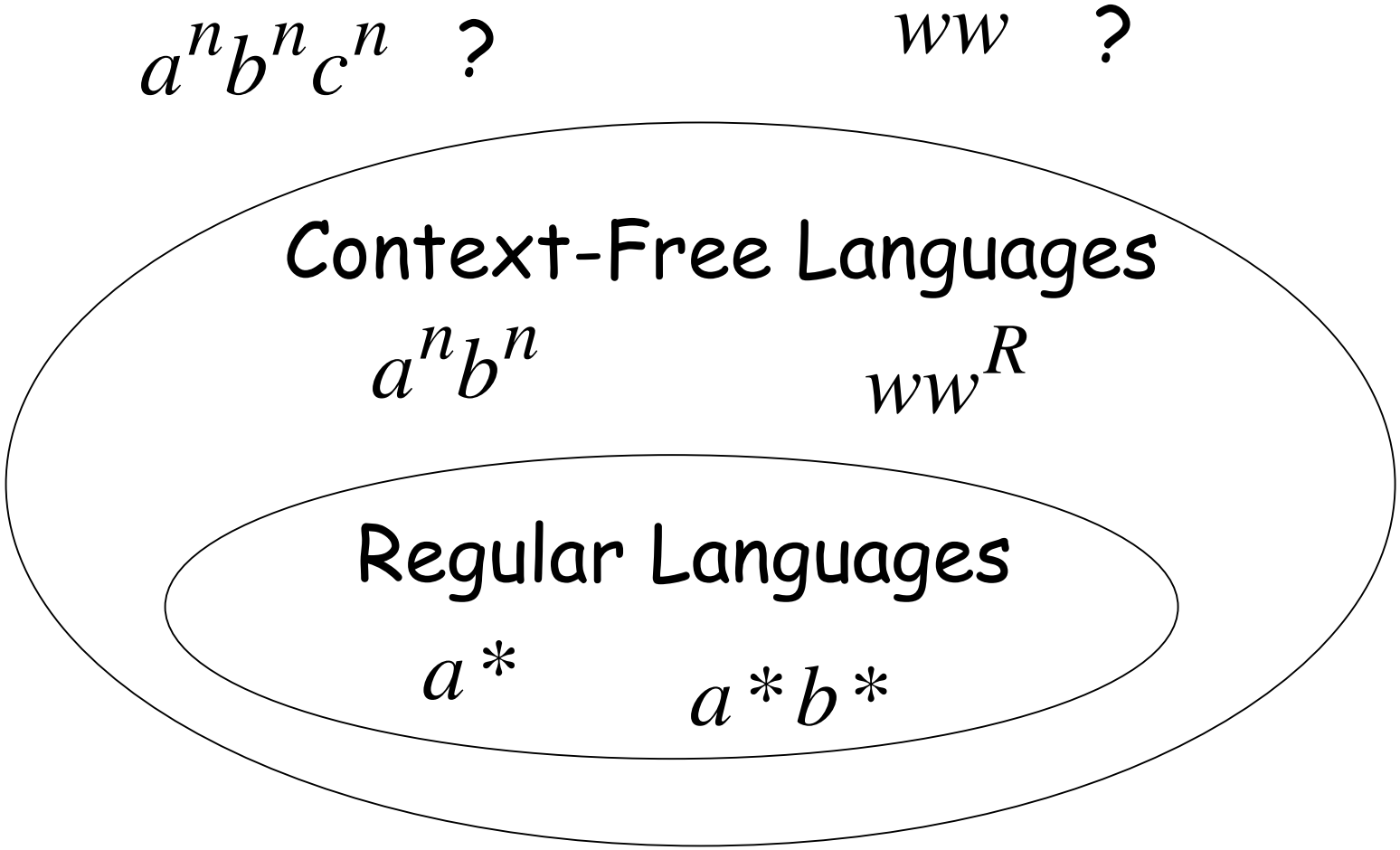
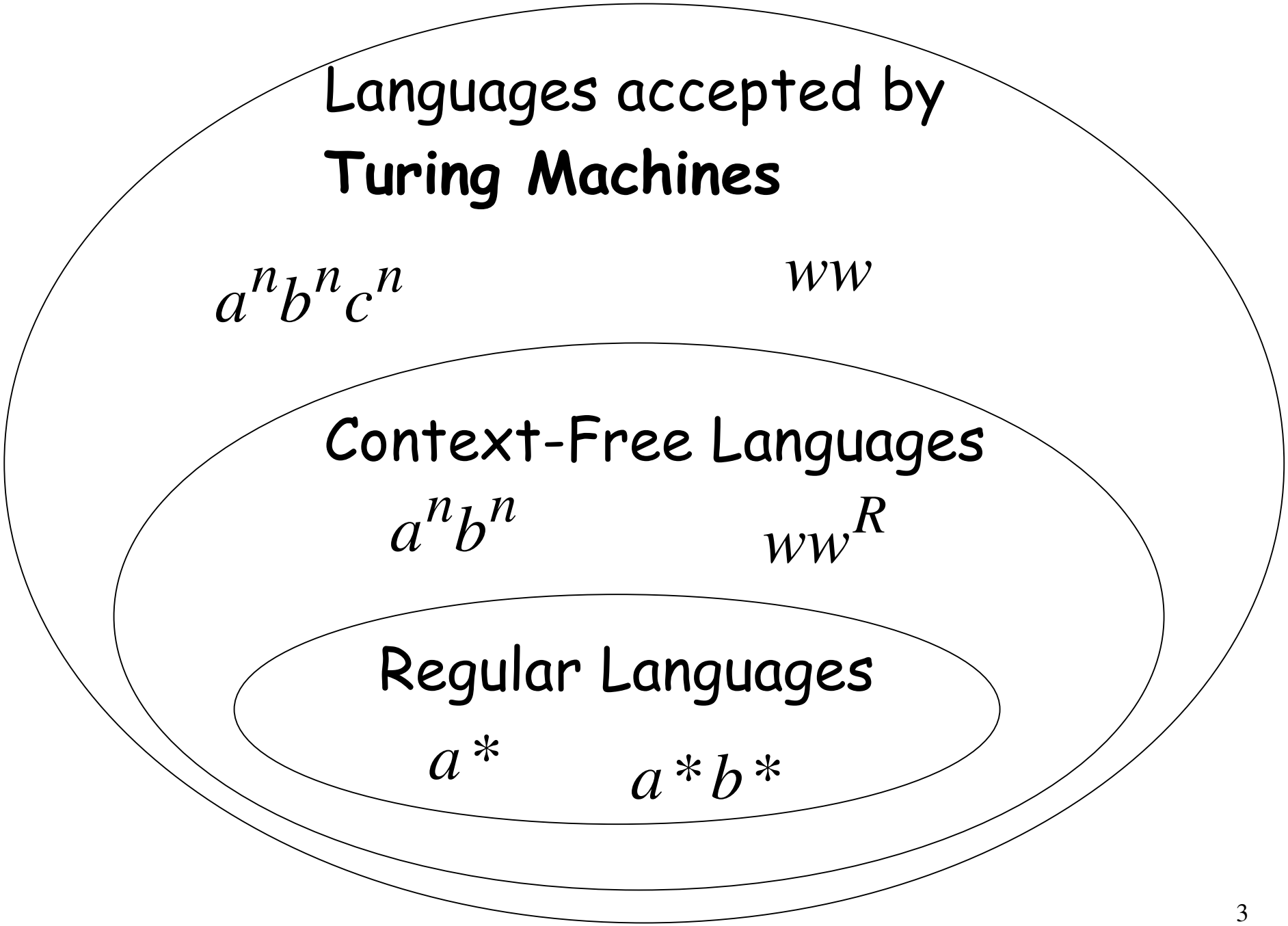
# Turing Machines

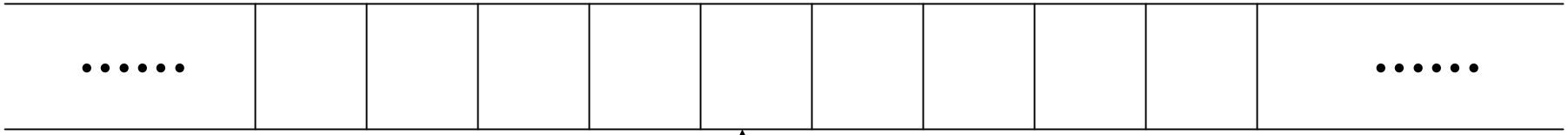
# The Language Hierarchy




Languages accepted by
Turing Machines
$a^n b^n c^n$
$ww$
Context-Free Languages
$a^n b^n$
$ww^R$
Regular Languages
$a*$
$a*b*$

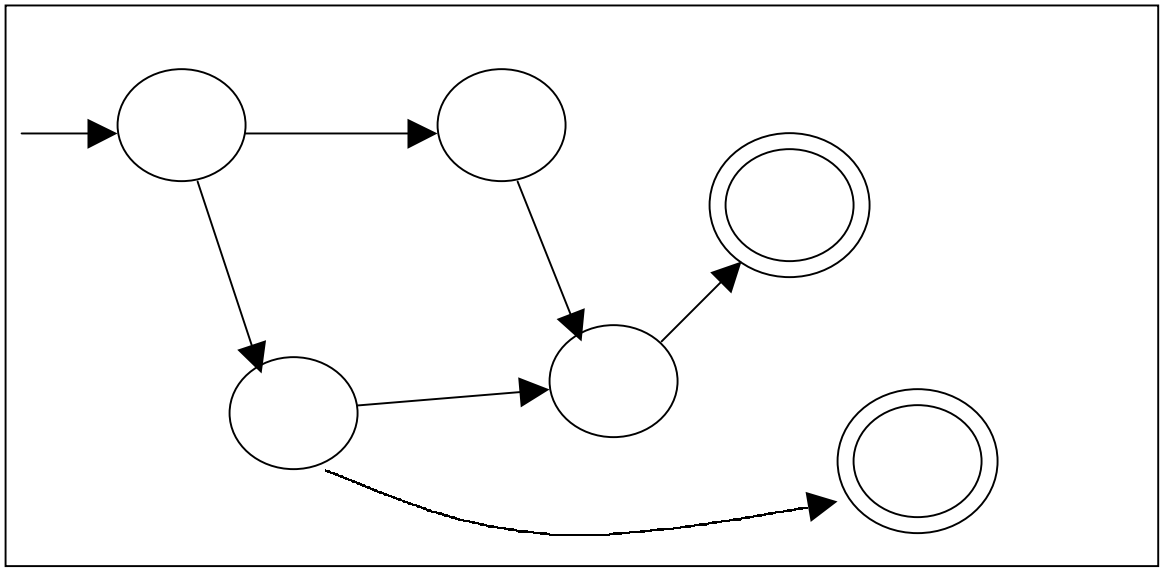
# A Turing Machine

Tape



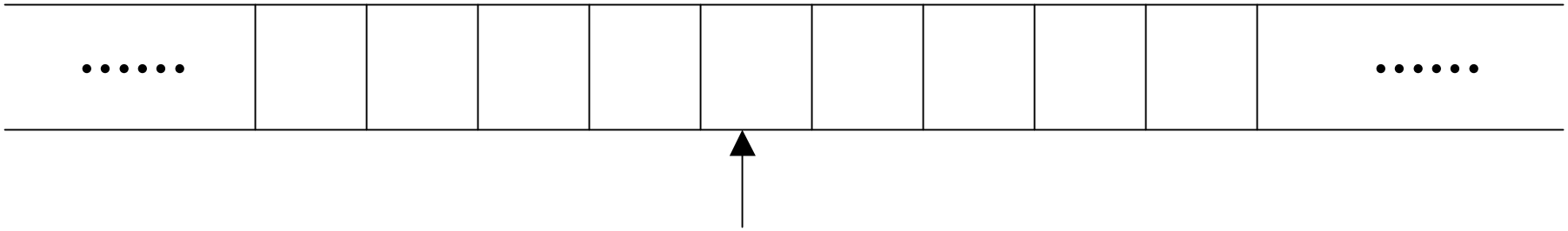

Read-Write head

Control Unit

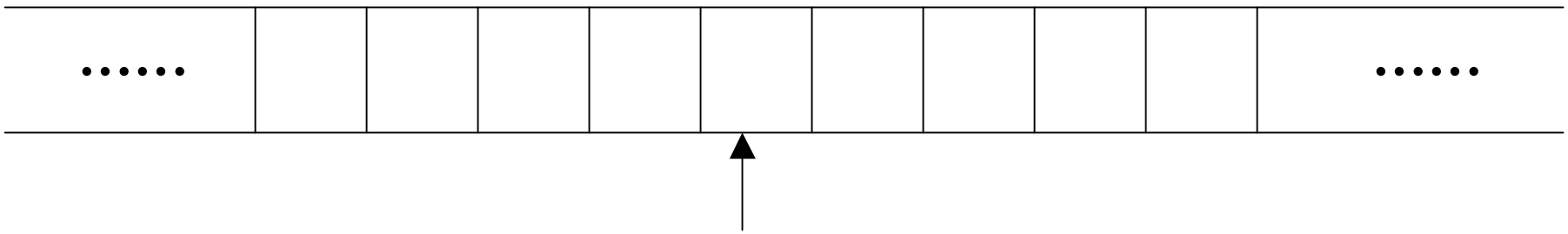


# The Tape

No boundaries (infinite length)




- The head moves Left or Right



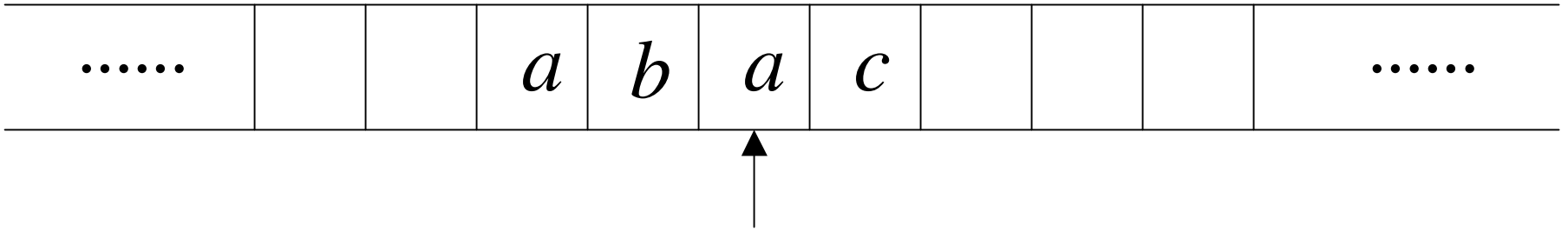

Read-Write head

The head at each time step:

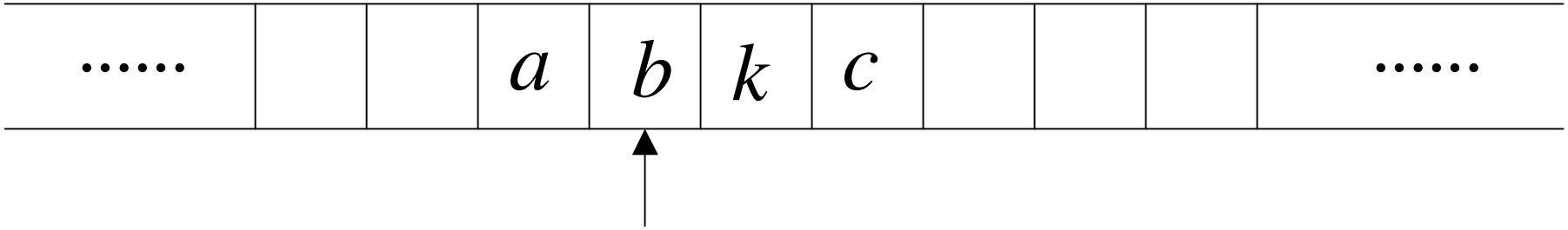
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0

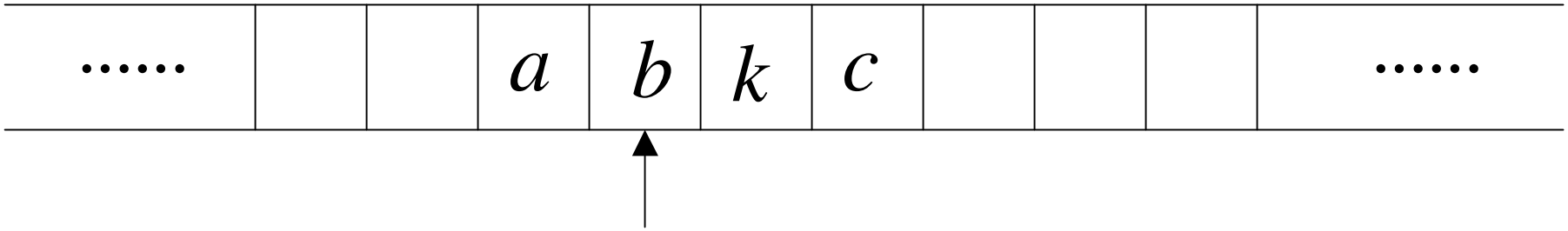



Time 1

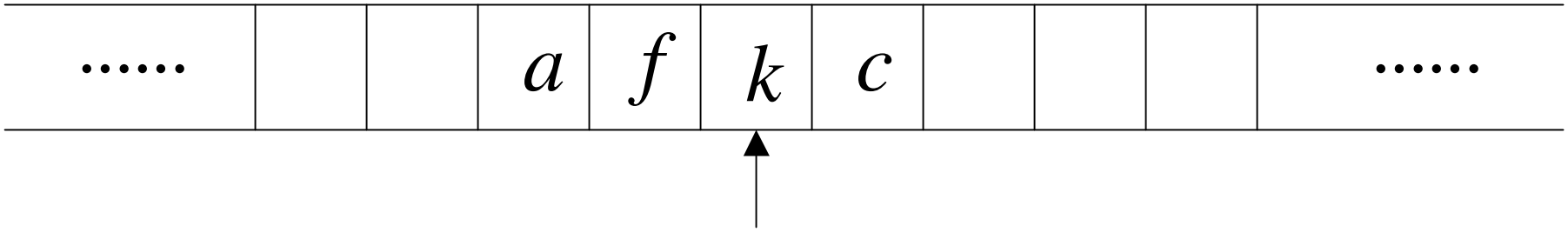



1. Reads $a$
2. Writes $k$
3. Moves Left

Time 1

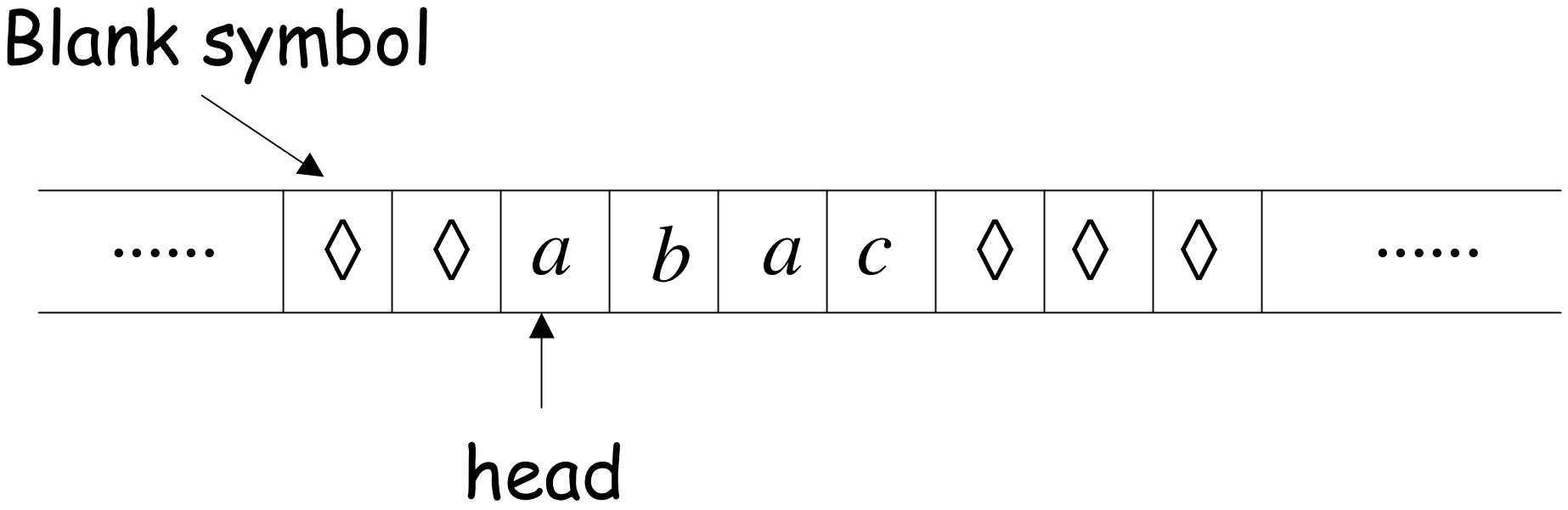



Time 2

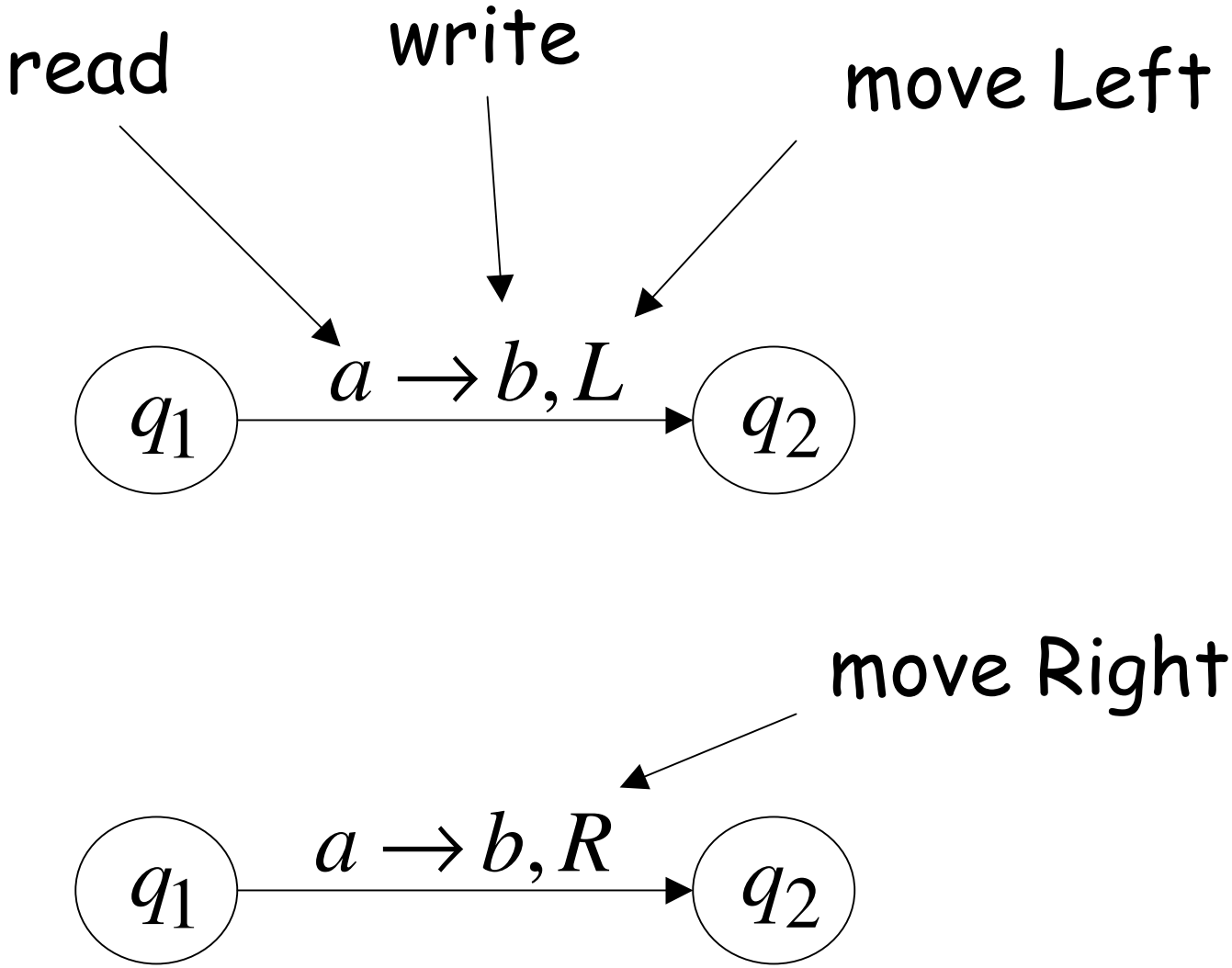



1. Reads $b$
2. Writes $f$
3. Moves Right

# The Input String

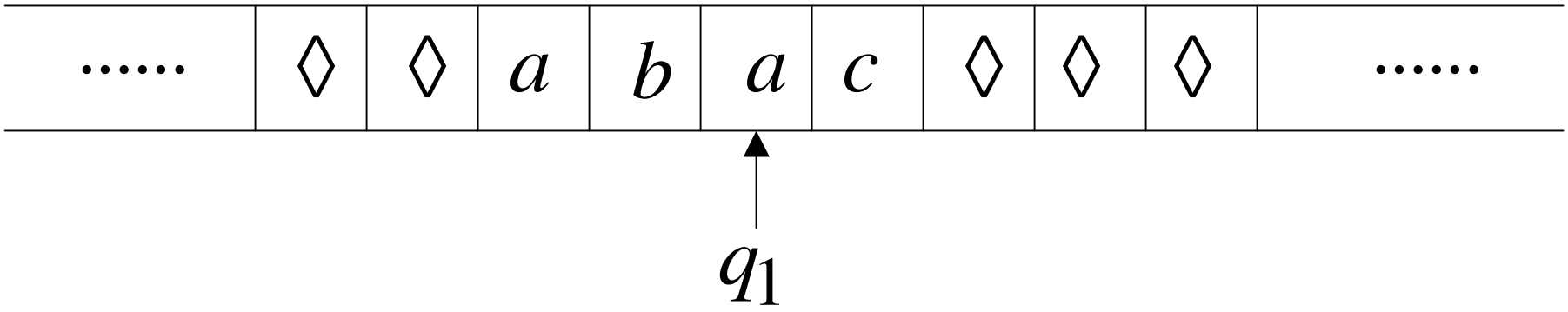


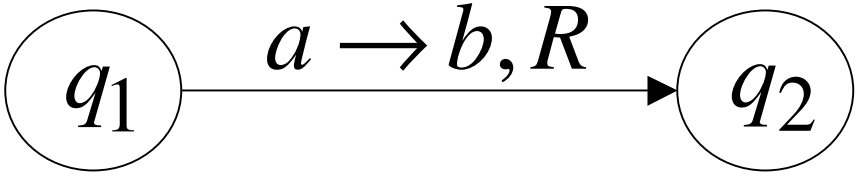
# The States



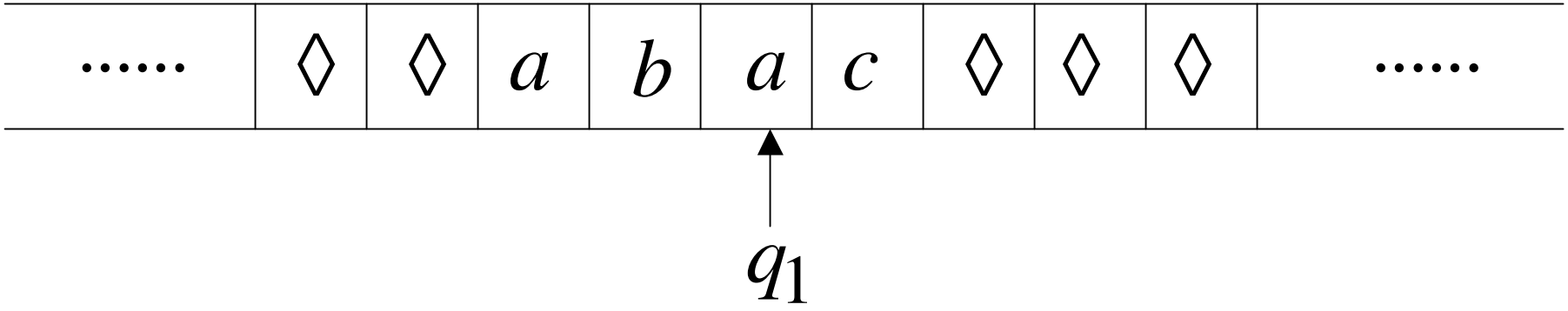
Example:

Time 1

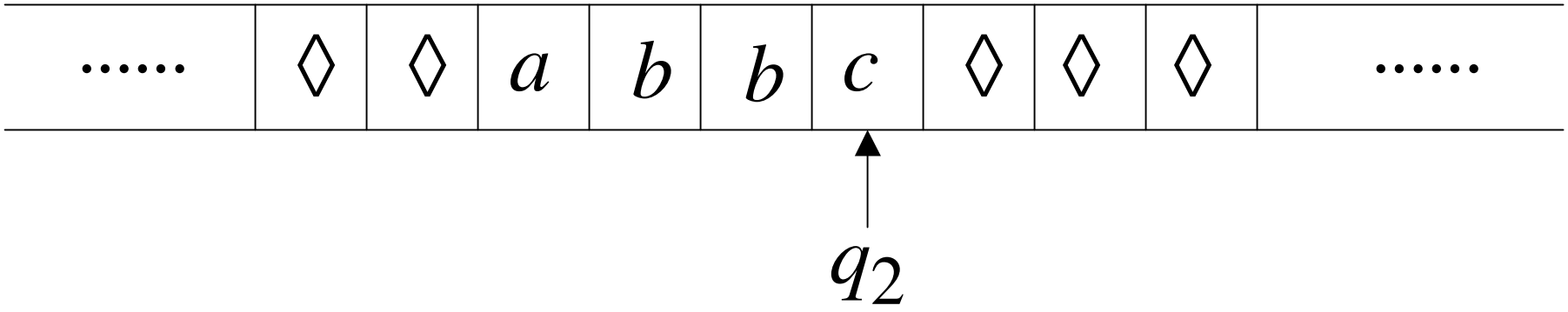


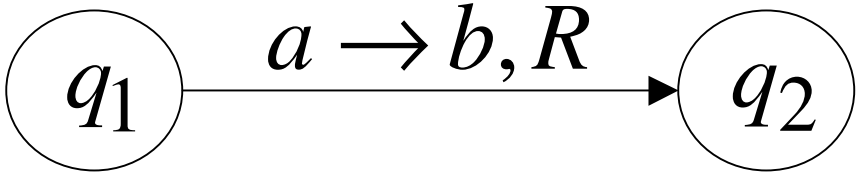



Time 1

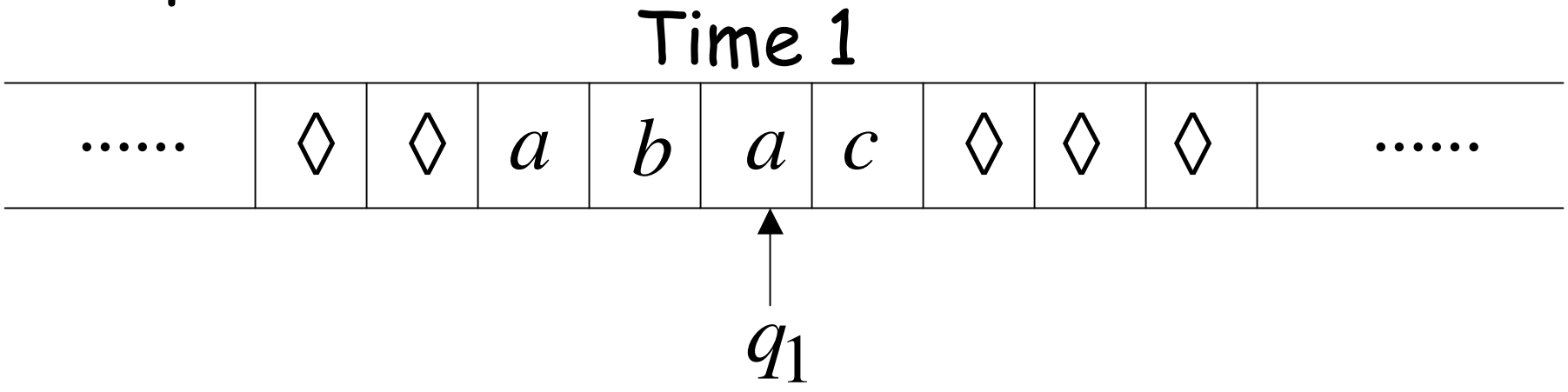


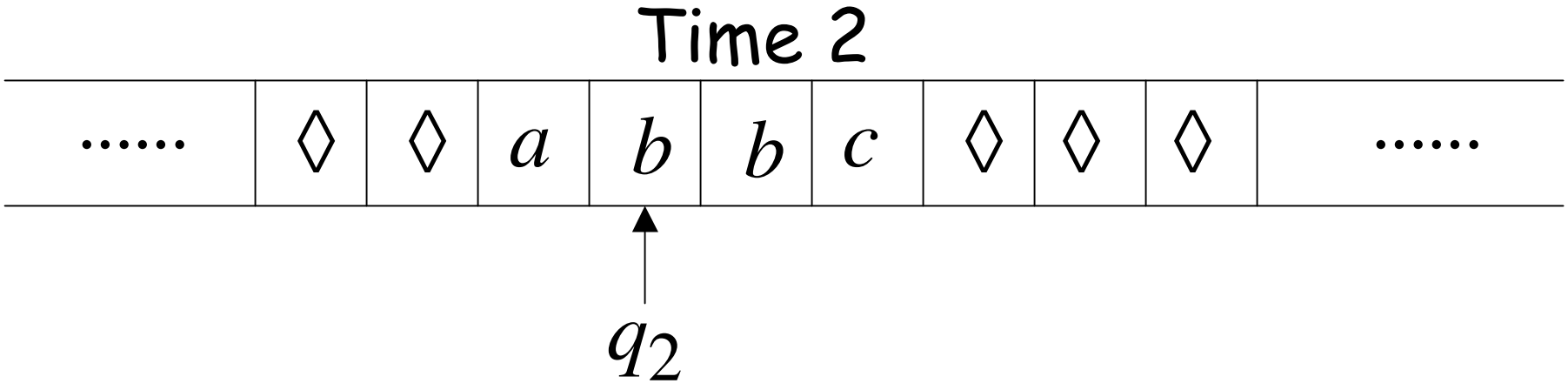

Time 2

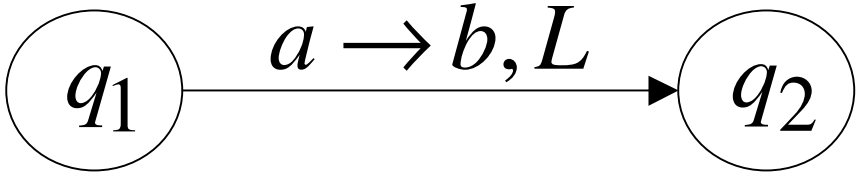


Example:

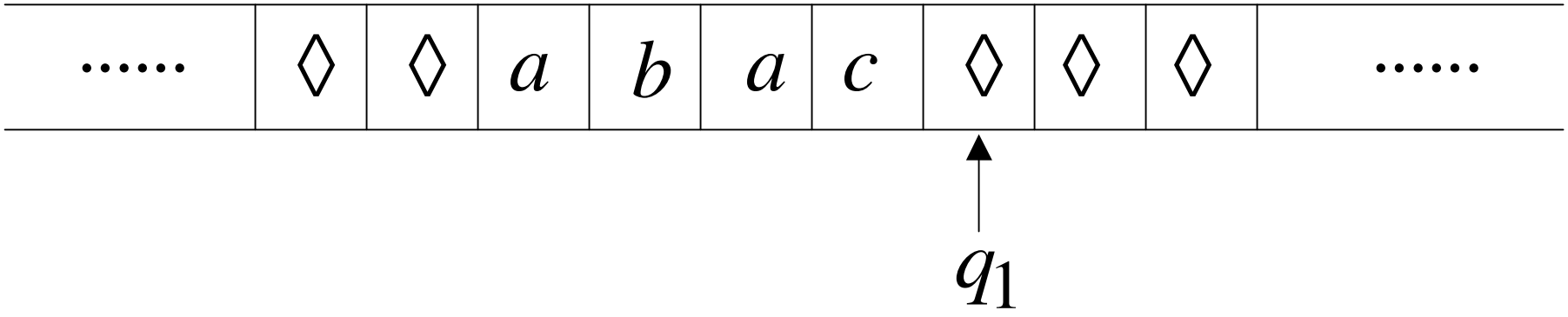


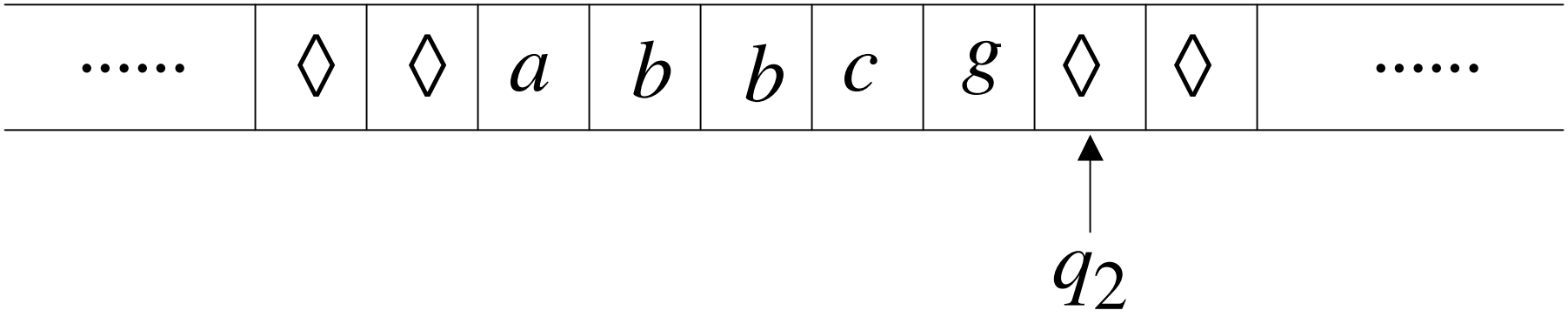



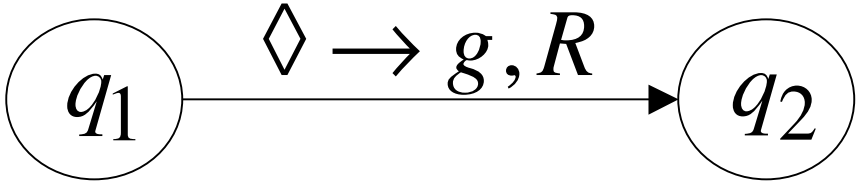
Example:

Time 1



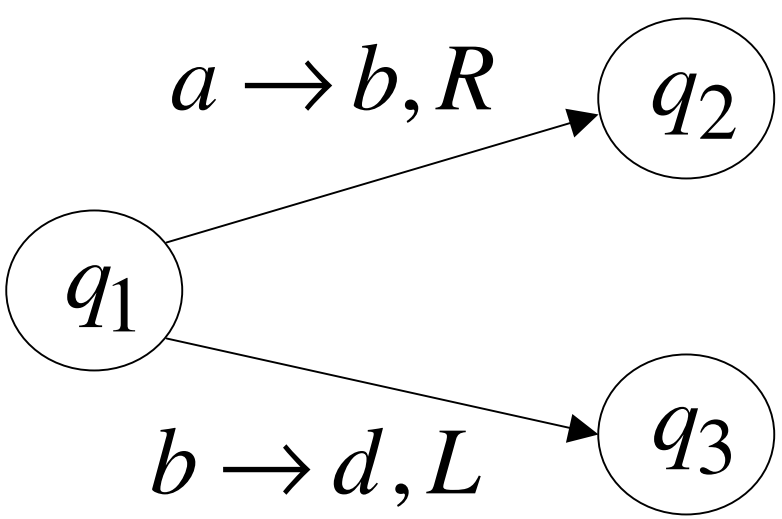

Time 2



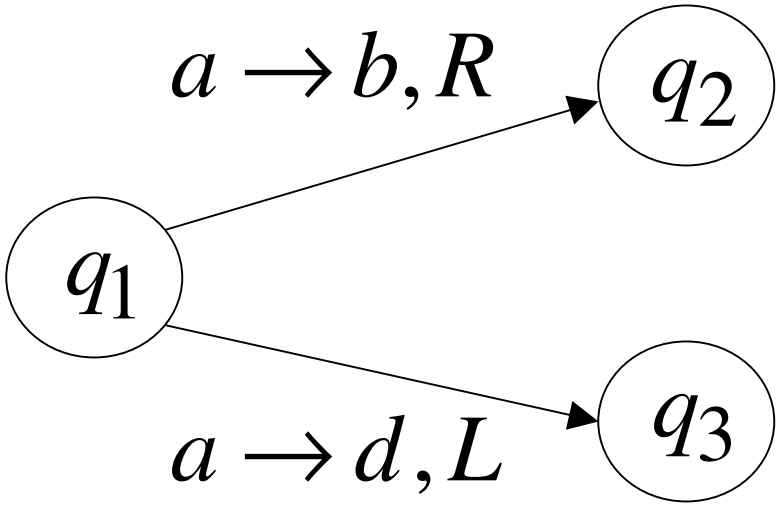



# Determinism

OK



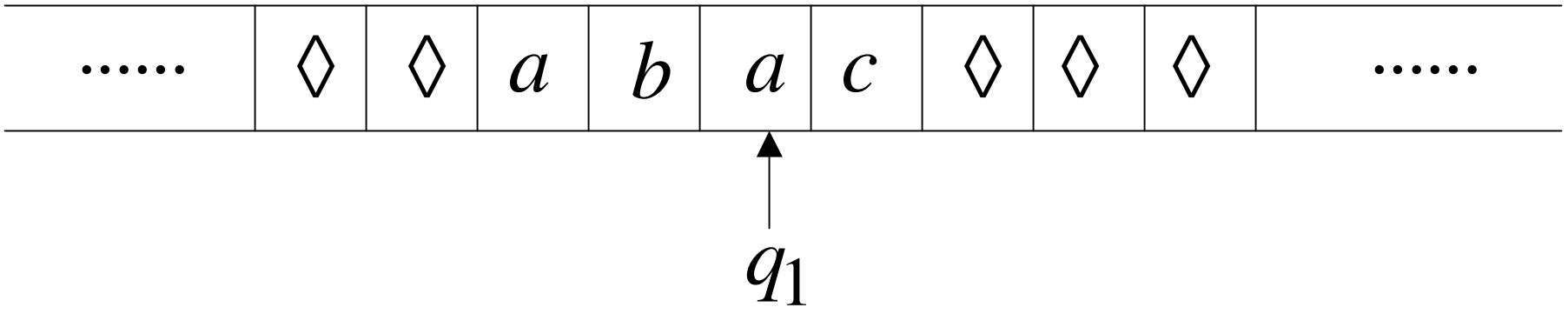

**NOT** ALLOWED

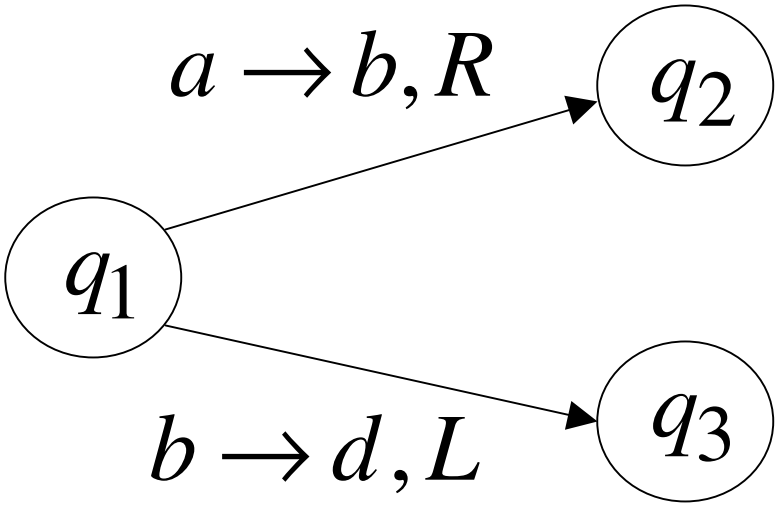



- For a read symbol only one transition is allowed
- No lambda transitions allowed

# Partial Transition Function

Example:



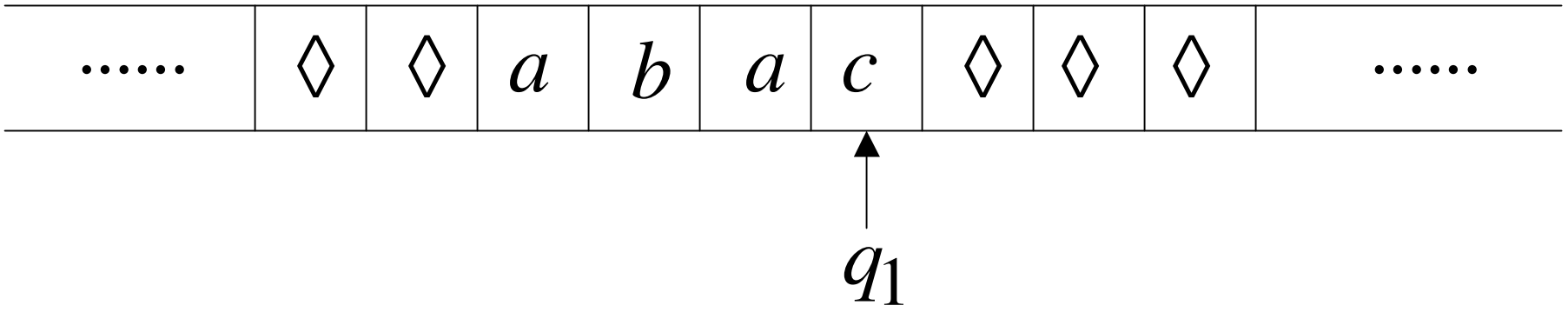

No transition
for input symbol $c$

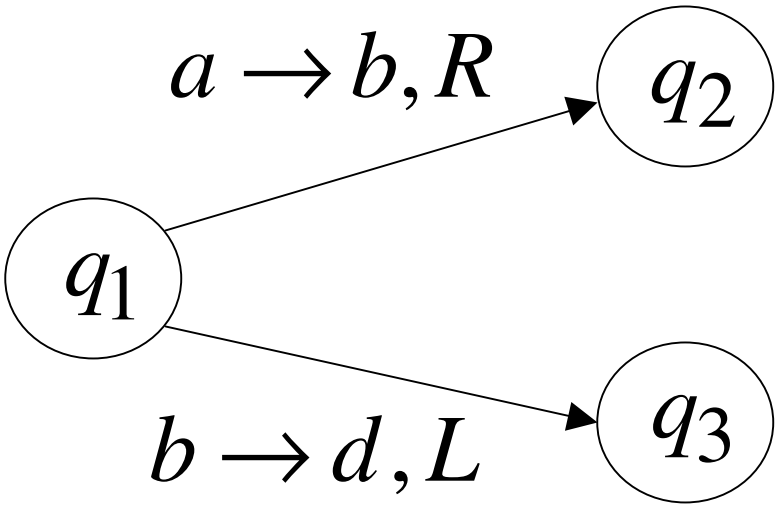
OK

# Halting

The machine ***halts***
if there are no possible transitions
to take

Example:

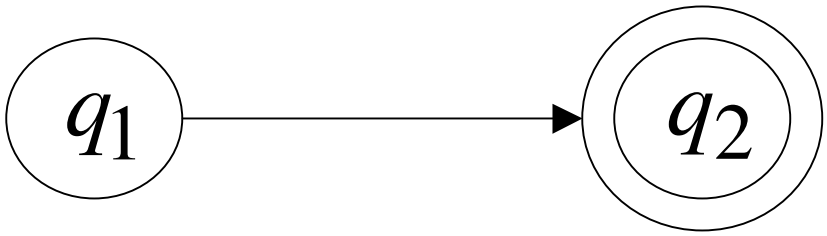



No possible transition

HALT!!!

# Final States

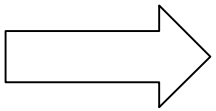


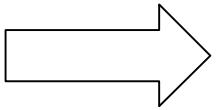

OK

**NOT** ALLOWED

- Final states have no outgoing transitions
- In a final state the machine halts

# Acceptance

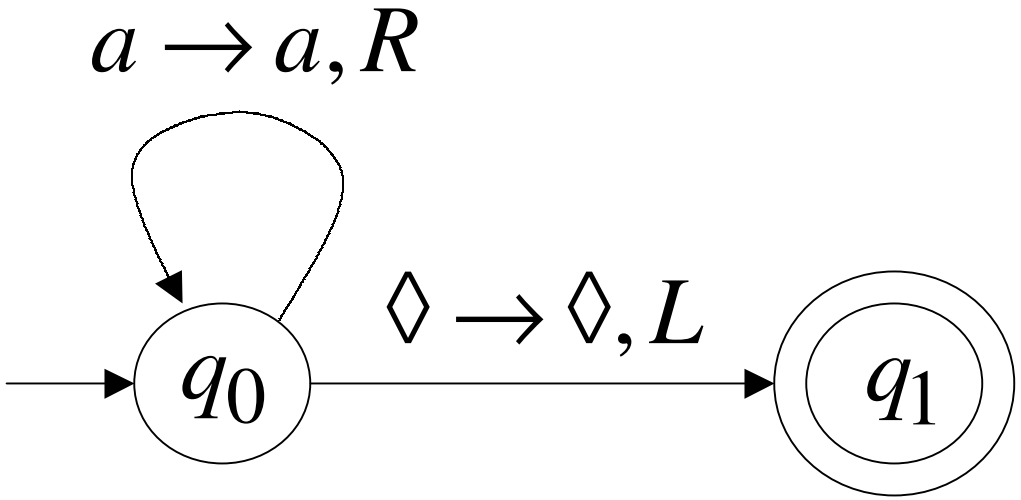
Accept Input  If machine halts in a final state

Reject Input 

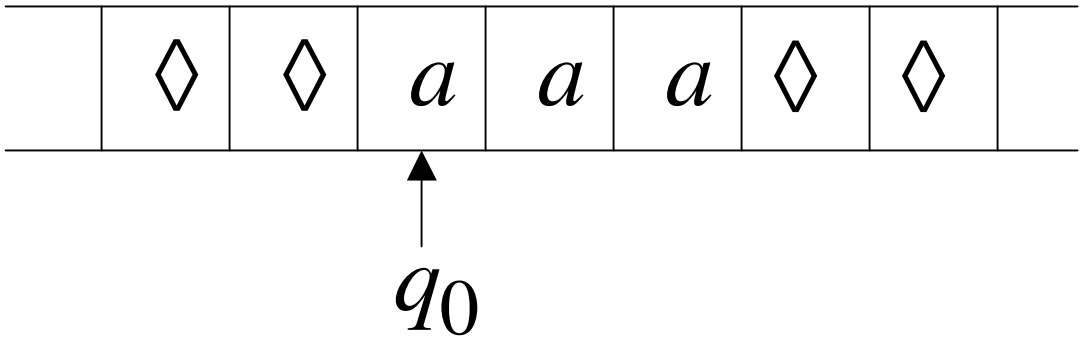
- If machine halts in a non-final state
- If machine enters an *infinite loop*

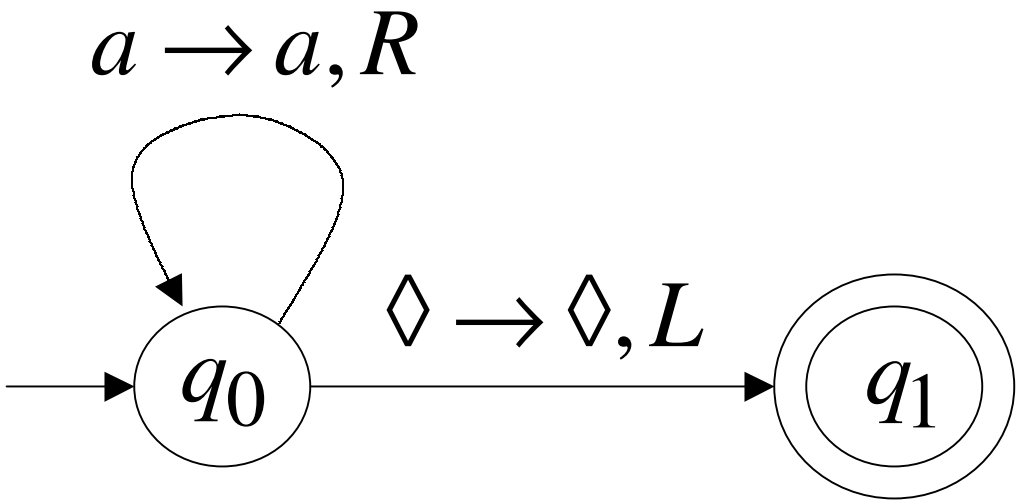
# Turing Machine Example

A turing machine for the language $aa^*$

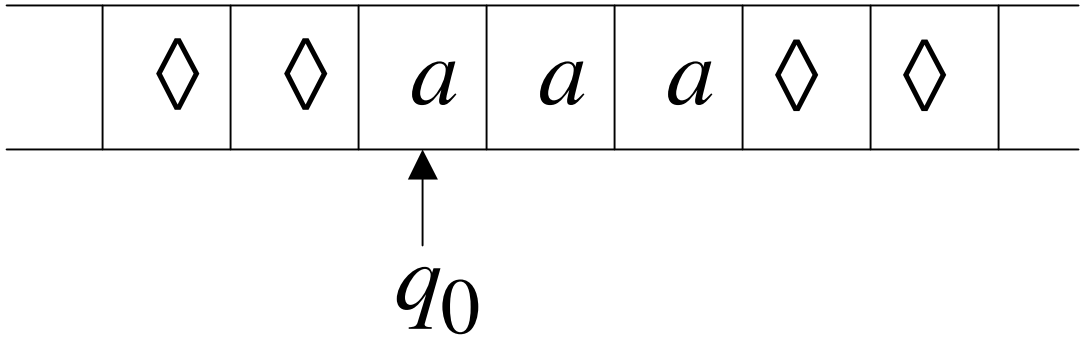


Time 0

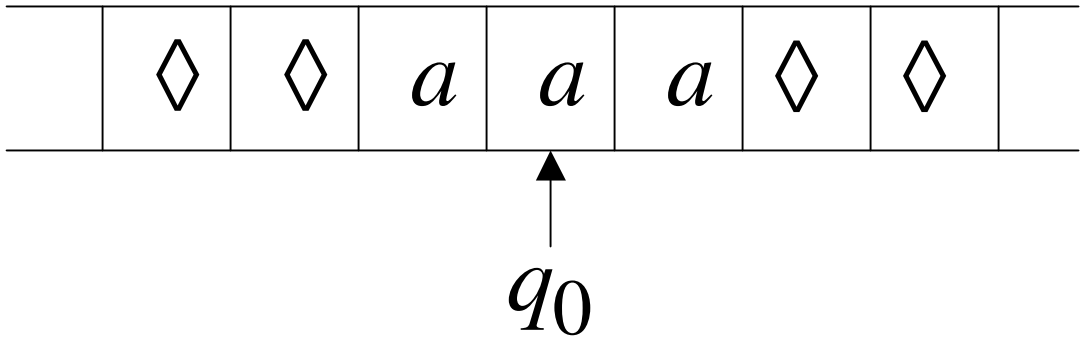


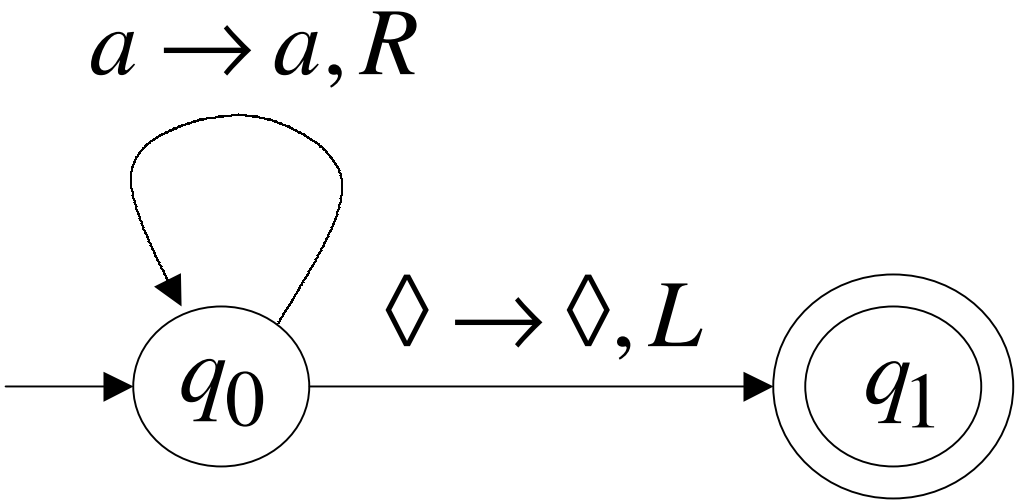



Time 0

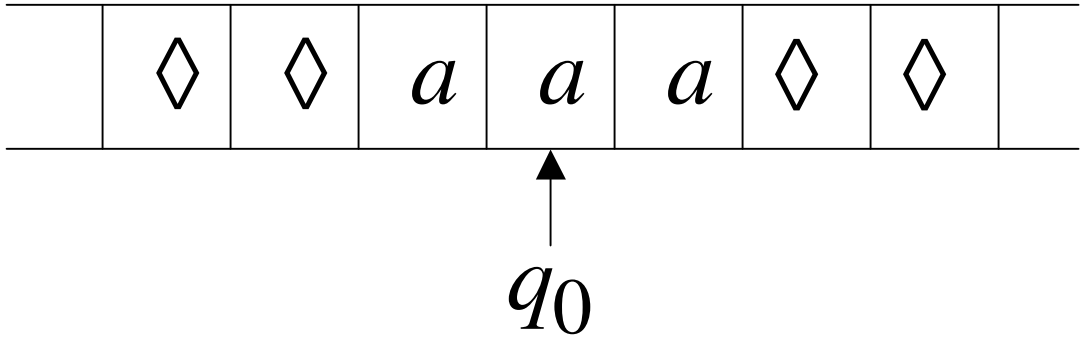



Time 1

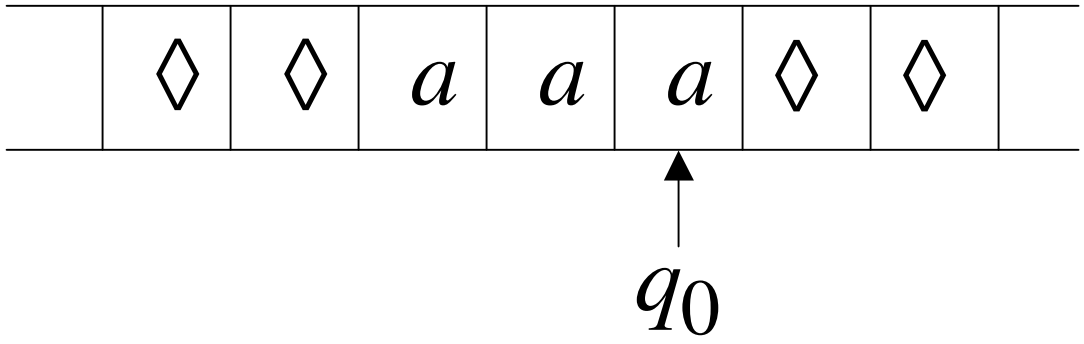


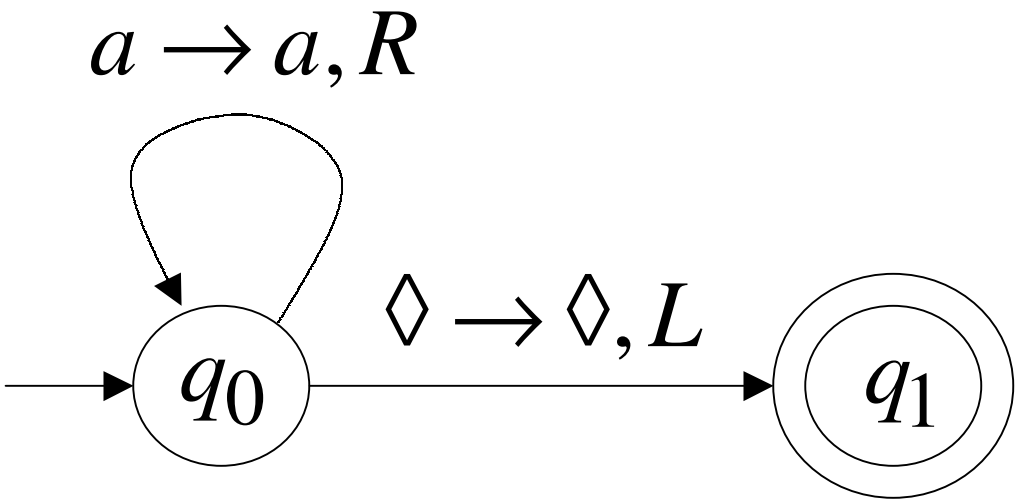



Time 1

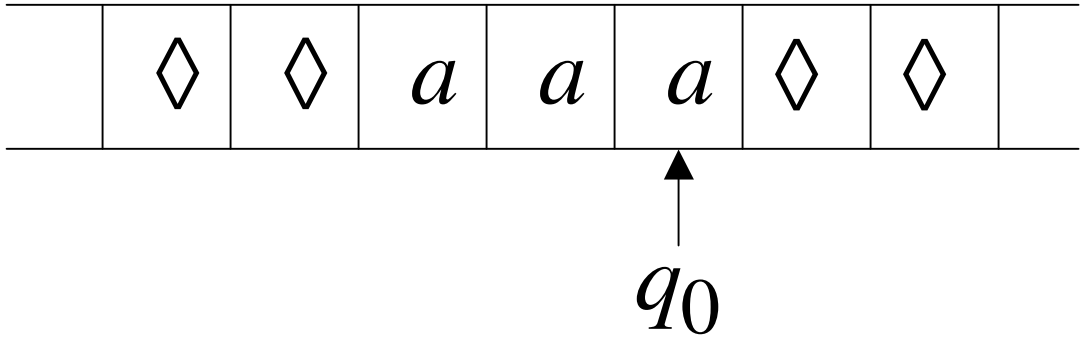



Time 2

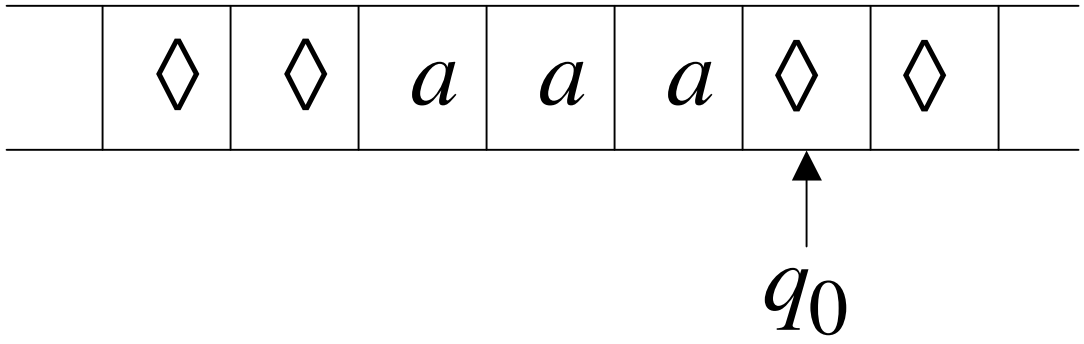


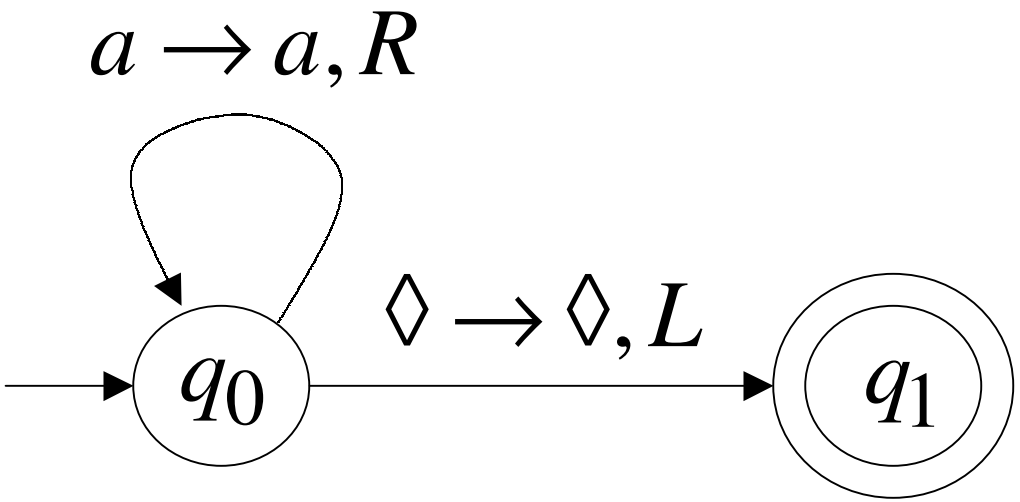



Time 2

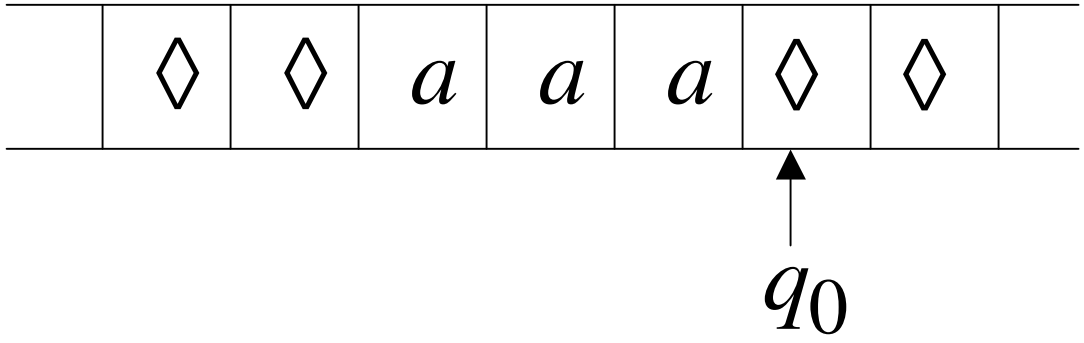



Time 3

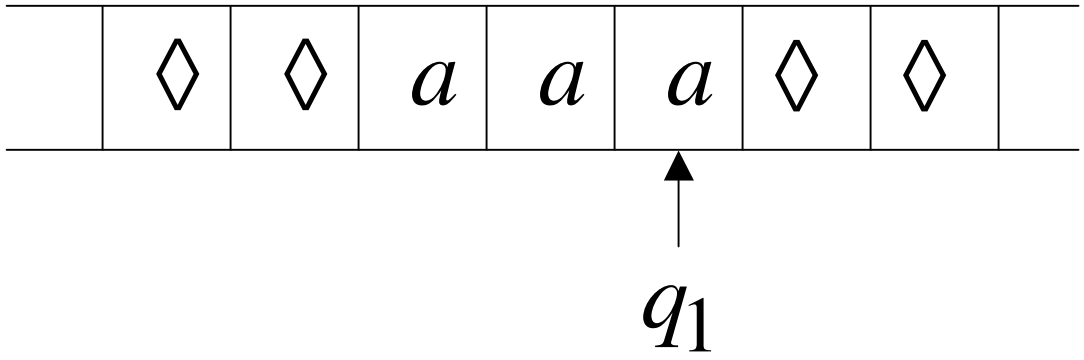


Time 3

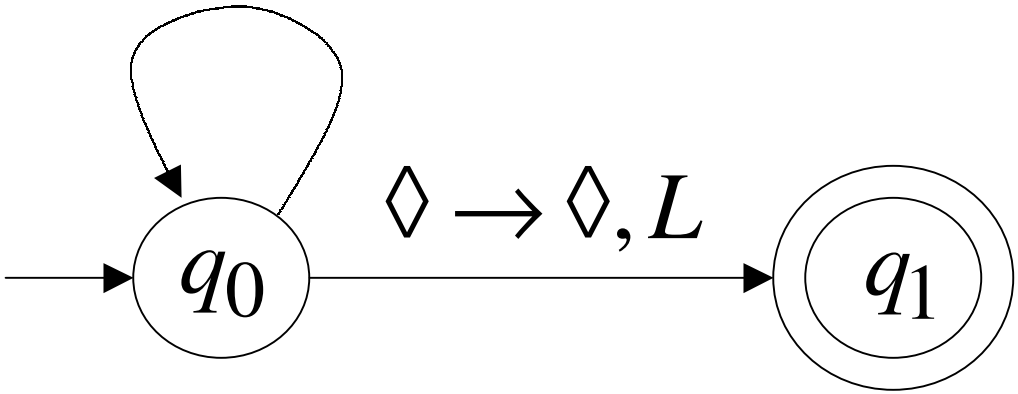



Time 4



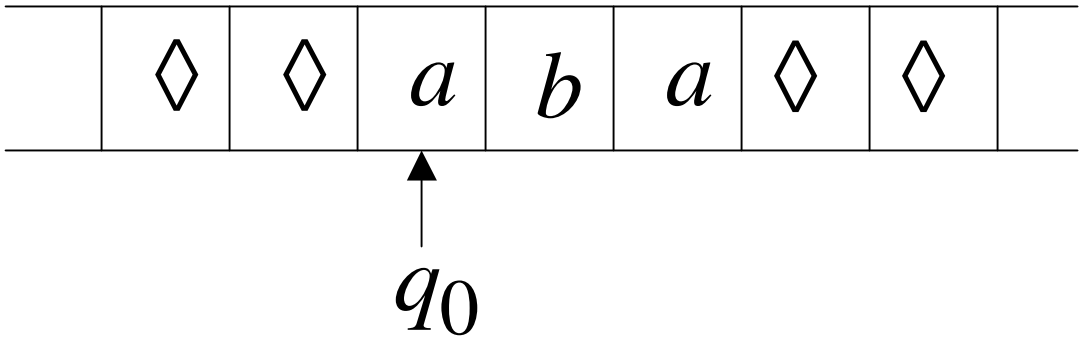

$a \to a, R$

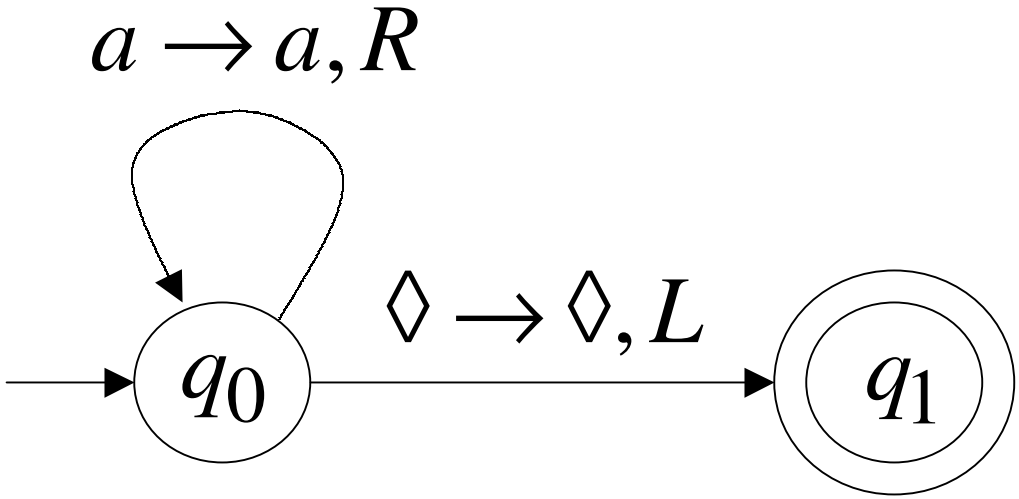
HALT & accept



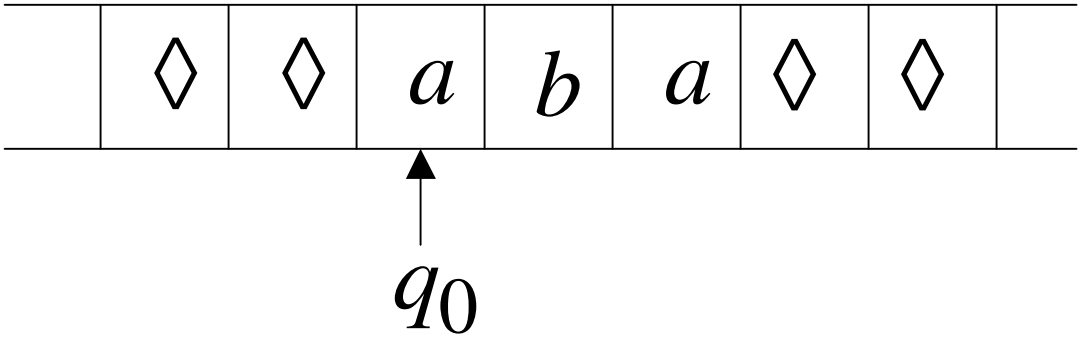
# Rejection Example

Time 0

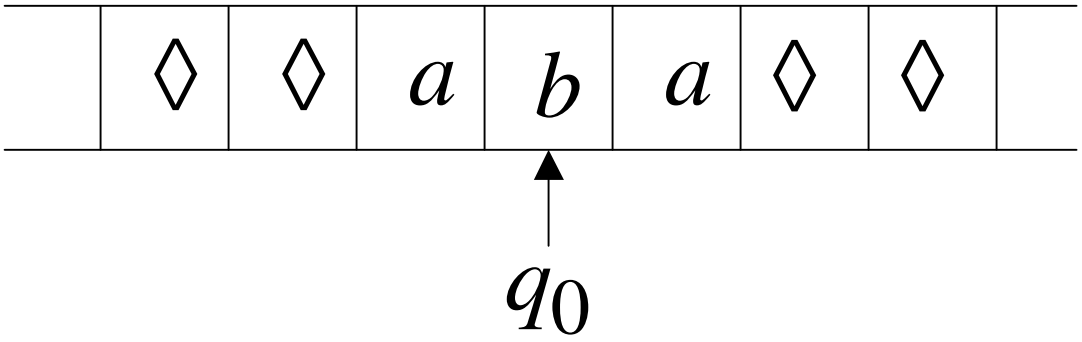


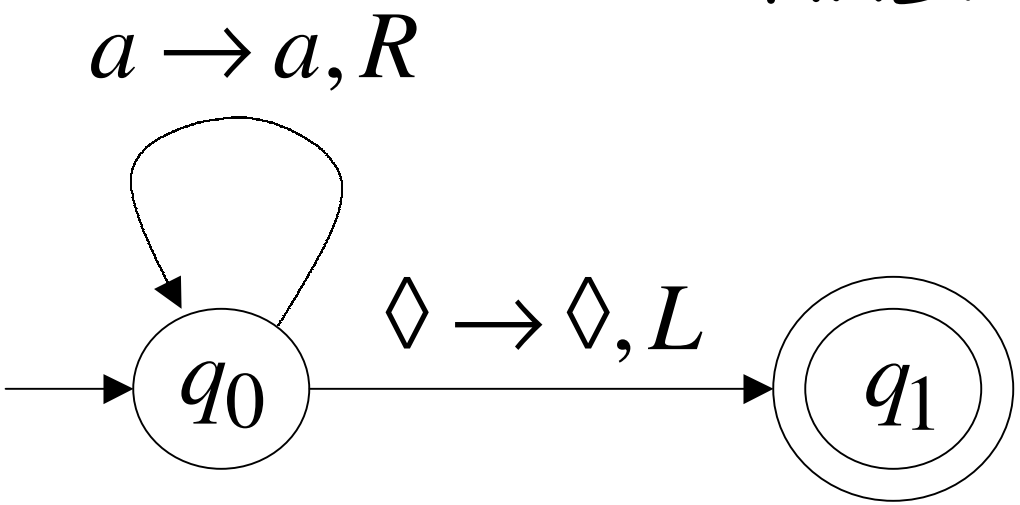
Time 0




Time 1

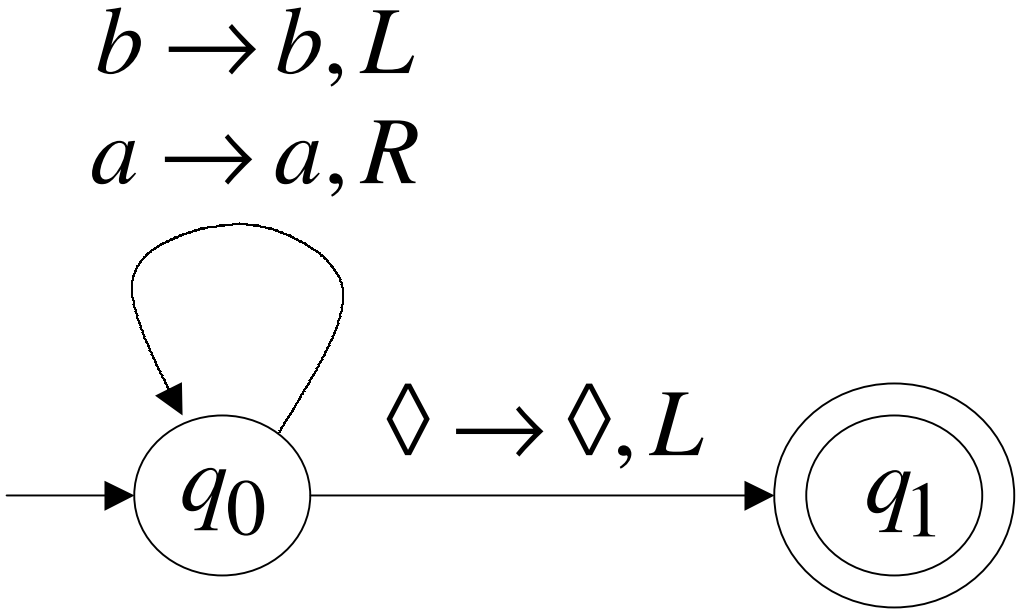



HALT & reject

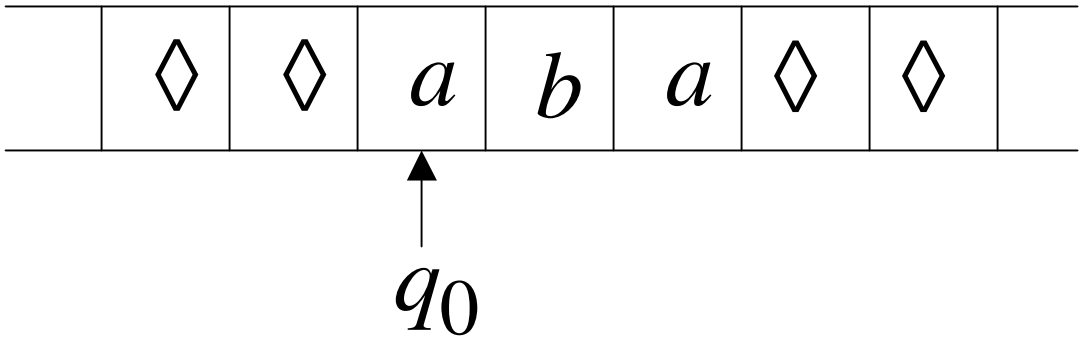


# Infinite Loop Example

Turing machine for the language $aa^*$

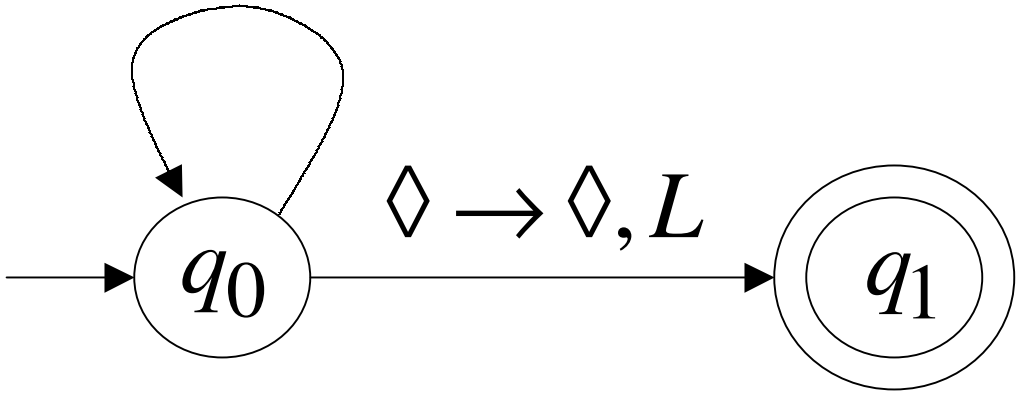


Time 0

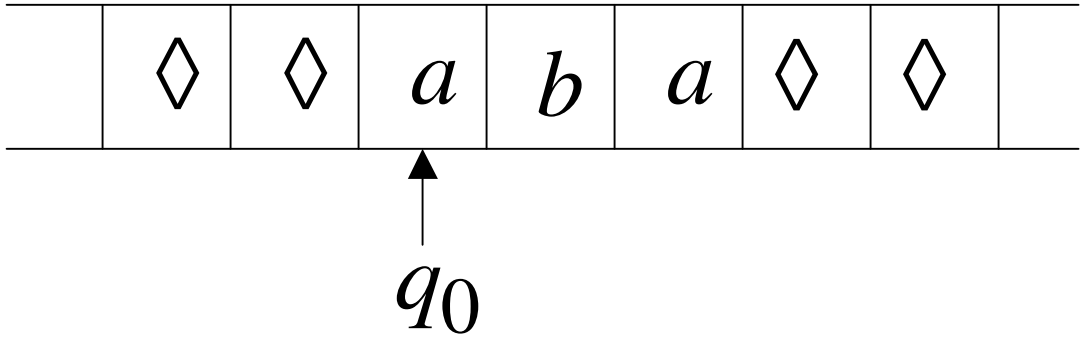



$b \to b, L$

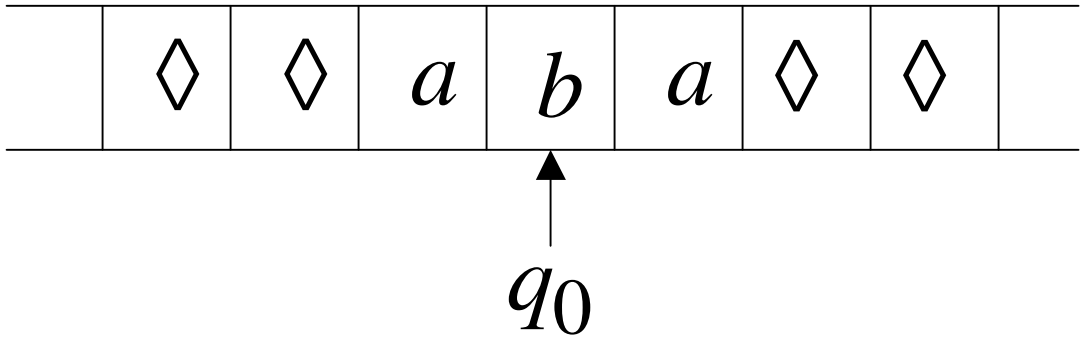
$a \to a, R$



Time 0

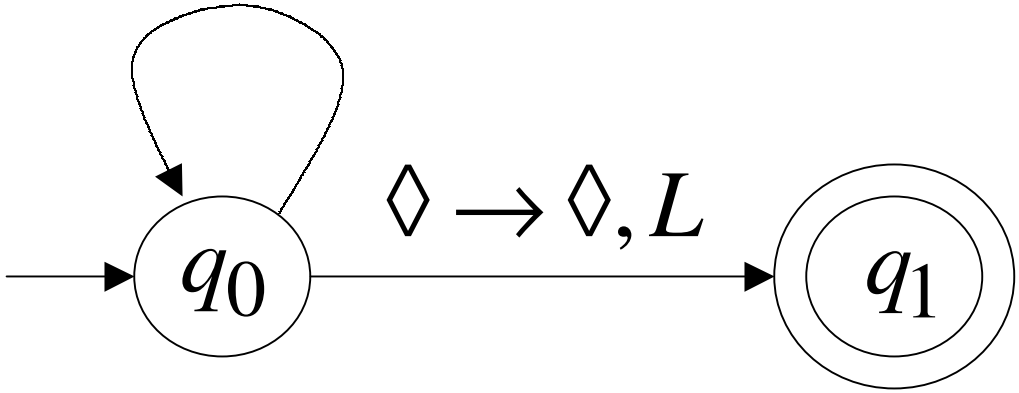



Time 1

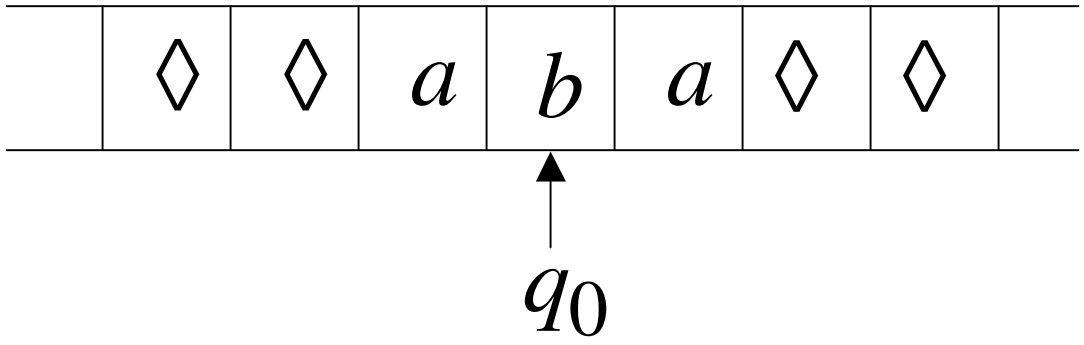



$b \to b, L$

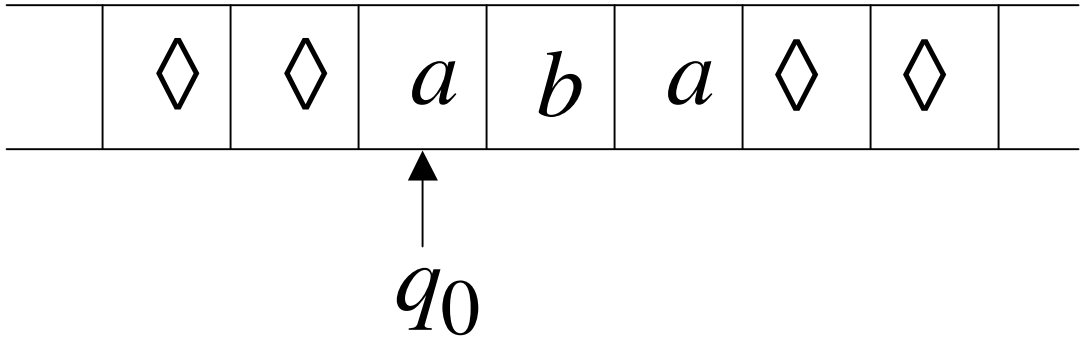
$a \to a, R$



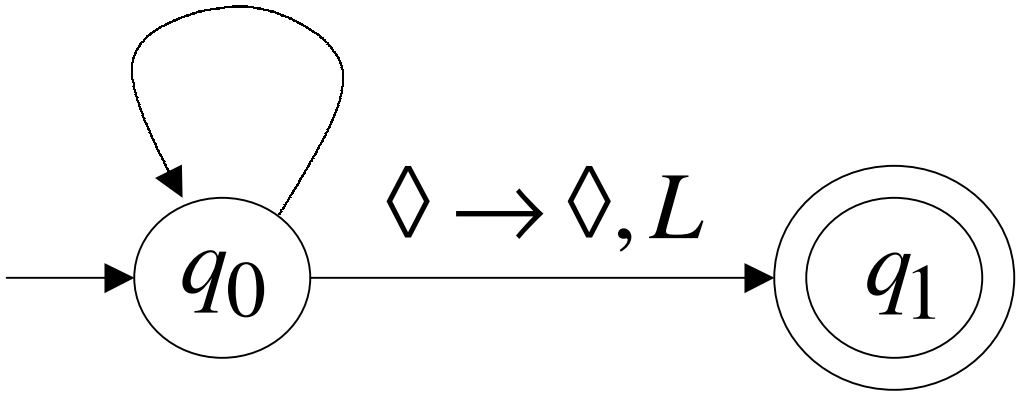
Time 1



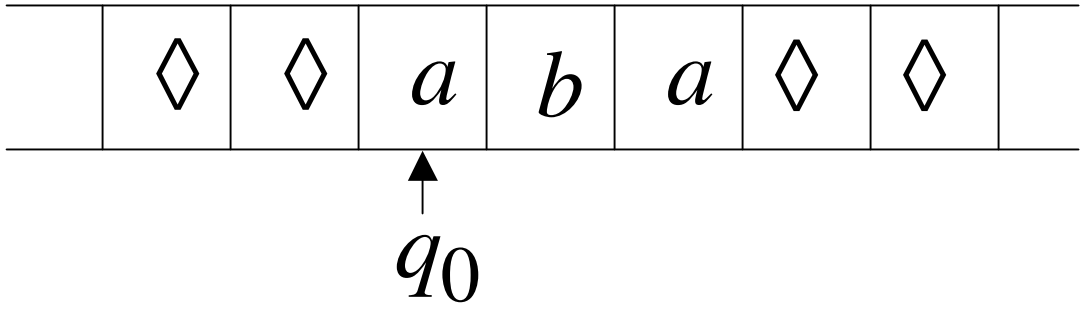

Time 2



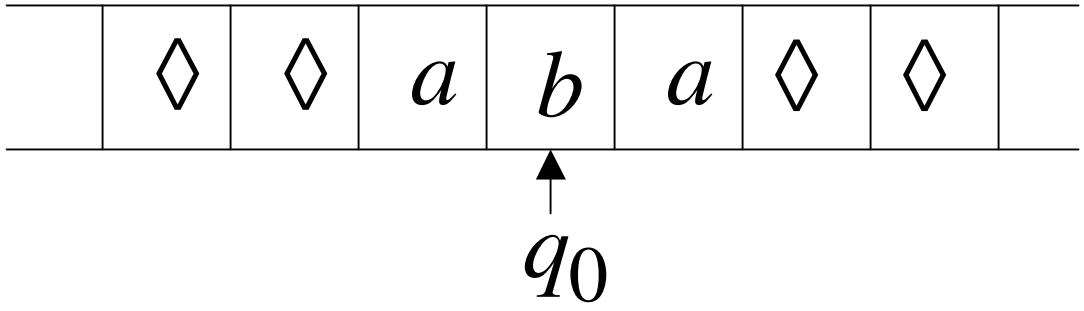

$$b \to b, L$$
$$a \to a, R$$



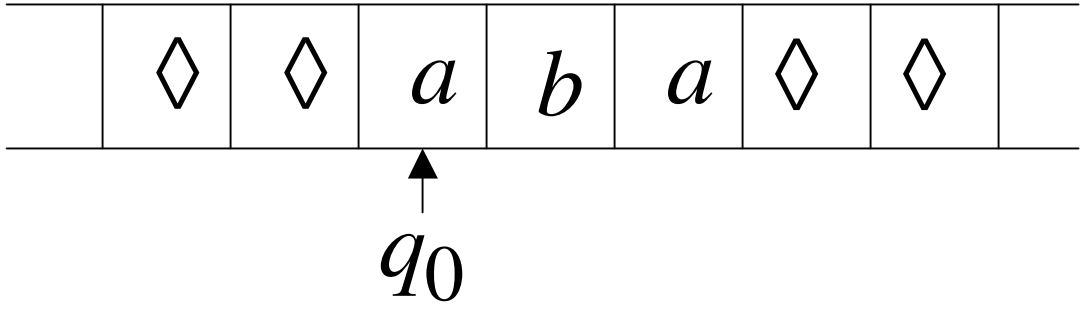
Time 2



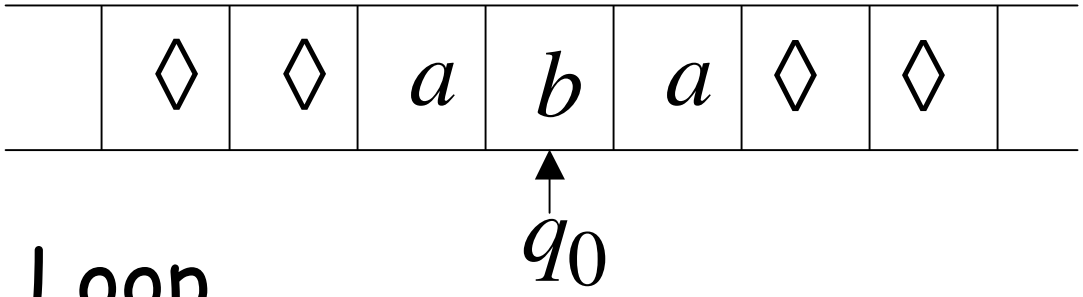

Time 3




Time 4




Time 5

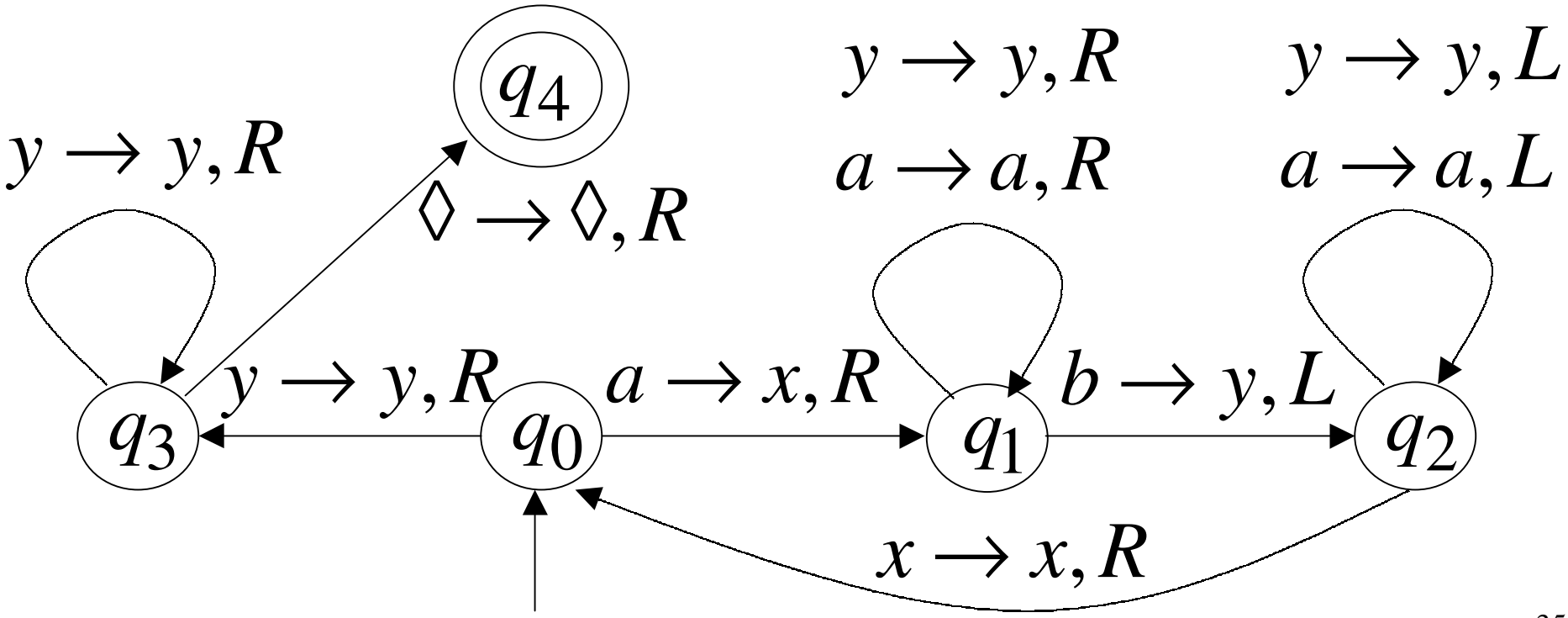



...Infinite Loop

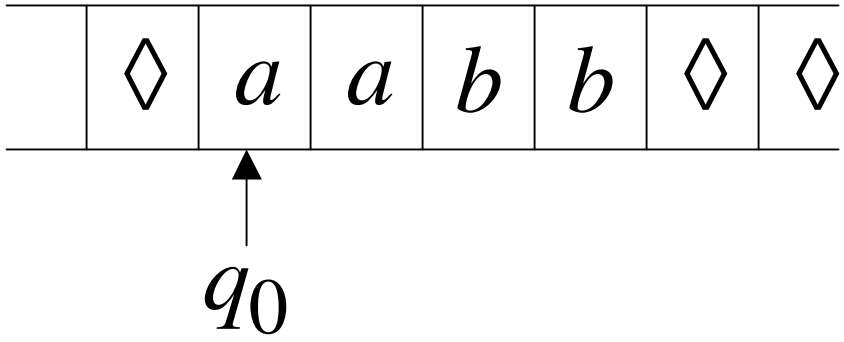
Because of the infinite loop
the final state cannot be reached
and
the input is not accepted

# Another Turing Machine Example

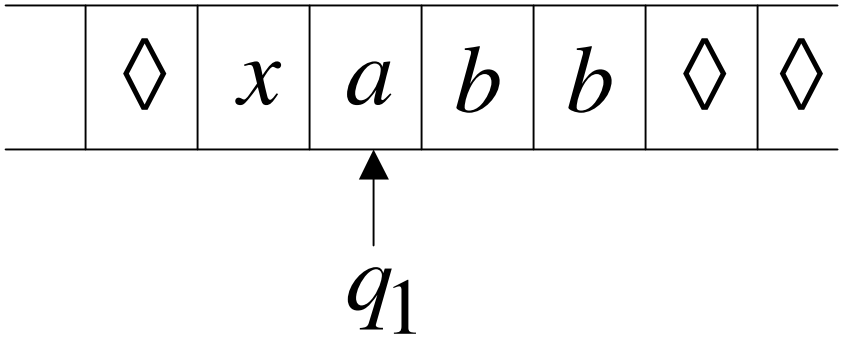
Turing machine for the language $\{a^n b^n\}$

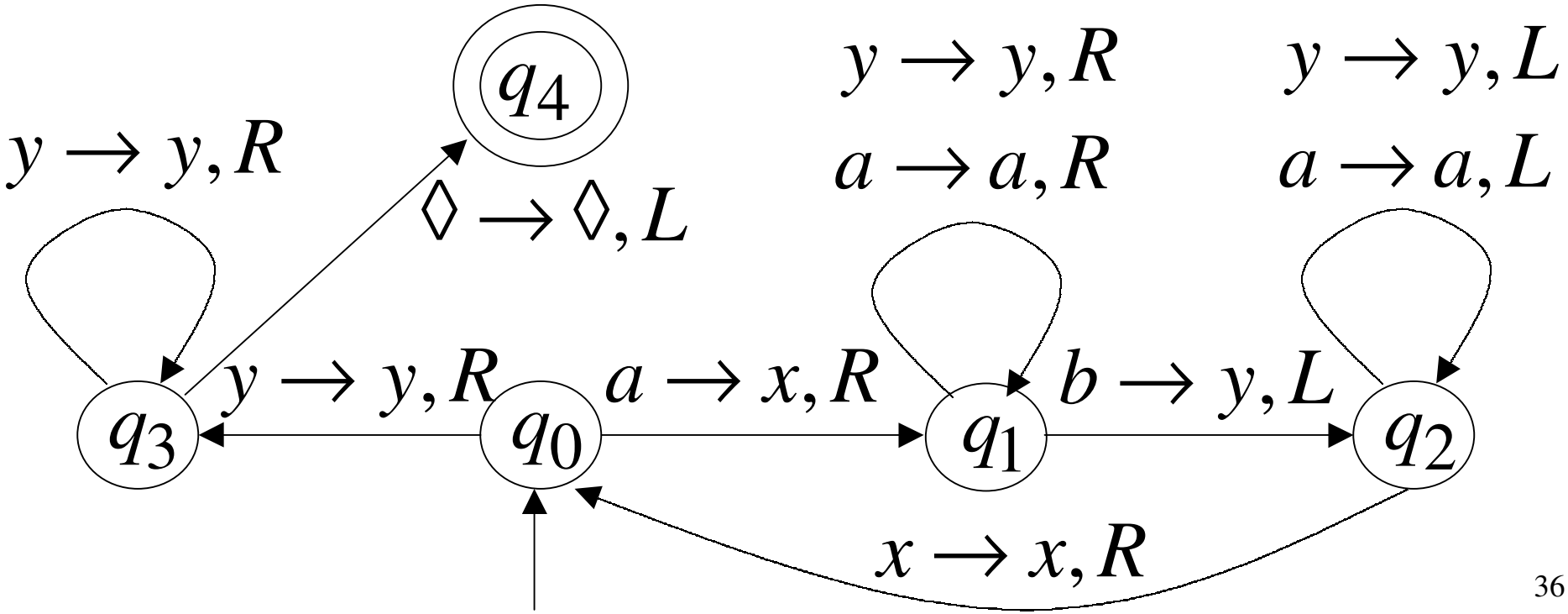


Time 0

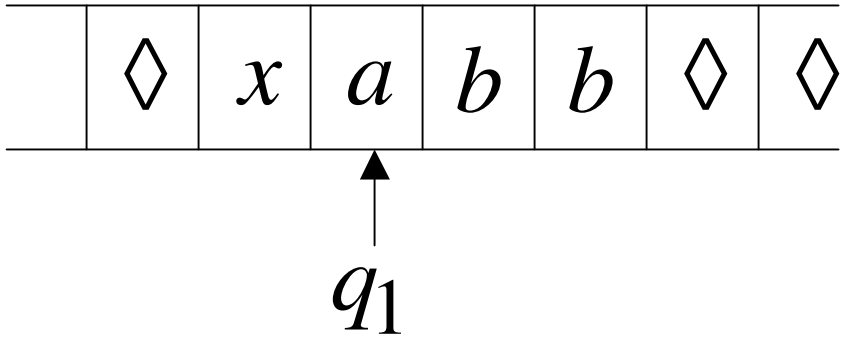

◊ a a b b ◊ ◊
$q_0$

Time 1

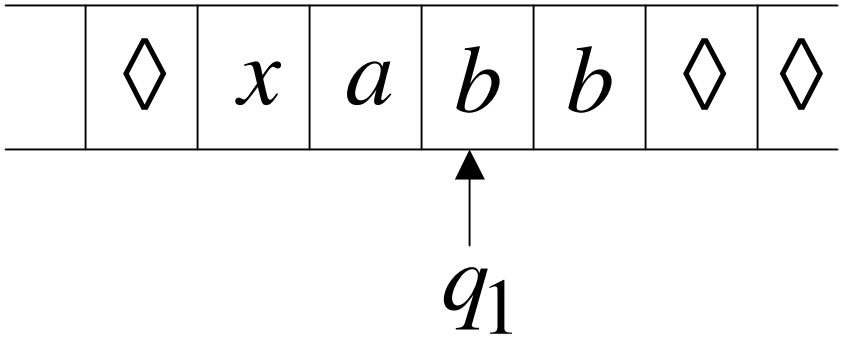

◊ x a b b ◊ ◊
$q_1$

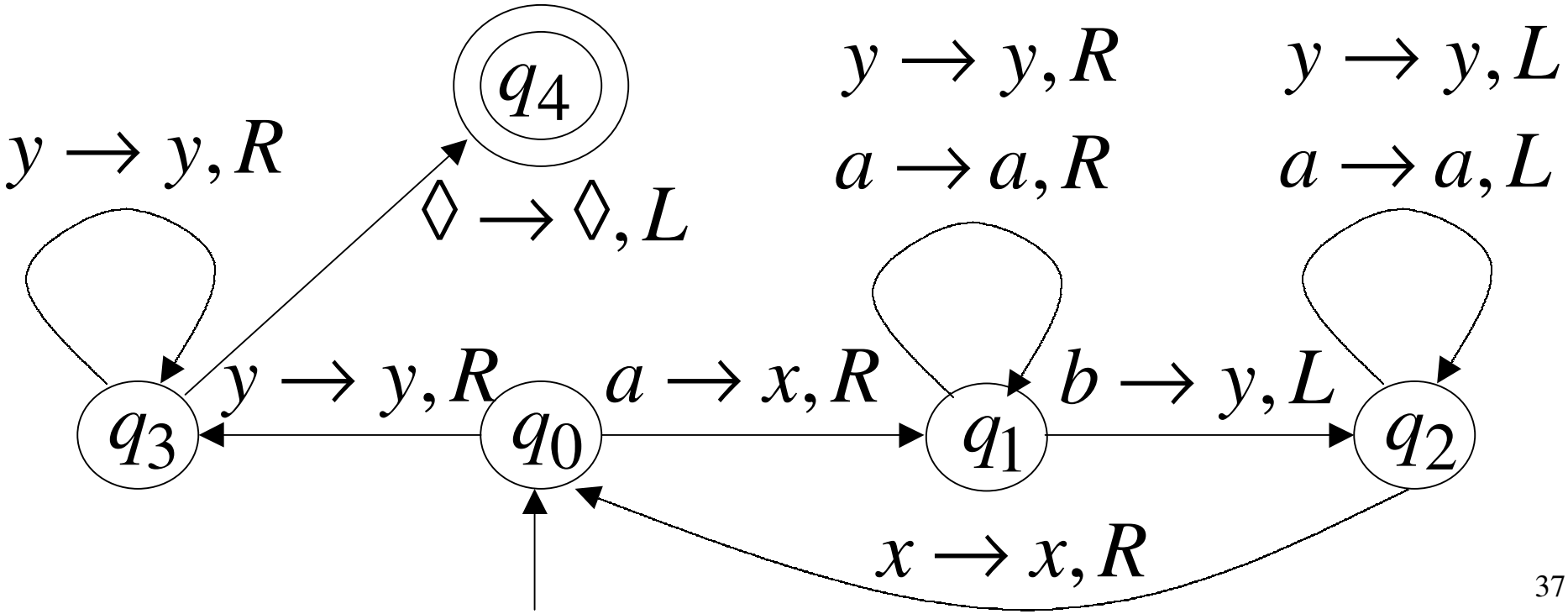

$q_4$
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$q_3$
$y \to y, R$
$q_0$
$a \to x, R$
$q_1$
$b \to y, L$
$q_2$
$x \to x, R$

Time 1

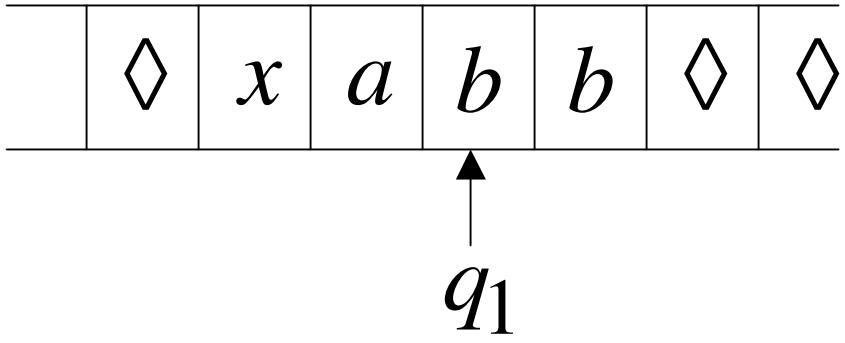

◊ x a b b ◊ ◊
$q_1$

Time 2

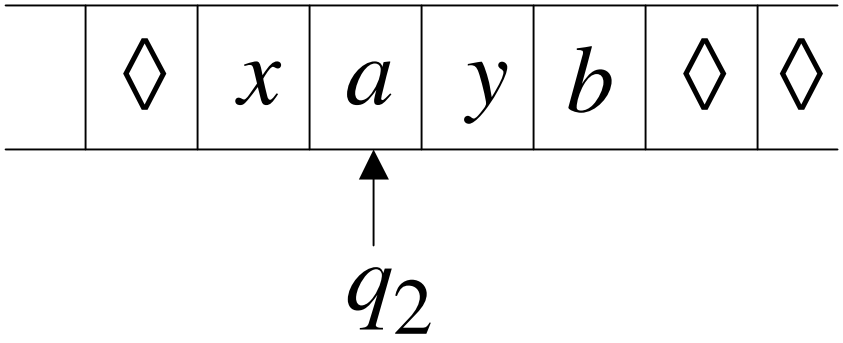

◊ x a b b ◊ ◊
$q_1$

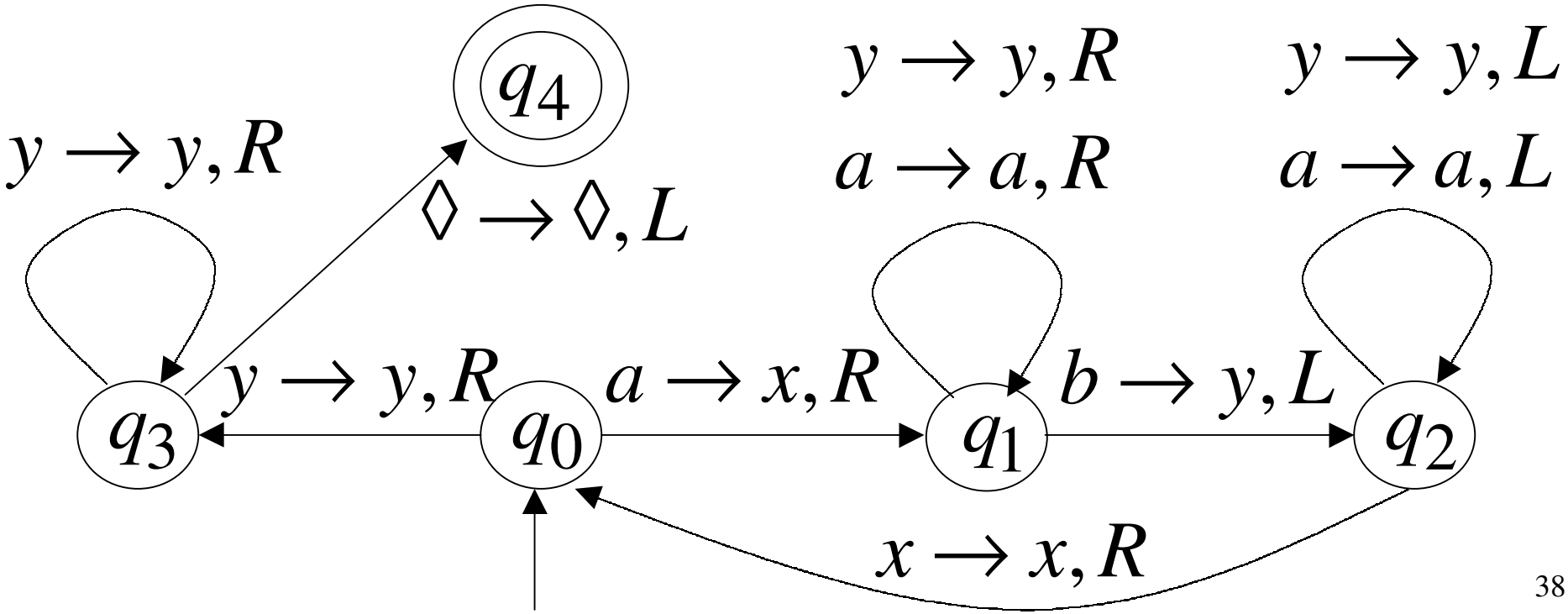

$q_4$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
$q_3$
$q_0$
$q_1$
$q_2$
$x \to x, R$

Time 2

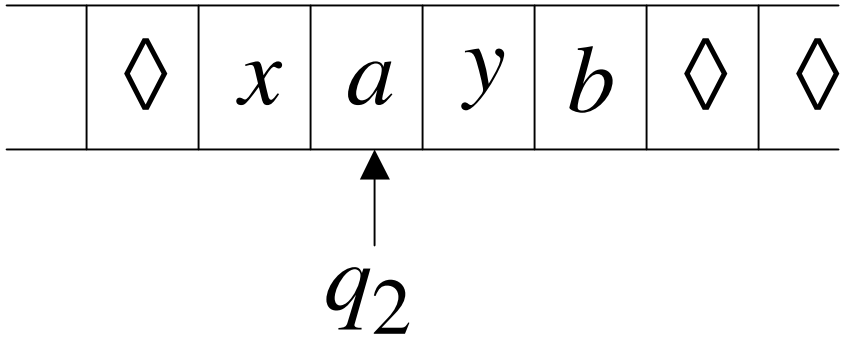

◊ x a b b ◊ ◊
$q_1$

Time 3

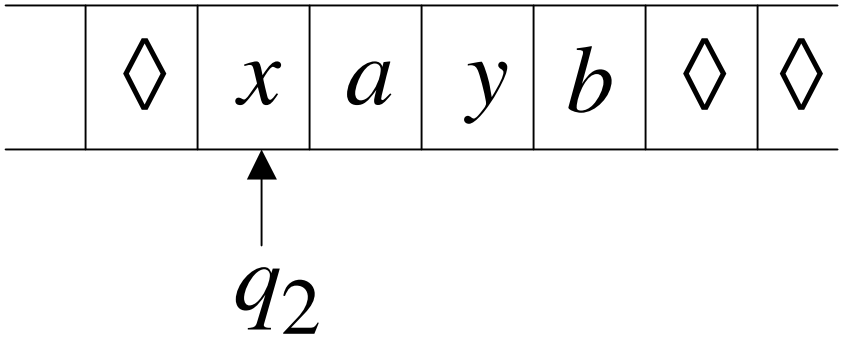

◊ x a y b ◊ ◊
$q_2$

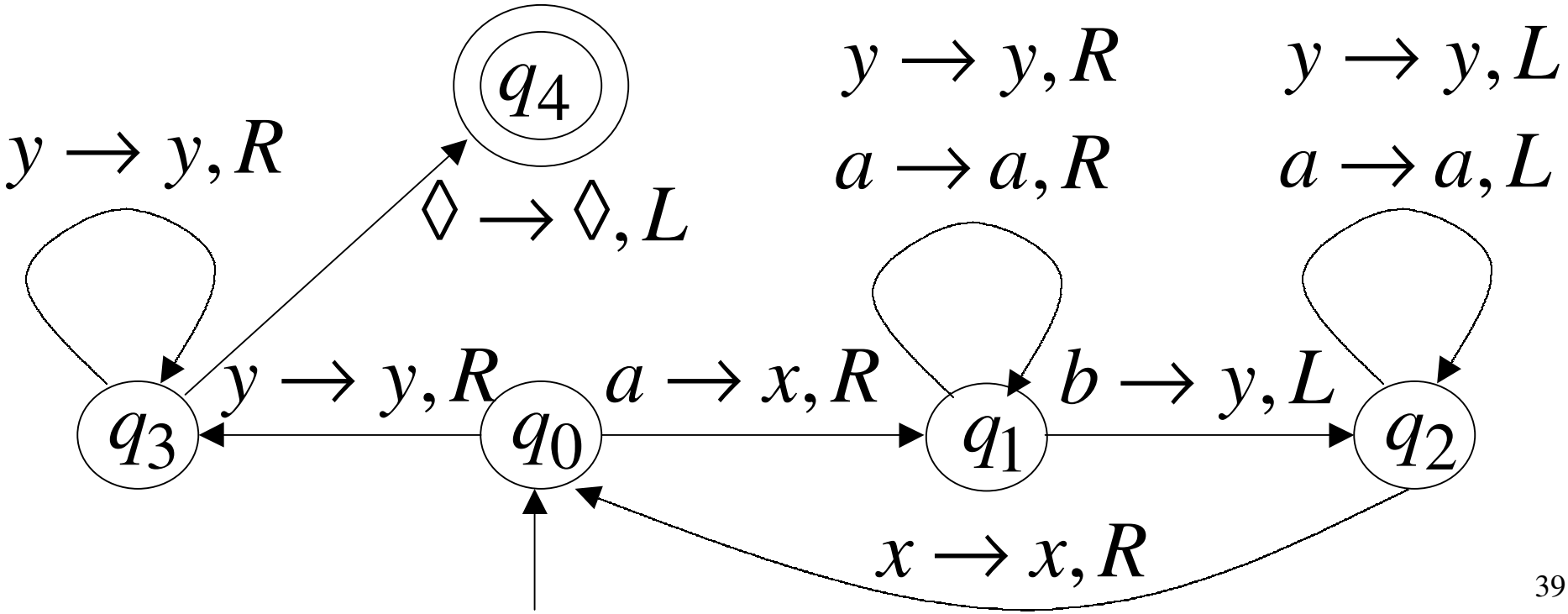

$q_4$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$q_3$
$y \to y, R$
$q_0$
$a \to x, R$
$q_1$
$b \to y, L$
$q_2$
$x \to x, R$

Time 3

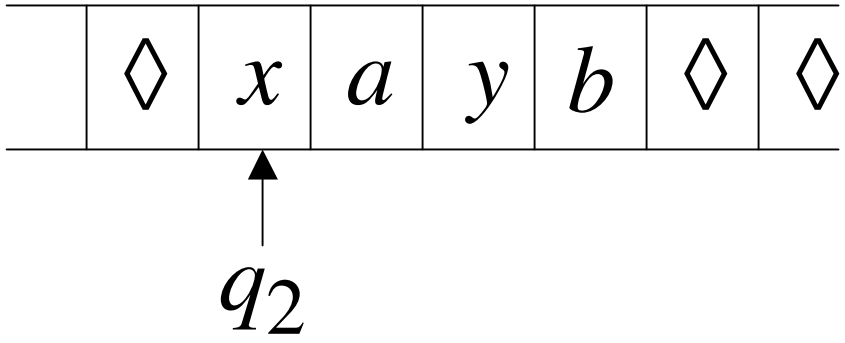

◊ x a y b ◊ ◊
$q_2$

Time 4

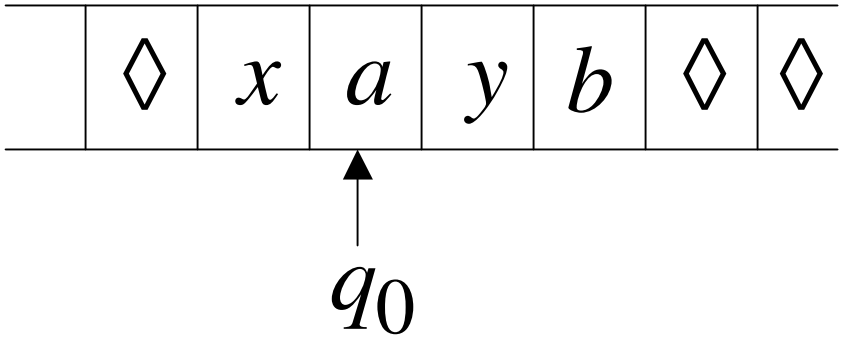

◊ x a y b ◊ ◊
$q_2$

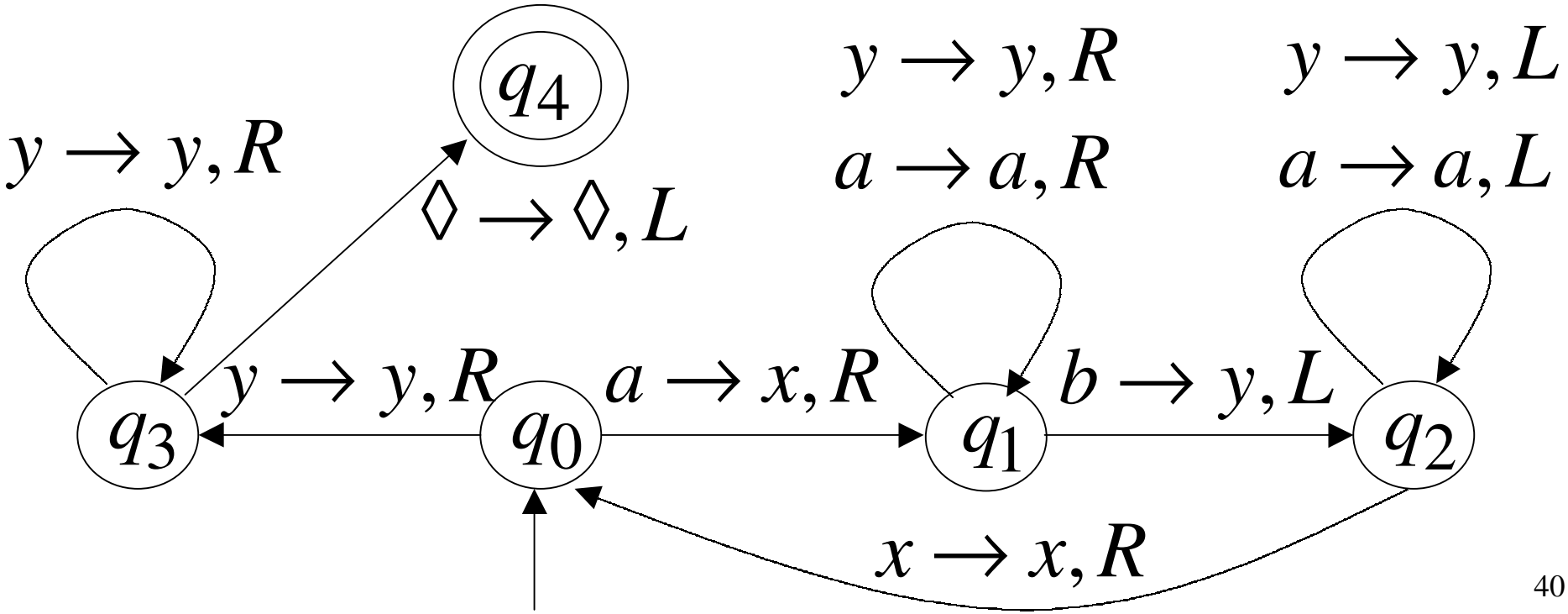

$q_4$
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
$q_3$
$q_0$
$q_1$
$q_2$
$x \to x, R$

Time 4

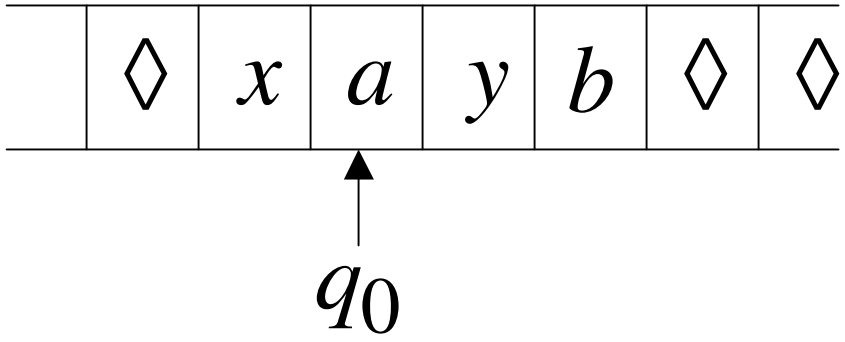

◊ x a y b ◊ ◊
$q_2$

Time 5

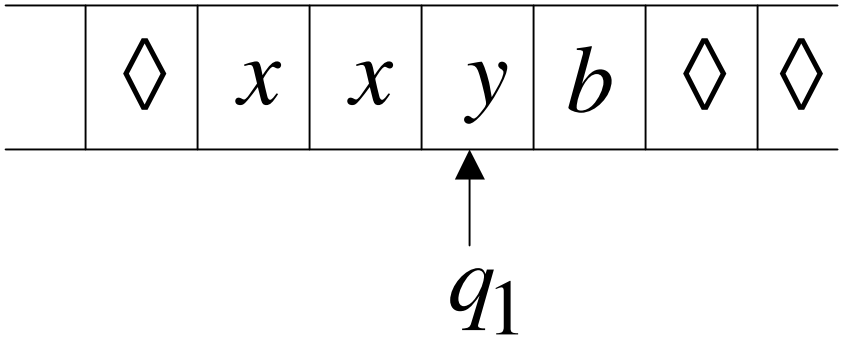

◊ x a y b ◊ ◊
$q_0$

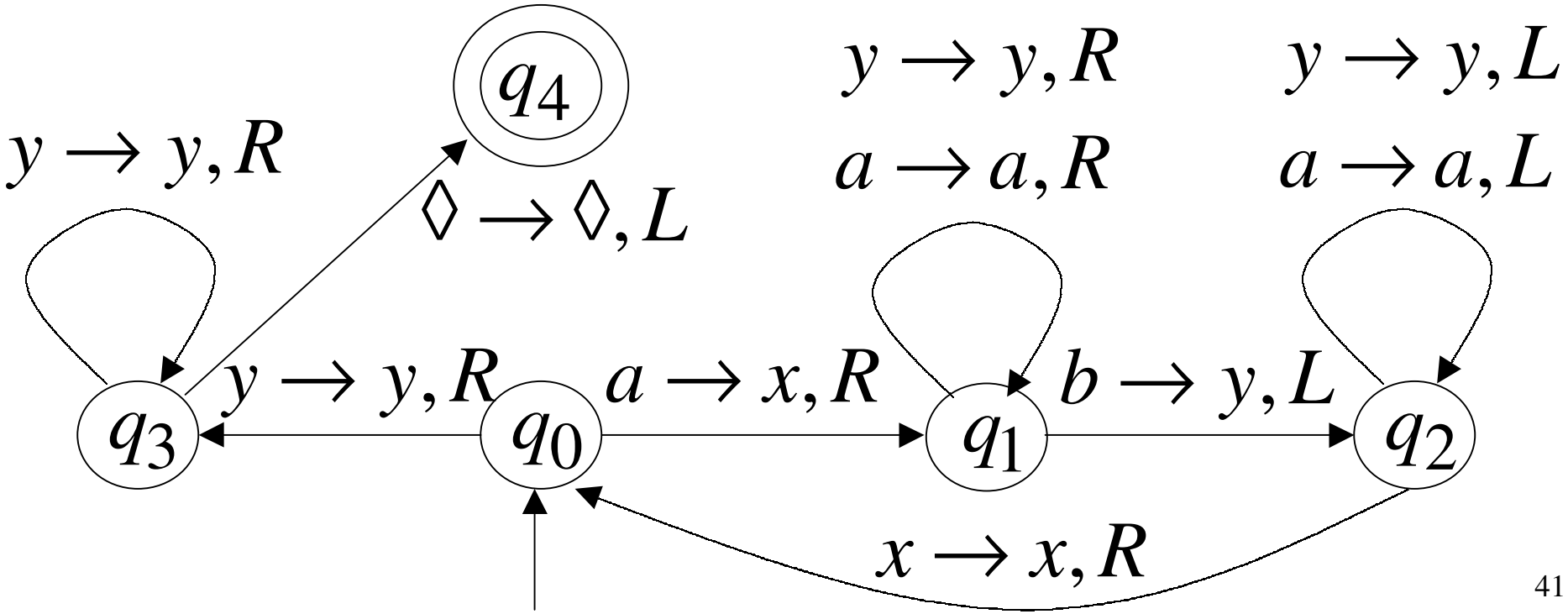

$q_4$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$q_3$
$y \to y, R$
$q_0$
$a \to x, R$
$q_1$
$b \to y, L$
$q_2$
$x \to x, R$

Time 5

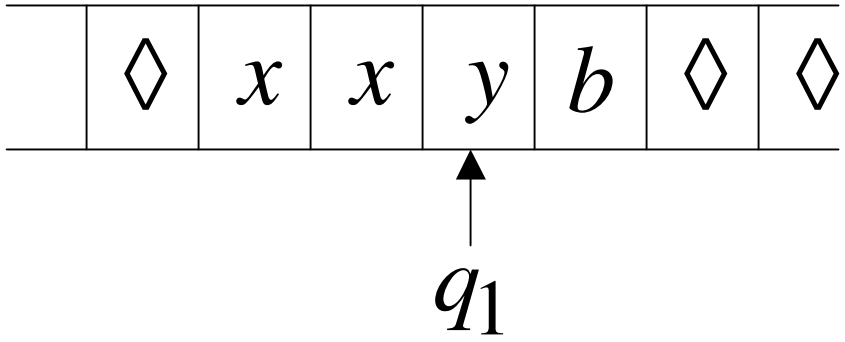

◊ x a y b ◊ ◊
$q_0$

Time 6

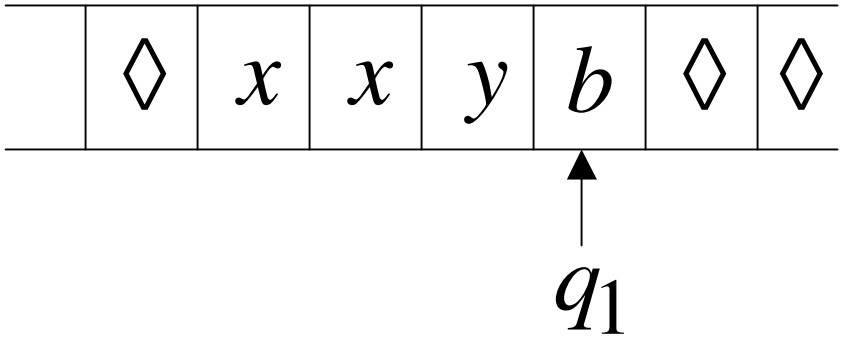

◊ x x y b ◊ ◊
$q_1$

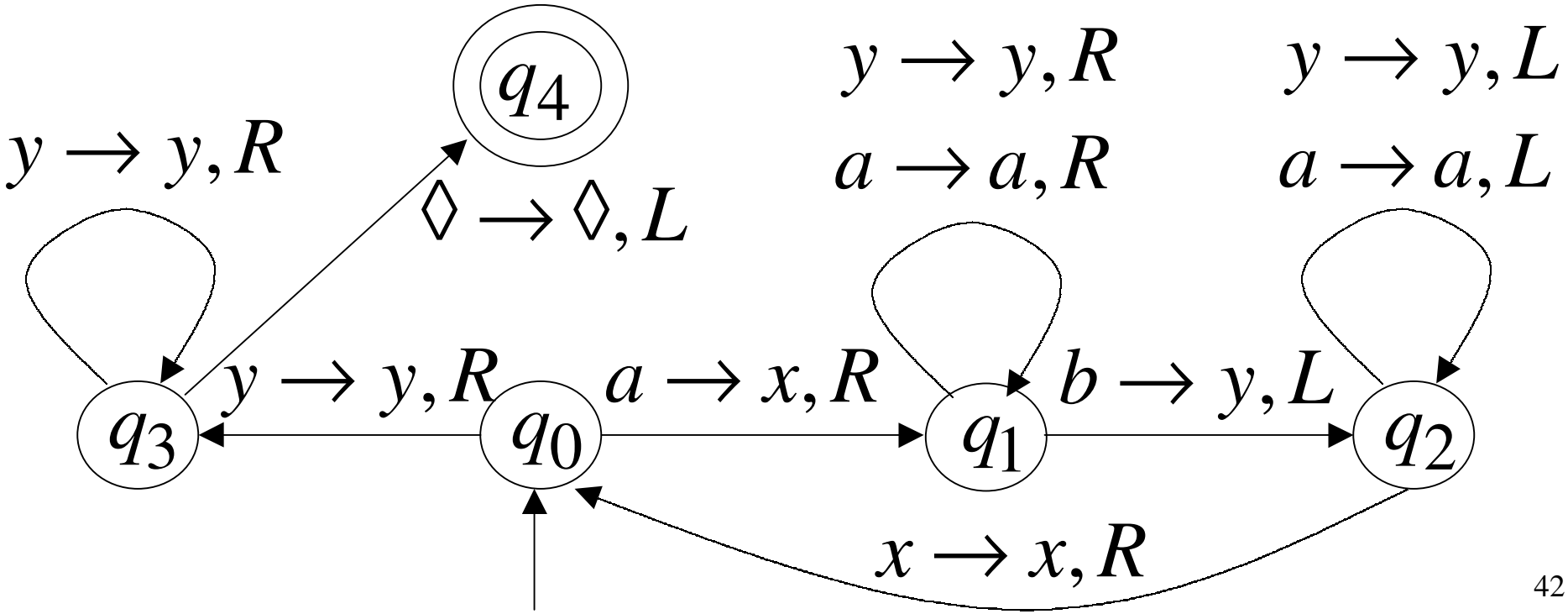

$q_4$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
$q_3$
$q_0$
$q_1$
$q_2$
$x \to x, R$

Time 6

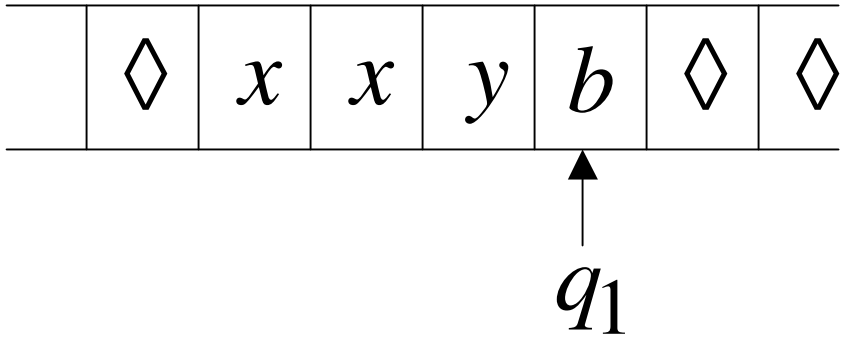

◊ x x y b ◊ ◊
$q_1$

Time 7

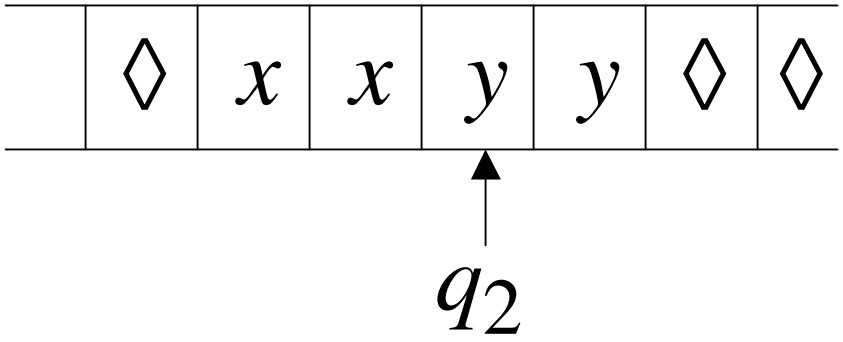

◊ x x y b ◊ ◊
$q_1$

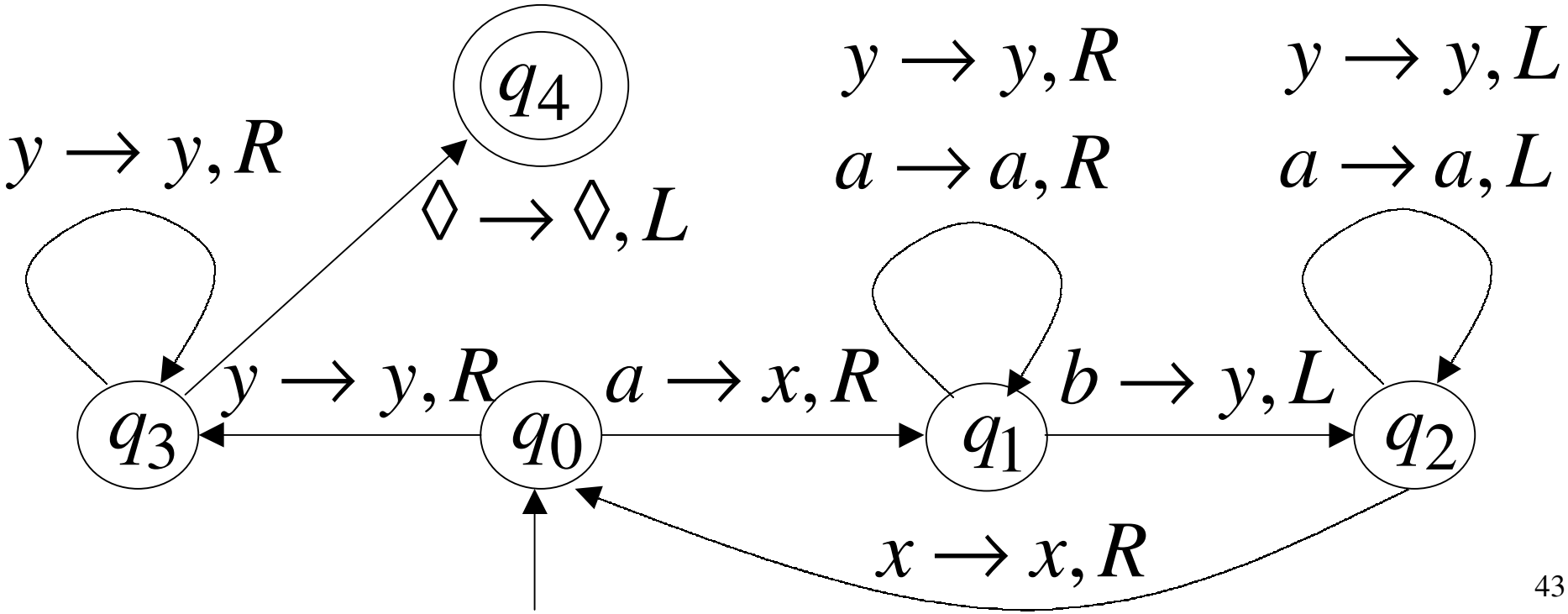

$q_4$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
$q_3$
$q_0$
$q_1$
$q_2$
$x \to x, R$

Time 7

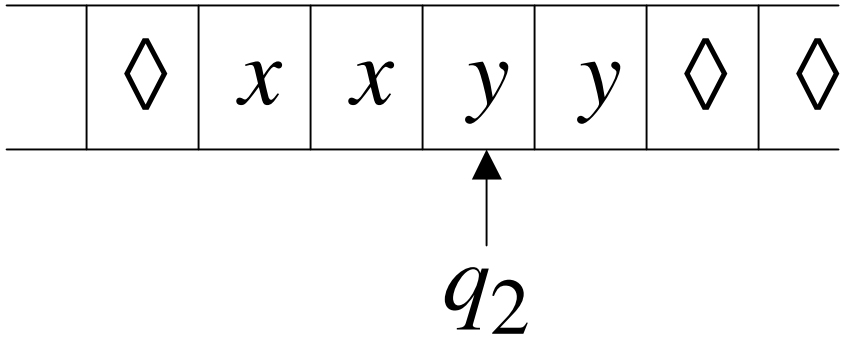

◊ x x y b ◊ ◊
$q_1$

Time 8

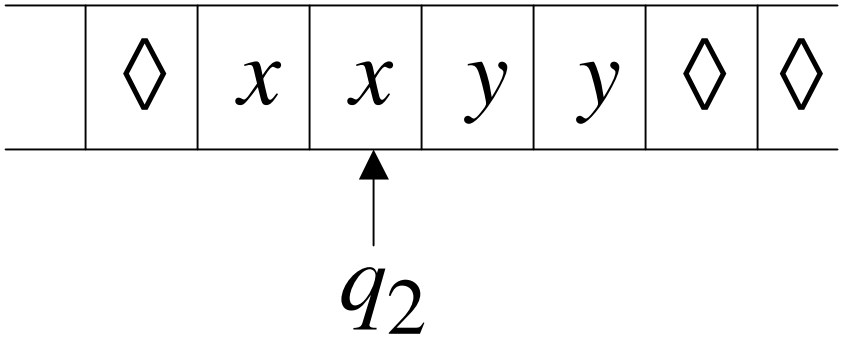

◊ x x y y ◊ ◊
$q_2$

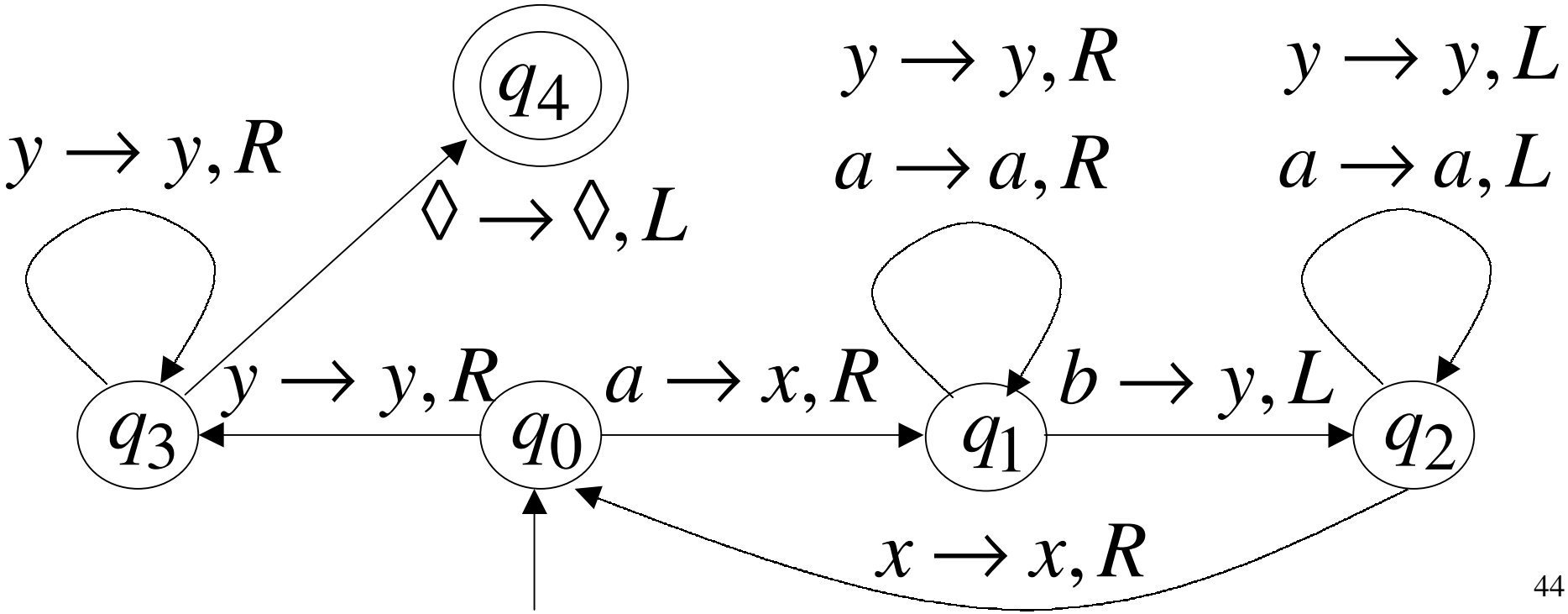

$q_4$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$q_3$
$y \to y, R$
$q_0$
$a \to x, R$
$q_1$
$b \to y, L$
$q_2$
$x \to x, R$

Time 8

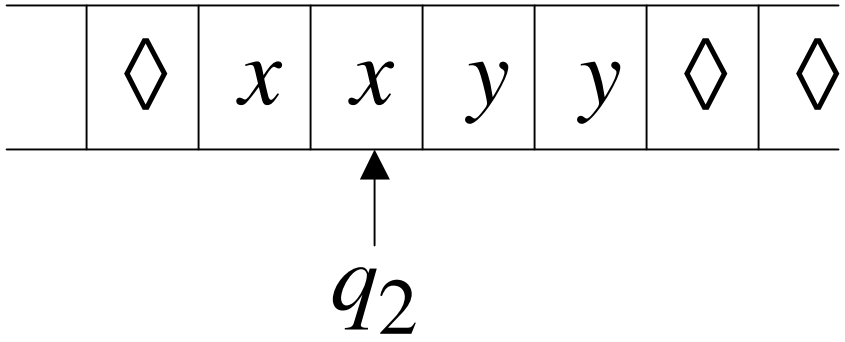

◊ x x y y ◊ ◊
$q_2$

Time 9

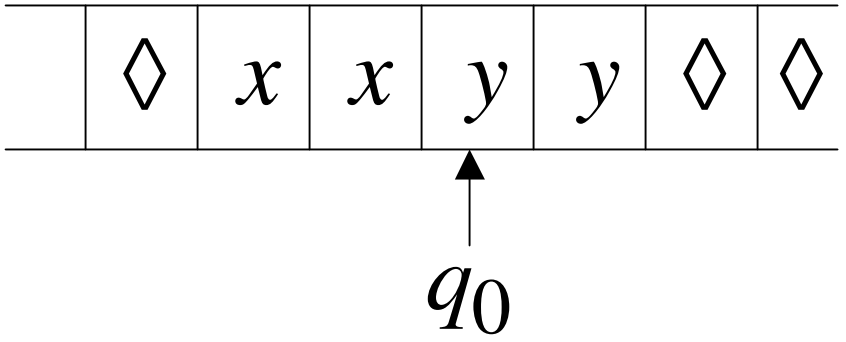

◊ x x y y ◊ ◊
$q_2$

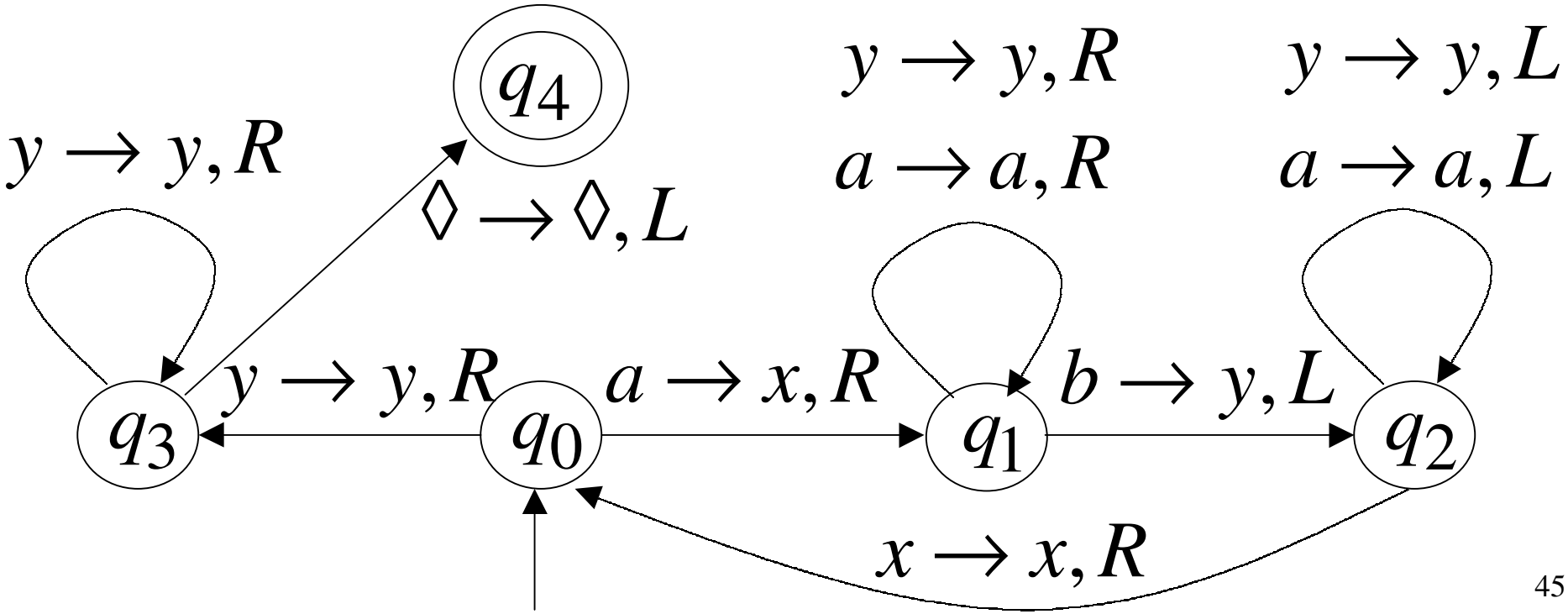

$q_4$
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
$q_3$
$q_0$
$q_1$
$q_2$
$x \to x, R$

Time 9

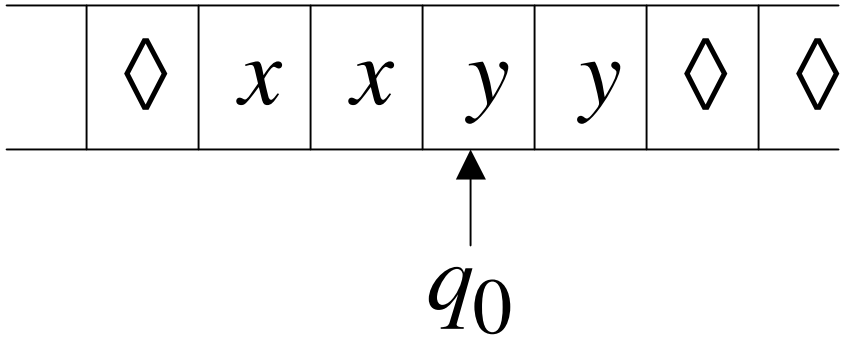



Time 10

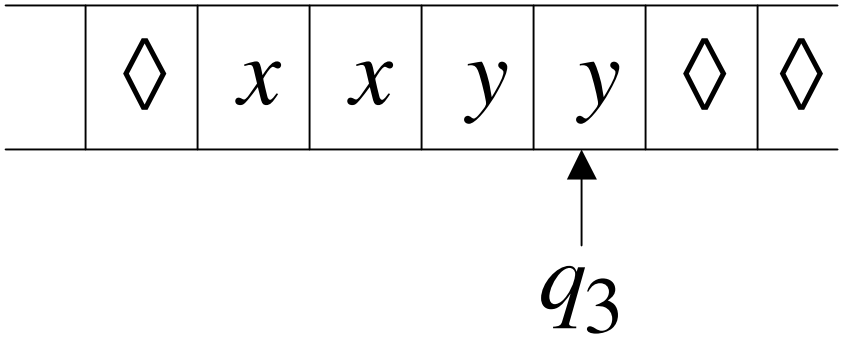


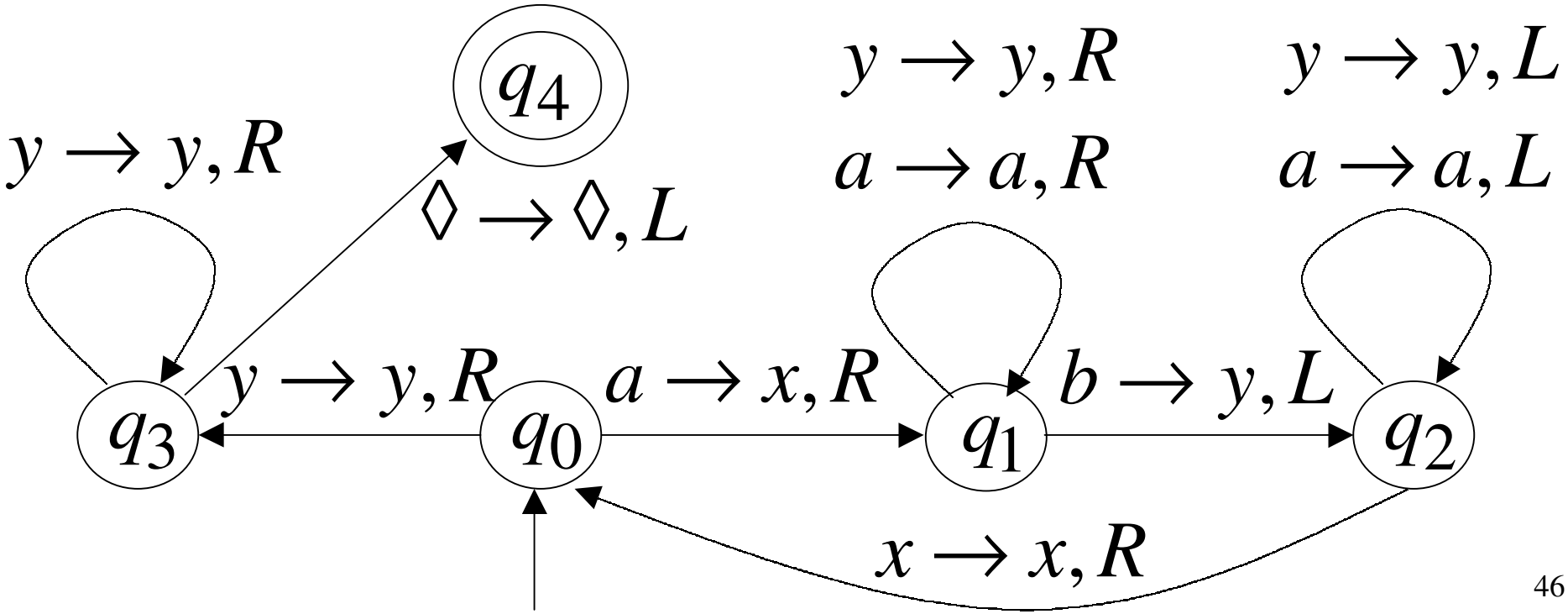



Time 10

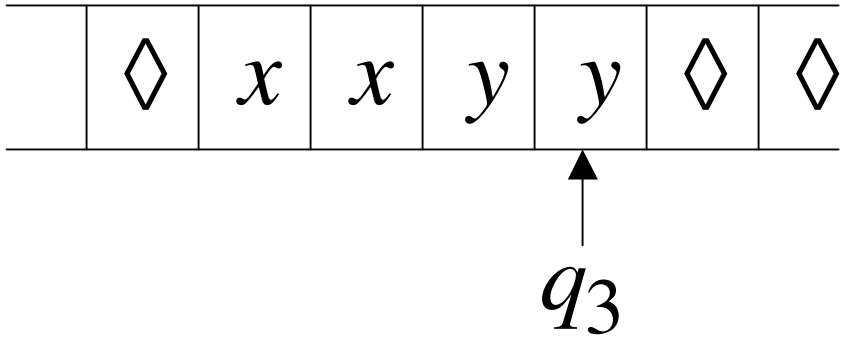



Time 11

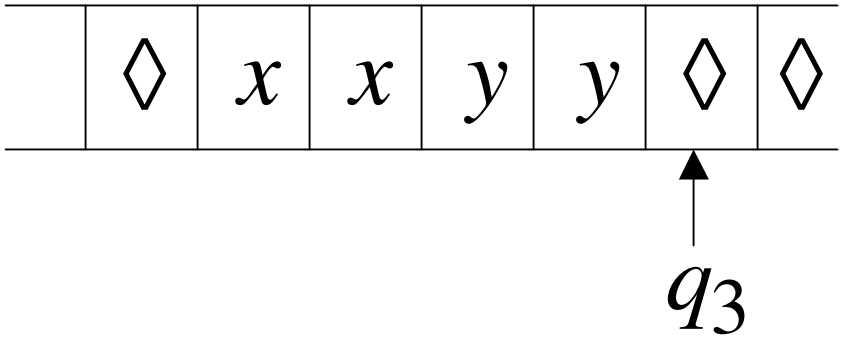


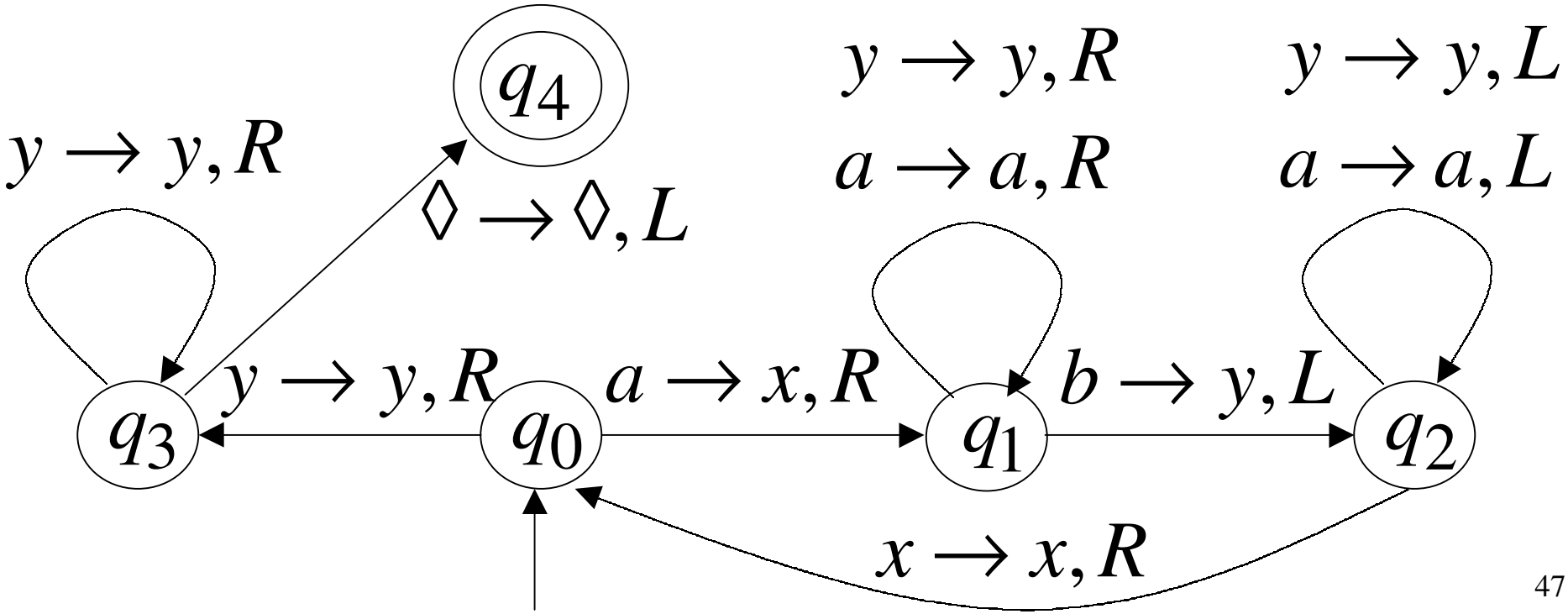



Time 11

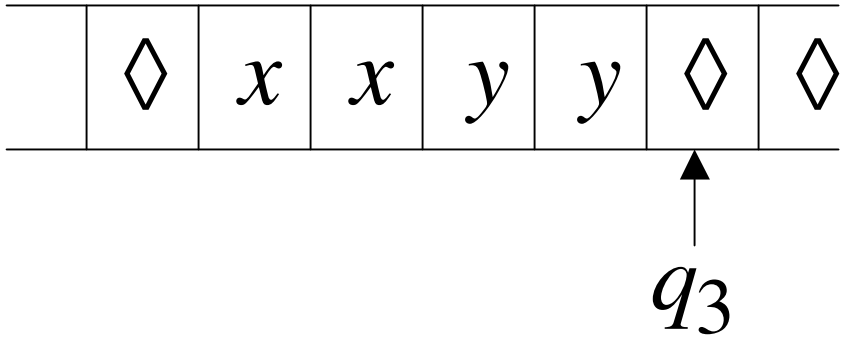



Time 12

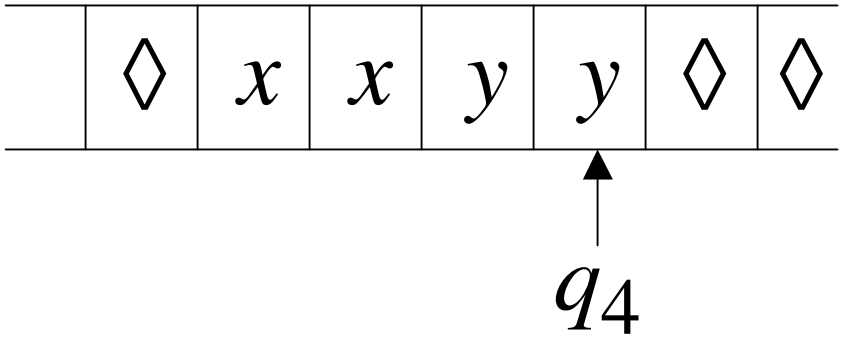


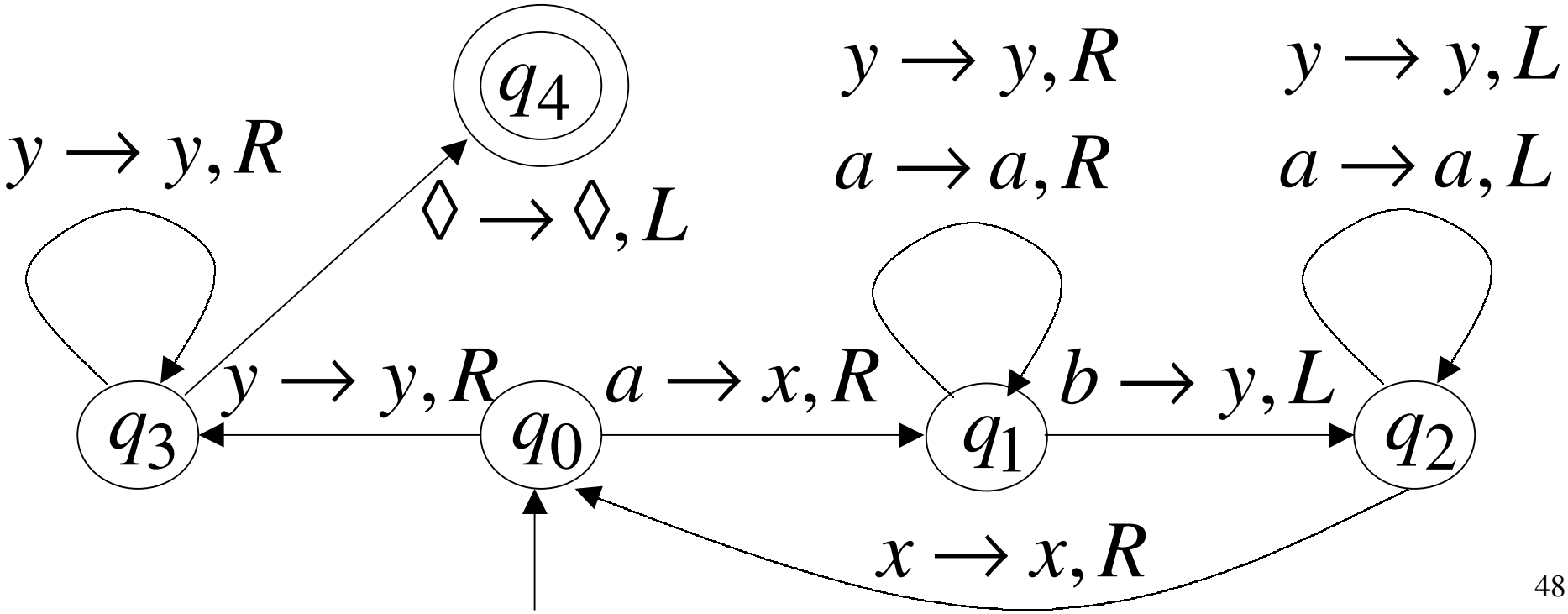
Time 12




Time 13




HALT & accept

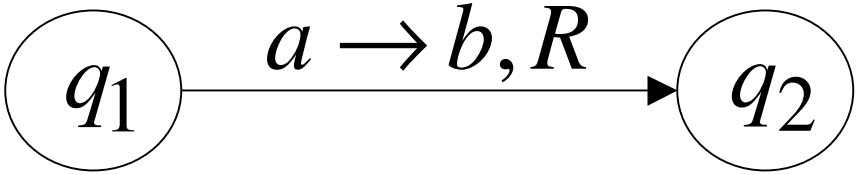


By modifying the
machine for the language $\{a^n b^n\}$

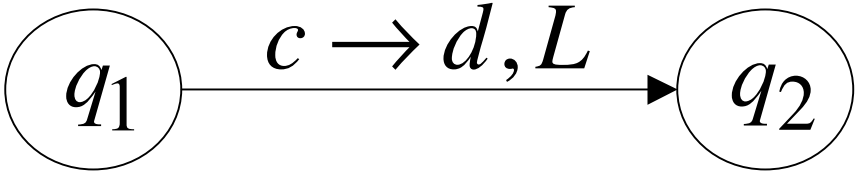
we can easily construct
a machine for language $\{a^n b^n c^n\}$

# Formal Definitions for Turing Machines

# Transition Function:

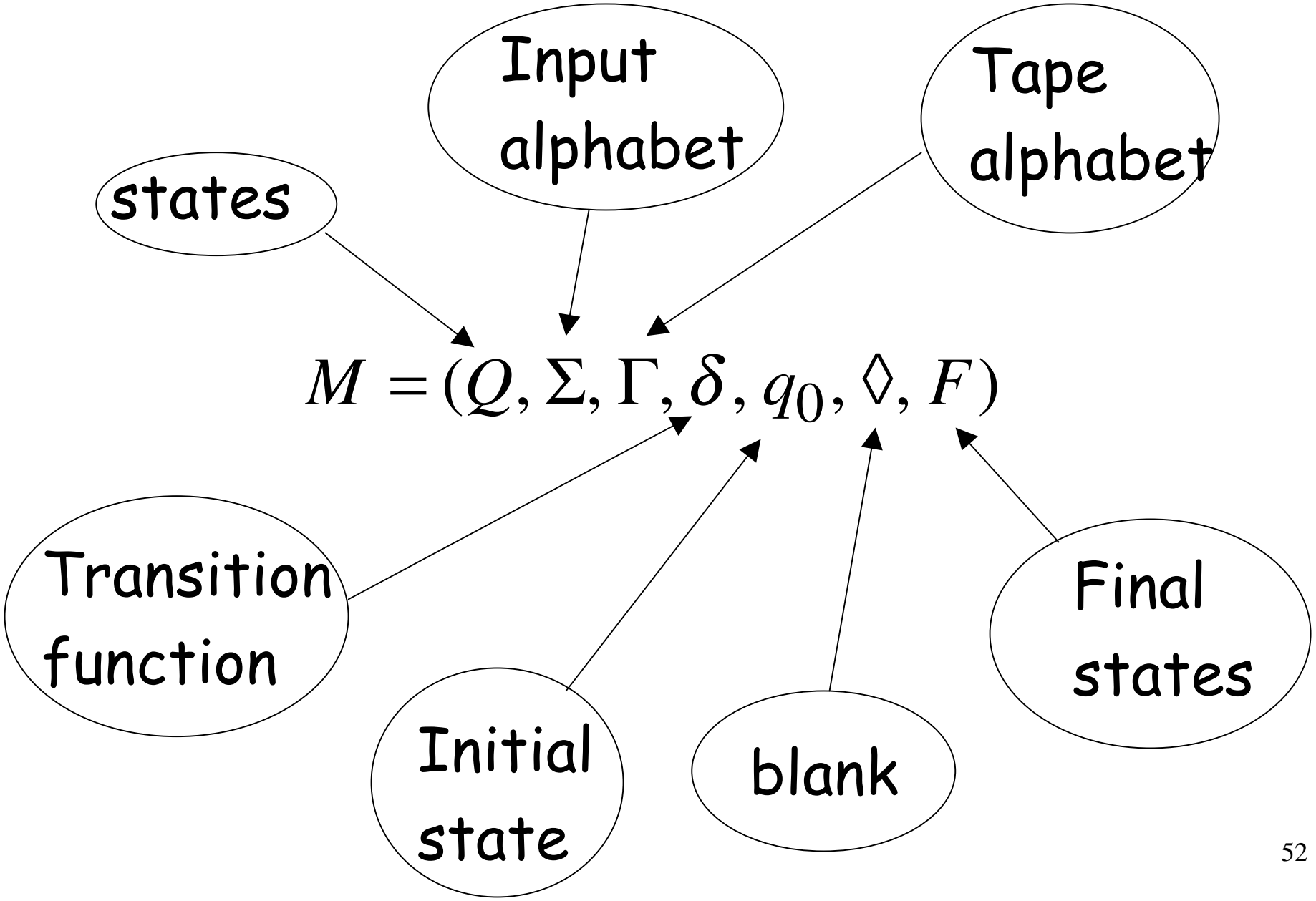



$$\delta(q_1, a) = (q_2, b, R)$$

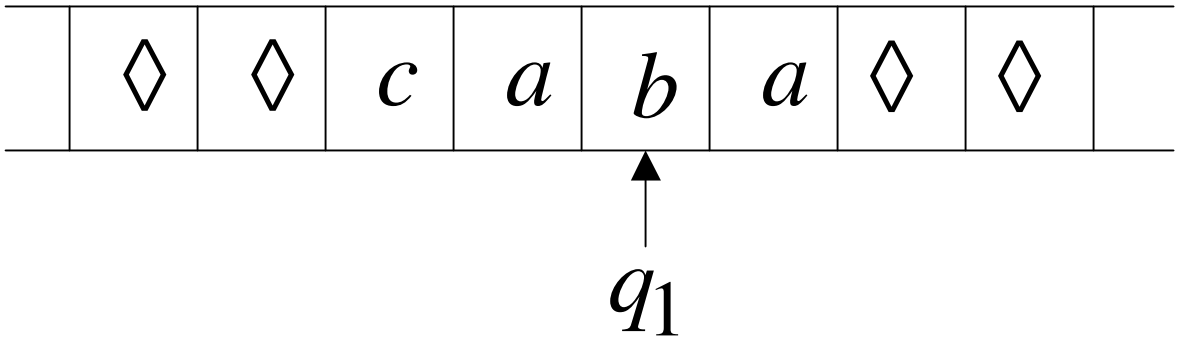



$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:



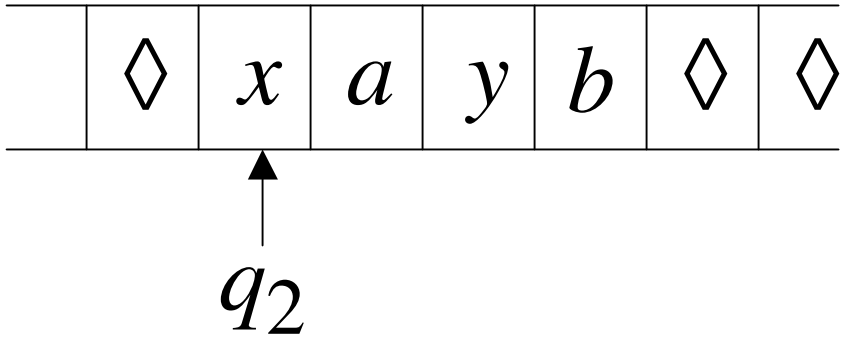
# Configuration



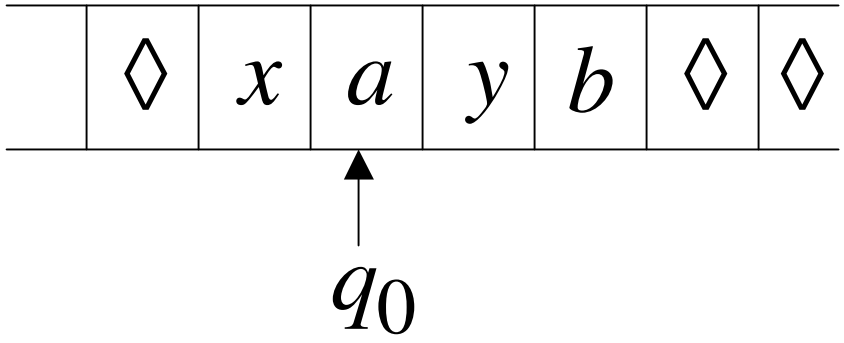

Instantaneous description:

$$ca\ q_1\ ba$$

Time 4

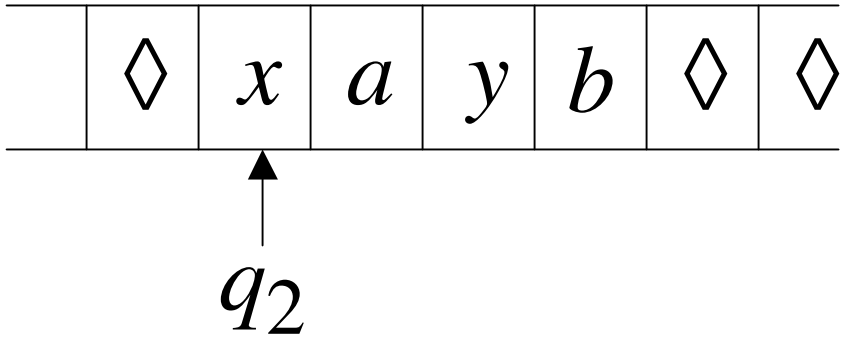



Time 5

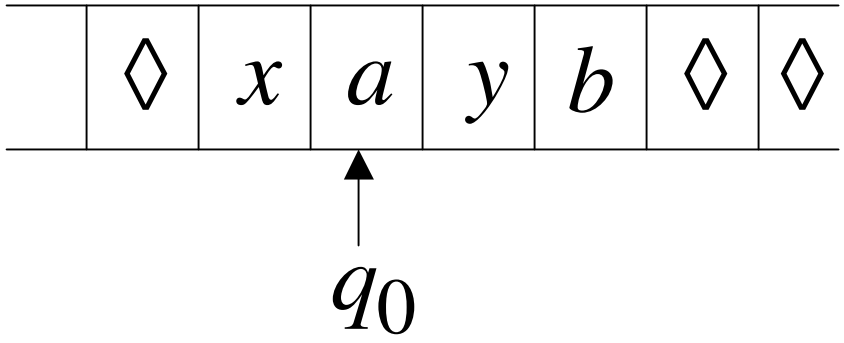



A move: $q_2\ xayb \succ x\ q_0\ ayb$

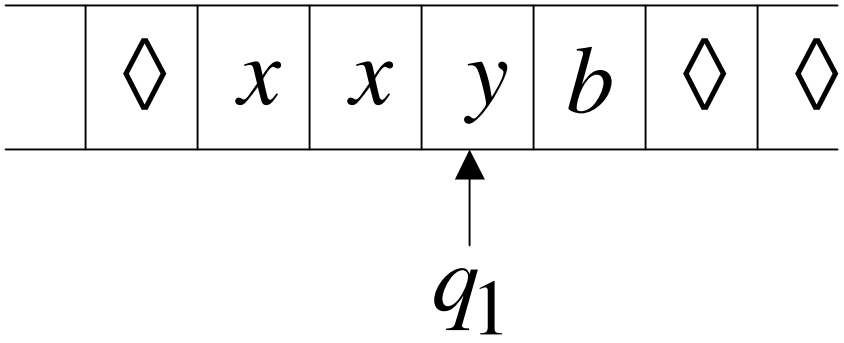
Time 4



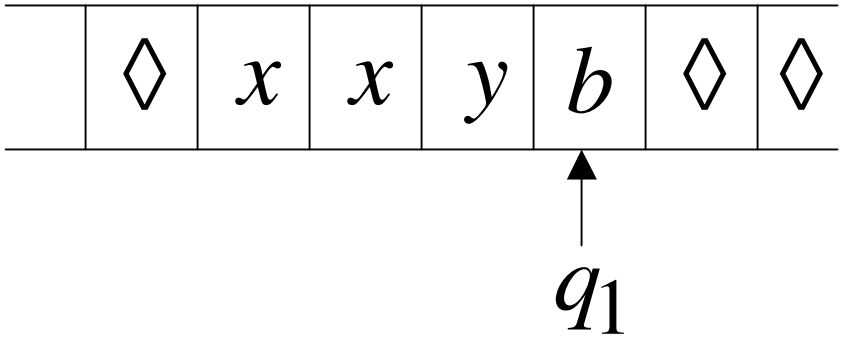

Time 5




Time 6




Time 7




$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

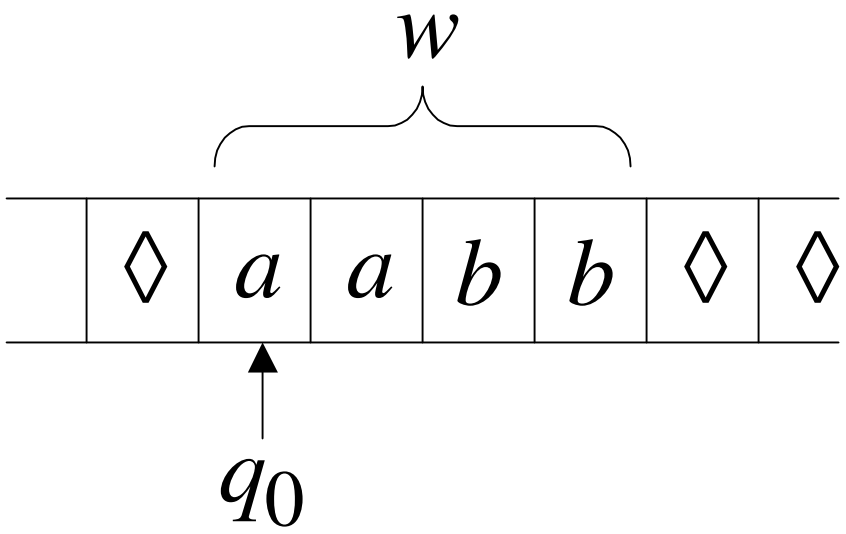
$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

Equivalent notation:

$$q_2\ xayb \overset{*}{\succ} xxy\ q_1\ b$$

Initial configuration: $q_0\ w$

Input string



# The Accepted Language

For any Turing Machine $M$

$$L(M) = \{w : \quad q_0\, w \overset{*}{\succ} x_1\, q_f\, x_2\}$$

Initial state

Final state

# Standard Turing Machine

The machine we described is the standard:

- Deterministic
- Infinite tape in both directions
- Tape is the input/output file