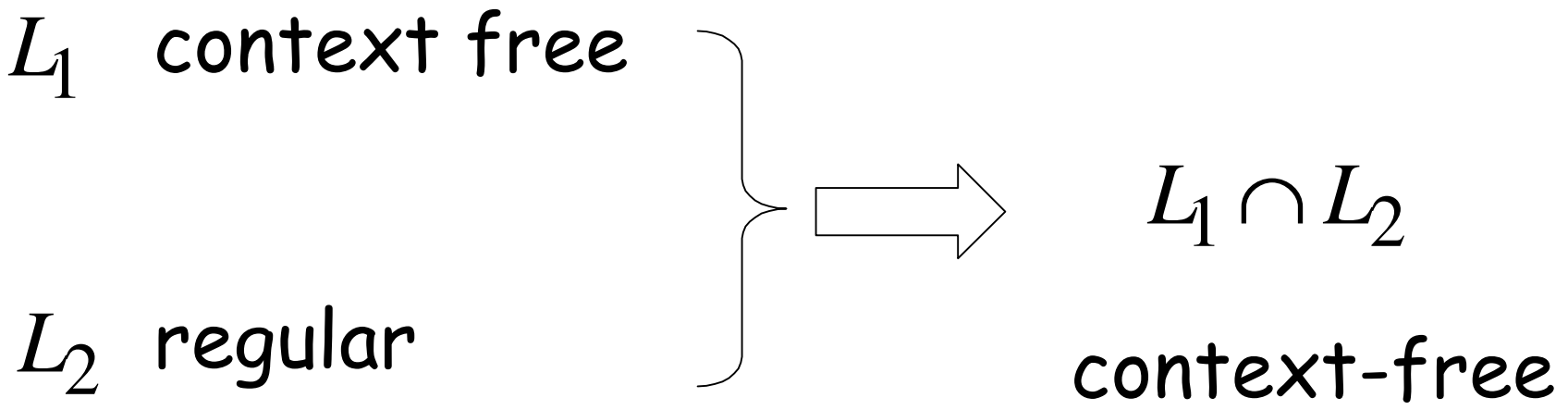
# Intersection of Context-free languages and Regular Languages

The intersection of

a context-free language and

a regular language

is a context-free language



Machine $M_1$

NPDA for $L_1$

context-free

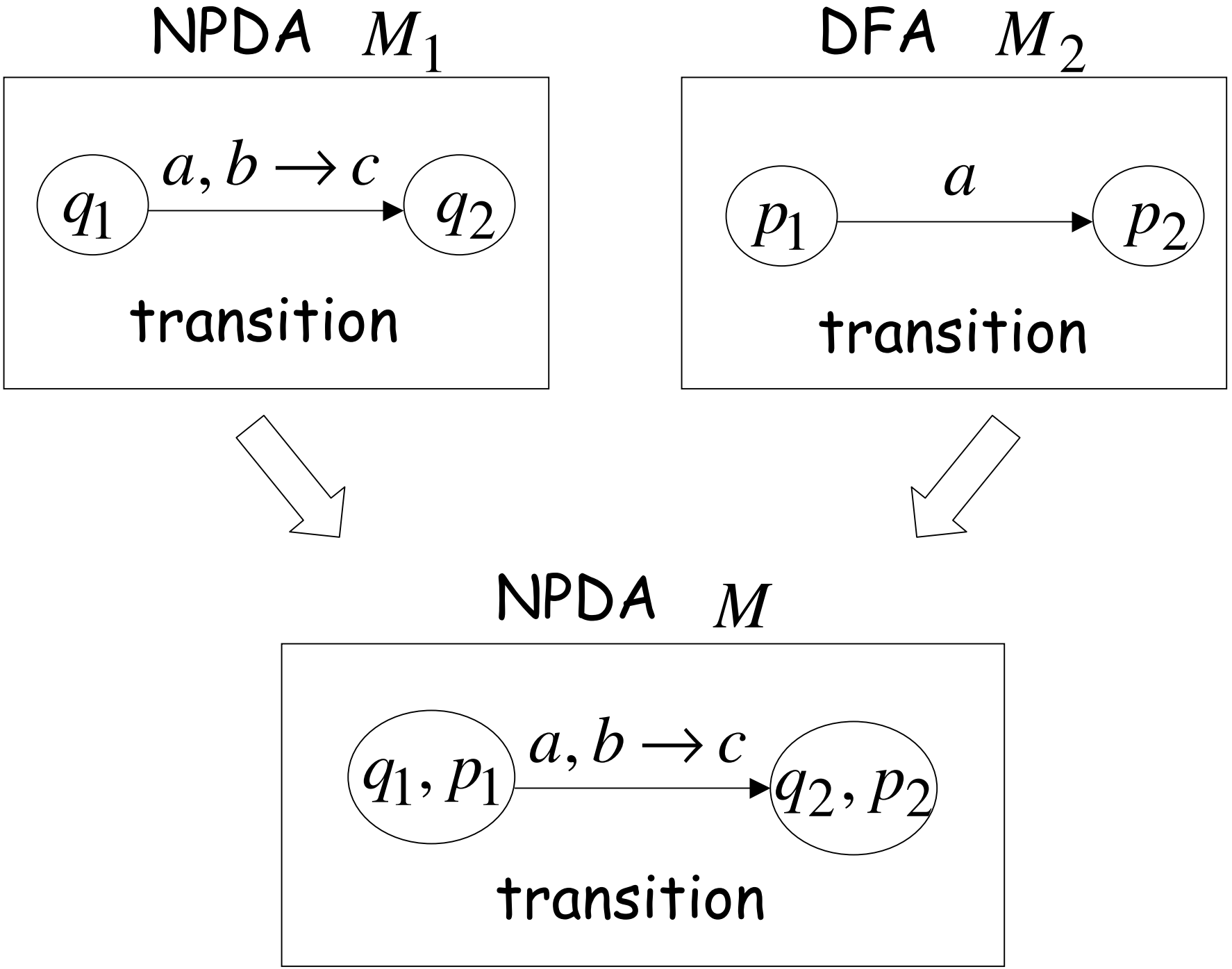
Machine $M_2$

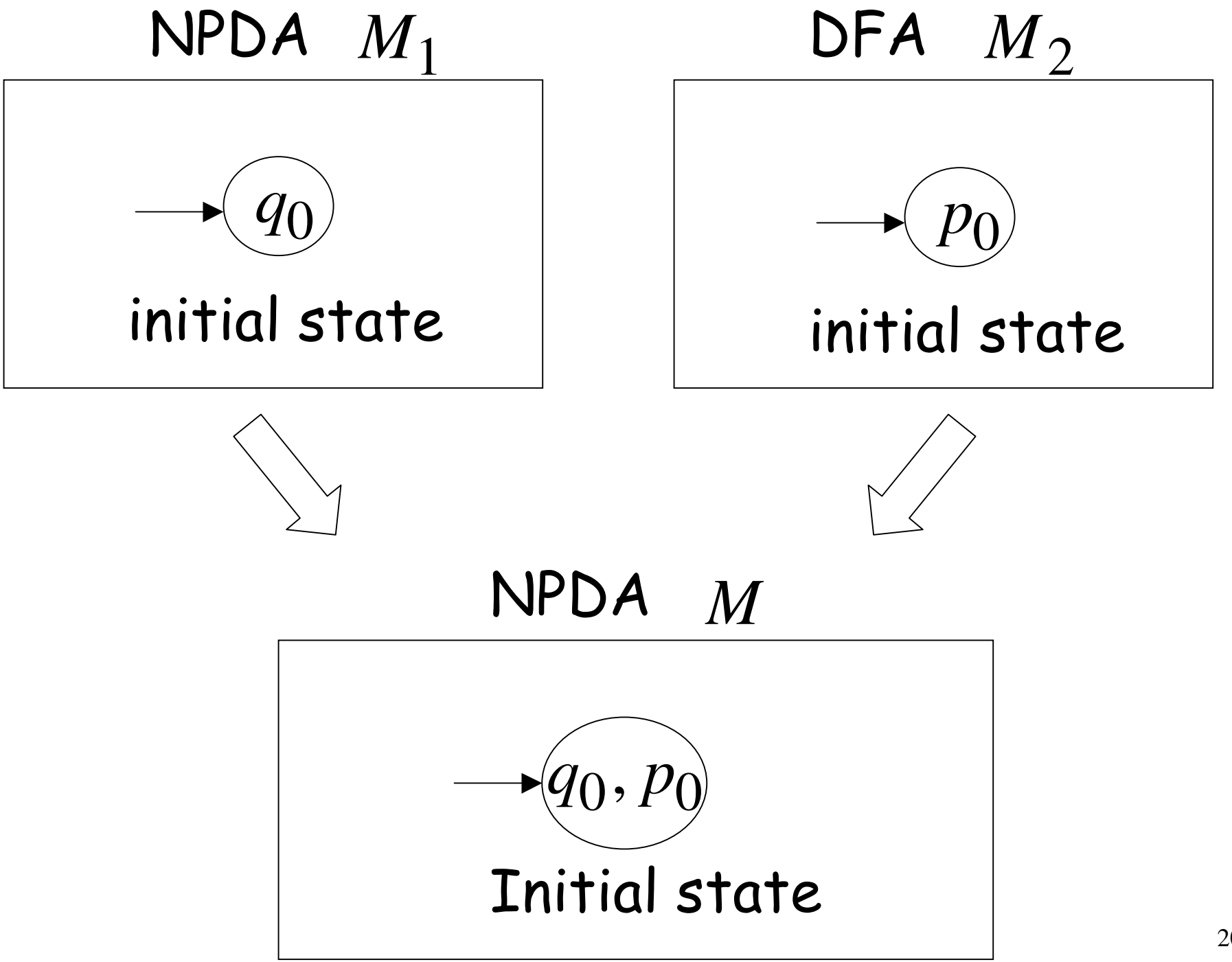
DFA for $L_2$

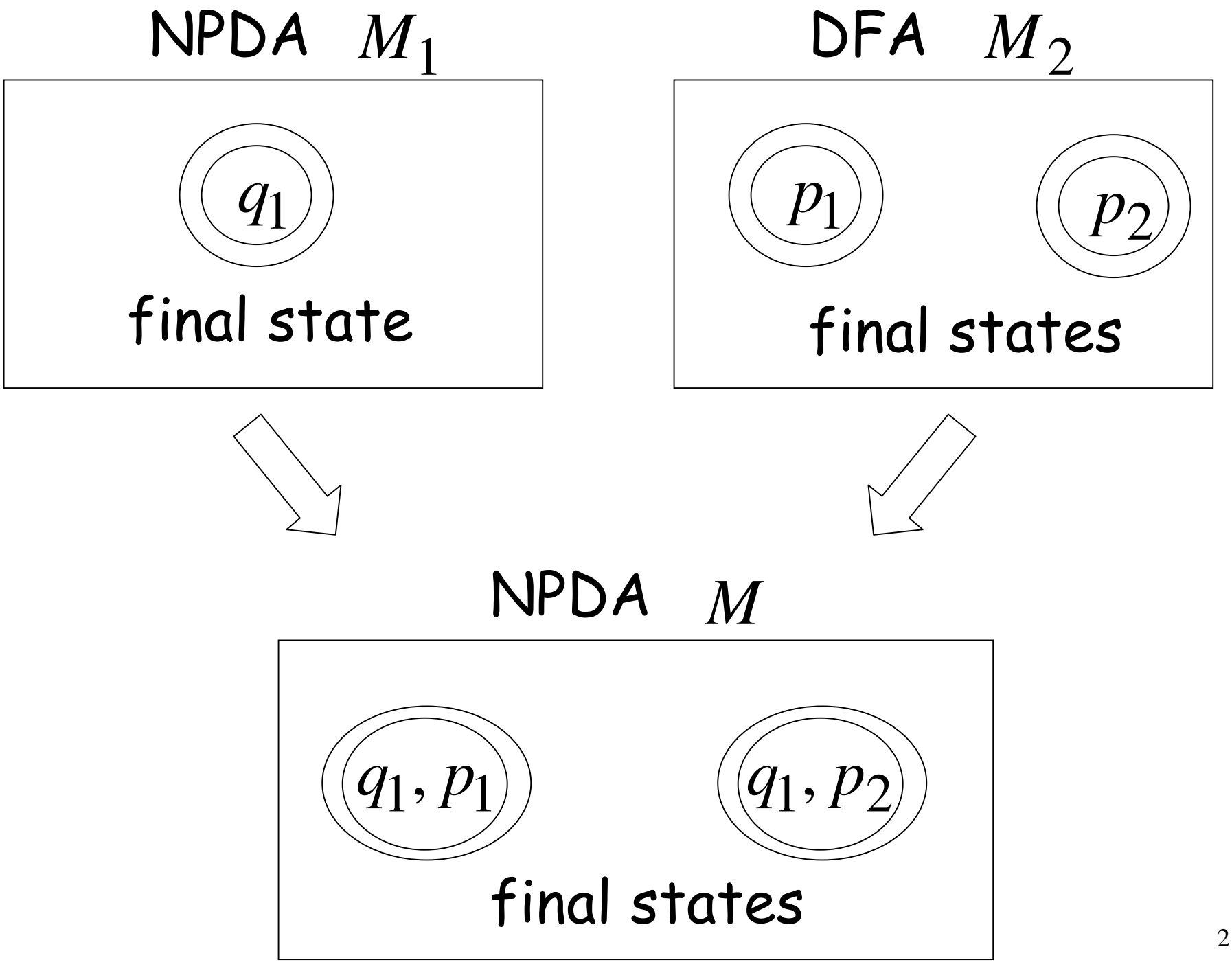
regular

Construct a new NPDA machine $M$ that accepts $L_1 \cap L_2$

$M$ simulates in parallel $M_1$ and $M_2$


NPDA $M_1$
$q_1$
$a, b \to c$
$q_2$
transition
DFA $M_2$
$p_1$
$a$
$p_2$
transition
NPDA $M$
$q_1, p_1$
$a, b \to c$
$q_2, p_2$
transition


NPDA $M_1$
$q_0$
initial state
DFA $M_2$
$p_0$
initial state
NPDA $M$
$q_0, p_0$
Initial state


NPDA $M_1$
$q_1$
final state
DFA $M_2$
$p_1$
$p_2$
final states
NPDA $M$
$q_1, p_1$
$q_1, p_2$
final states

$M$ simulates in parallel $M_1$ and $M_2$

$M$ accepts string $w$ if and only if

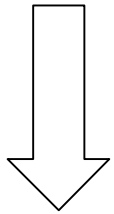
$M_1$ accepts string $w$ and

$M_2$ accepts string $w$

$$L(M) = L(M_1) \cap L(M_2)$$

Therefore: $L(M_1) \cap L(M_2)$ is context-free

(since $M$ is NPDA)



$L_1 \cap L_2$ is context-free

# Example Applications of Regular Closure

Prove that: $L = \{a^n b^n : \ n \neq 100\}$

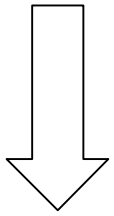
is context-free

We know: $\{a^n b^n\}$

is context-free

We also know:

$L_1 = \{a^{100} b^{100}\}$ is regular



$\overline{L_1} = \{(a+b)^*\} - \{a^{100} b^{100}\}$ is regular

$\{a^n b^n\}$

is context-free

$\overline{L_1} = \{(a+b)^*\} - \{a^{100} b^{100}\}$

is regular

$\{a^n b^n\} \cap \overline{L_1}$

is context-free

$L = \{a^n b^n : \quad n \neq 100\}$

$= \{a^n b^n\} \cap \overline{L_1}$

is context-free

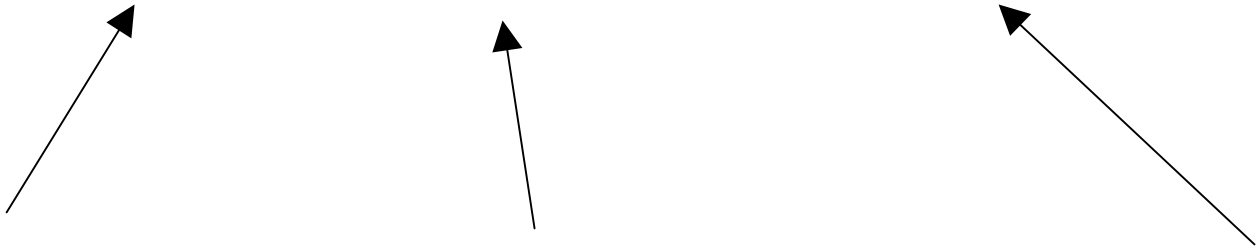
# Another application

Prove that: $L = \{w : \ n_a = n_b = n_c\}$

is **not** context-free

If $L = \{w: \ n_a = n_b = n_c\}$ is context-free

Then $L \cap \{a*b*c*\} = \{a^n b^n c^n\}$



context-free regular Is context-free

Impossible!!!

Therefore, $L$ is **not** context free

# Decidable Properties of Context-Free Languages

Membership Question:

for grammar $G$

find if string $w \in L(G)$

Membership Question:

for grammar $G$

find if string $w \in L(G)$

Membership Algorithms:

Parsers

Exhaustive search parser

**CYK** parsing algorithm

# Empty Language Question:

for grammar $G$

find if $L(G) = \emptyset$

Empty Language Question:

for grammar $G$

find if $L(G) = \emptyset$

Algorithm:

Remove useless variables

Check if start variable $S$ is useless

Infinite Language Question:

for grammar $G$

find if $L(G)$ is infinite

Infinite Language Question:

for grammar $G$

find if $L(G)$ is infinite

Algorithm:

- Remove useless variables
- Remove unit and lambda productions
- Create the dependency graph for variables
- If there is a loop in the dependency graph the language is infinite

Example:

$$S \to AB$$

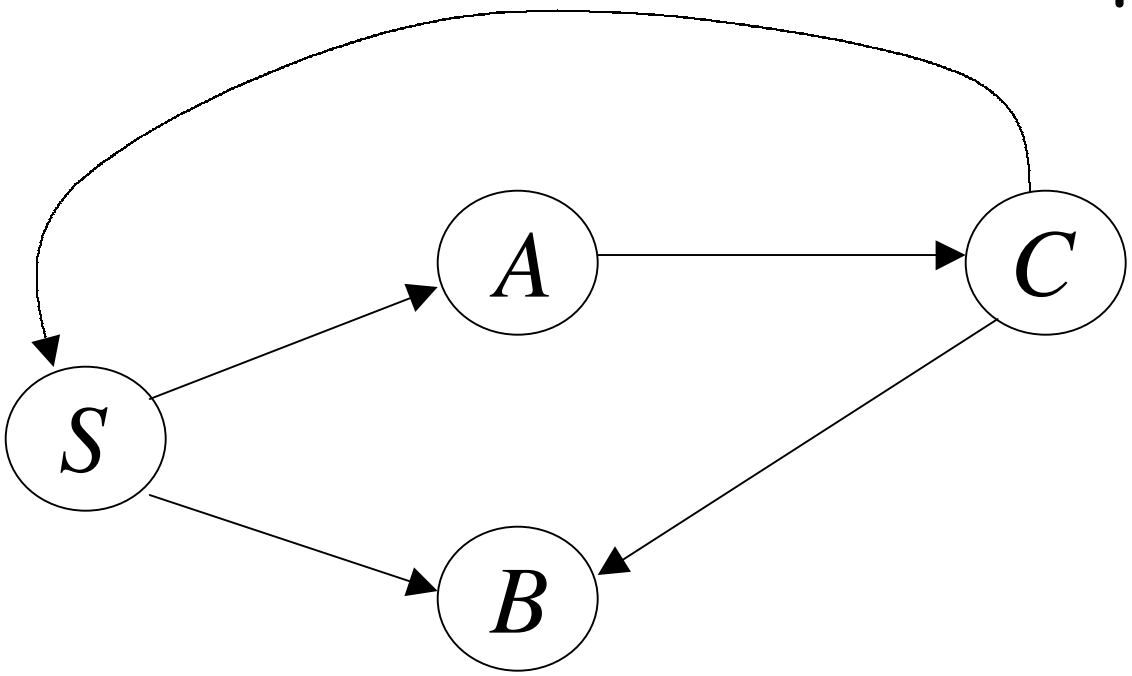
$$A \to aCb \mid a$$

$$B \to bB \mid bb$$

$$C \to cBS$$

Dependency graph

infinite



$$S \to AB$$

$$A \to aCb \mid a$$

$$B \to bB \mid bb$$

$$C \to cBS$$

$$S \Rightarrow AB \Rightarrow aCbB \Rightarrow acBSbB \Rightarrow acbbSbbb$$

$$S \overset{*}{\Rightarrow} acbbSbbb \overset{*}{\Rightarrow} (acbb)^2 S (bbb)^2$$

$$\overset{*}{\Rightarrow} (acbb)^i S (bbb)^i$$