Languages and Finite Automata

or

how to talk to machines...

Languages

A language is a set of strings

String:

A sequence of letters (a word)

Examples: "cat", "dog", "house", …

Defined over an alphabet:

set of symbols (letters)

$$\Sigma = \{a, b, c, \ldots, z\}$$

Alphabets and Strings

We will use small alphabets

$$\Sigma = \{a, b\}$$

Strings

a

ab

abba

baba

aaabbbaabab

$u = ab$

$v = bbbaaa$

$w = abba$

String Operations

$w = a_1 a_2 \cdots a_n$ $\qquad$ $v = b_1 b_2 \cdots b_m$

$w = abba$ $\qquad$ $v = bbbaaa$

Concatenation $\qquad$ $wv = a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m$

$wv = abbabbbaaa$

Reverse $\qquad$ $v^R = b_m \cdots b_2 b_1$

$v^R = aaabbb$

String Length

$$w = a_1 a_2 \cdots a_n$$

Length: $|w| = n$

Examples: $|abba| = 4$

$|aa| = 2$

$|a| = 1$

Recursive Definition of Length

For any letter: $|a| = 1$

For any string wa: $|wa| = |w| + 1$

Example:

$$\begin{aligned} |abba| &= |abb| + 1 \\ &= |ab| + 1 + 1 \\ &= |a| + 1 + 1 + 1 \\ &= 1 + 1 + 1 + 1 \\ &= 4 \end{aligned}$$

Length of Concatenation

$$|uv| = |u| + |v|$$

Example: $u = aab, \quad |u| = 3$

$$v = abaab, \quad |v| = 5$$

$$|uv| = |aababaaab| = 8$$

$$|uv| = |u| + |v| = 3 + 5 = 8$$

Proof of Concatenation Length

Claim: $|uv| = |u| + |v|$

Proof:

By induction on the length $|v|$

Induction basis: $|v| = 1$

v is only one symbol

From definition of length: $|uv| = |u| + 1 = |u| + |v|$

Inductive hypothesis: $|uv| = |u| + |v|$

for all v with $|v| = 1,2,\ldots,n$

Inductive step:

we will prove $|uv| = |u| + |v|$

for $|v| = n + 1$

Inductive Step

Write $v = wa$, where $|w| = n, \quad |a| = 1$

From definition of length: $|uv| = |uwa| = |uw| + 1$

$|wa| = |w| + 1$

From inductive hypothesis: $|uw| = |u| + |w|$

Thus: $|uv| = |u| + |w| + 1 = |u| + |wa| = |u| + |v|$

Empty String

A string with no letters: λ

Observations: $|\lambda| = 0$

$$\lambda w = w\lambda = w$$

$$\lambda abba = abba\lambda = abba$$

Substring

Definition:

A substring of a string w is any sequence of consecutive characters

Example: *abbab*

Substrings

ab

abba

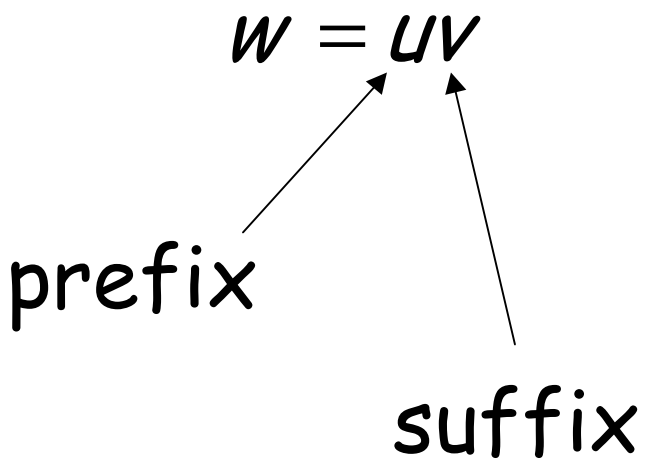
b

bbab

Prefix and Suffix

abbab

Prefixes	Suffixes
λ	*abbab*
a	*bbab*
ab	*bab*
abb	*ab*
abba	*b*
abbab	λ



Another Operation

$$w^n = \underbrace{ww\cdots w}_{n}$$

Example: $(abba)^2 = abbaabba$

Definition

for any w: $w^0 = \lambda$

$$(abba)^0 = \lambda$$

The * Operation

Σ^* : the set of all possible strings from alphabet Σ

Example: $\Sigma = \{a, b\}$

$$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

$$\Sigma^+ = \Sigma^* - \lambda$$

$$\Sigma^+ = \{a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

Language

A language is any subset of Σ^*

Examples: $\Sigma = \{a, b\}$

$$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

$\{\lambda\}$, $\{a, aa, aab\}$

$\{\lambda, abba, baba, aa, ab, aaaaaa\}$

A string is called "sentence"

Another Example

An infinite language

$$L = \{a^n b^n : n \geq 0\}$$

λ

ab

$aabb$

$aaaaabbbbb$

$\in L$

$abb \notin L$

Operations on Languages

The usual set operations

$$\{a, ab, aaaa\} \cup \{bb, ab\} = \{a, ab, bb, aaaa\}$$

$$\{a, ab, aaaa\} \cap \{bb, ab\} = \{ab\}$$

$$\{a, ab, aaaa\} - \{bb, ab\} = \{a, aaaa\}$$

Complement: $\overline{L} = \Sigma^* - L$

$$\overline{\{a, ba\}} = \{\lambda, b, aa, ab, bb, aaa, \ldots\}$$

Reverse

Definition: $L^R = \{w^R : w \in L\}$

Examples: $\{ab, aab, baba\}^R = \{ba, baa, abab\}$

$$L = \{a^n b^n : n \geq 0\}$$

$$L^R = \{b^n a^n : n \geq 0\}$$

Concatenation

Definition: $L_1 L_2 = \{xy : x \in L_1, y \in L_2\}$

Example: $\{a, ab, ba\}\{b, aa\}$

$= \{ab, aaa, abb, abaa, bab, baaa\}$

Another Operation

Definition: $L^n = \underbrace{LL\cdots L}_{n}$

Example: $\{a,b\}^3 = \{a,b\}\{a,b\}\{a,b\} =$

$\{aaa, aab, aba, abb, baa, bab, bba, bbb\}$

Special case: $L^0 = \{\lambda\}$

$\{a, bba, aaa\}^0 = \{\lambda\}$

Example

$$L = \{a^n b^n : n \geq 0\}$$

$$L^2 = \{a^n b^n a^m b^m : n, m \geq 0\}$$

$$aabbaaabbb \in L^2$$

Star-Closure (Kleene *)

Definition: $L^* = L^0 \cup L^1 \cup L^2 \cdots$

Example:

$$\{a, bb\}^* = \begin{Bmatrix} \lambda, \\ a, bb, \\ aa, abb, bba, bbbb, \\ aaa, aabb, abba, abbbb, \ldots \end{Bmatrix}$$

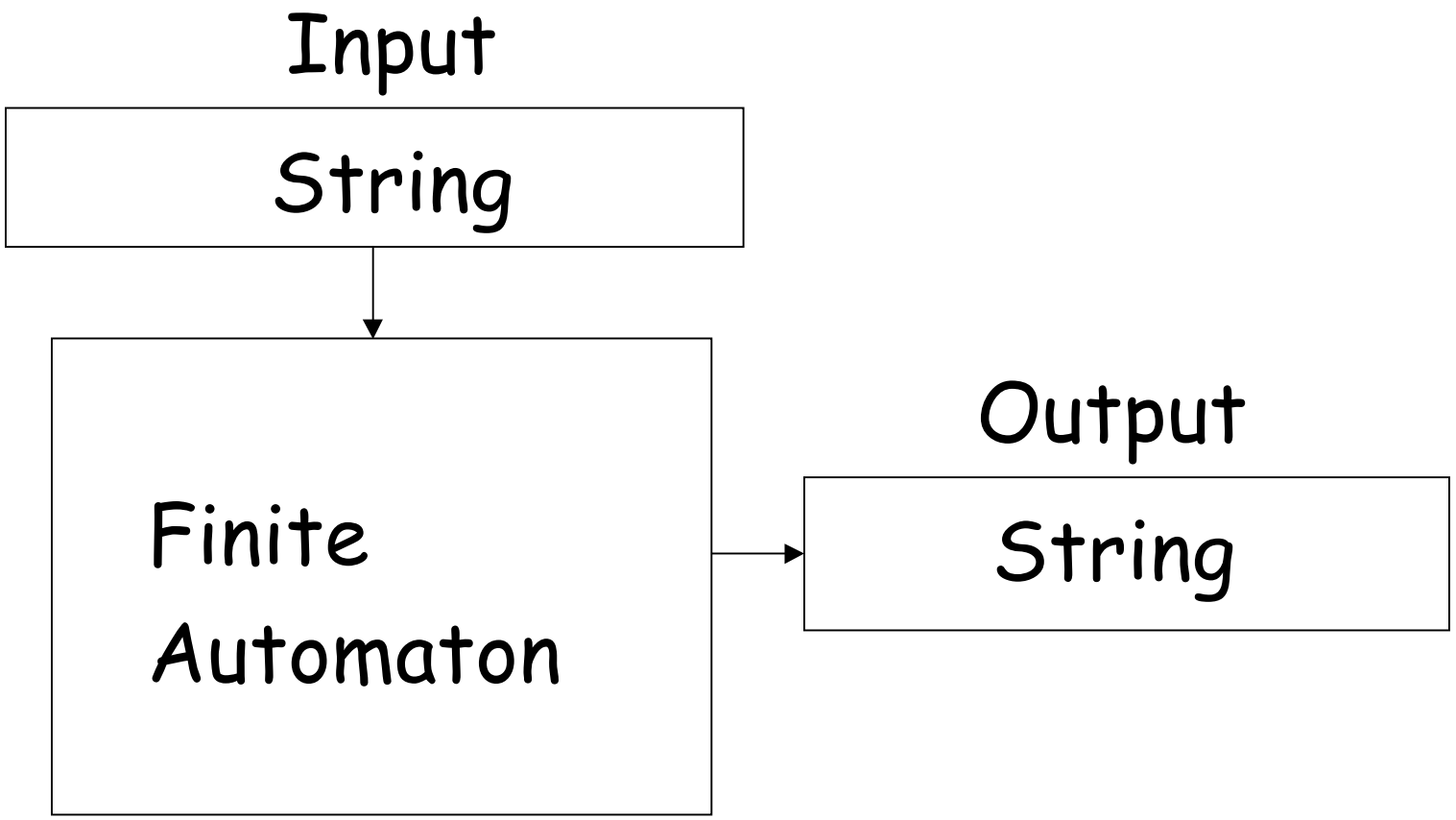
Positive Closure

Definition: $L^+ = L^1 \cup L^2 \cup \cdots$

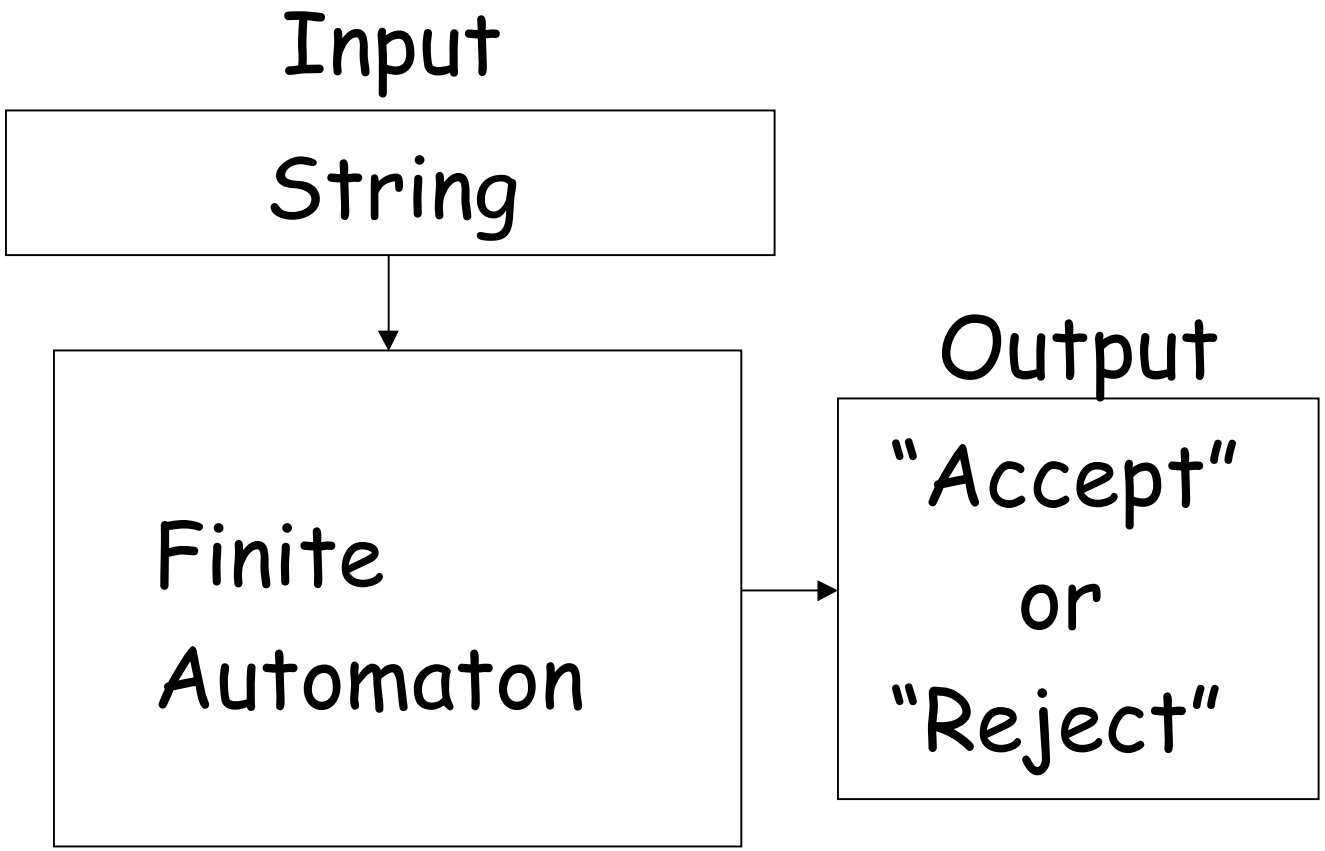
$$= L^* - \{\lambda\}$$

$$\{a, bb\}^+ = \begin{Bmatrix} a, bb, \\ aa, abb, bba, bbbb, \\ aaa, aabb, abba, abbbb, \ldots \end{Bmatrix}$$

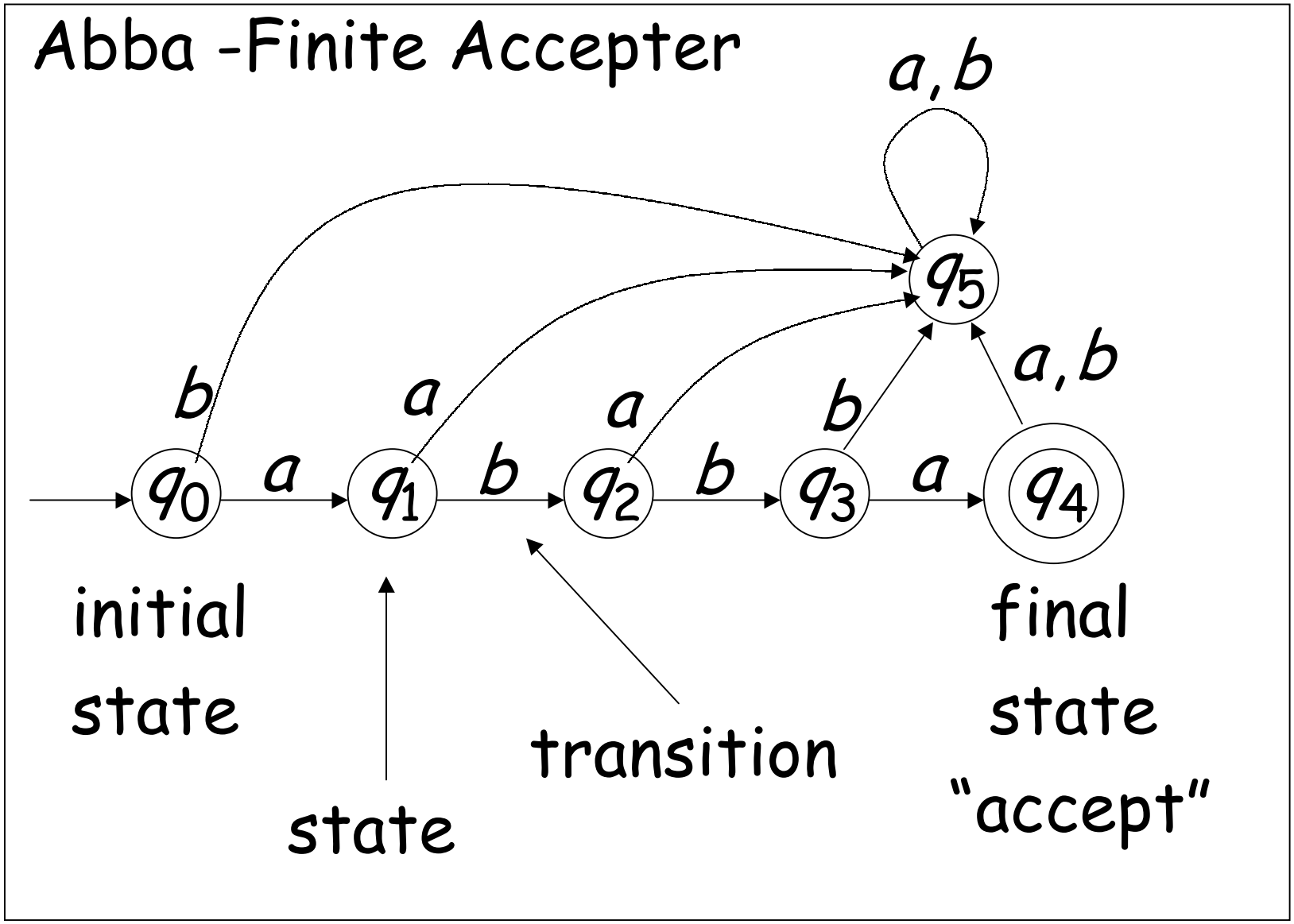
Finite Automata



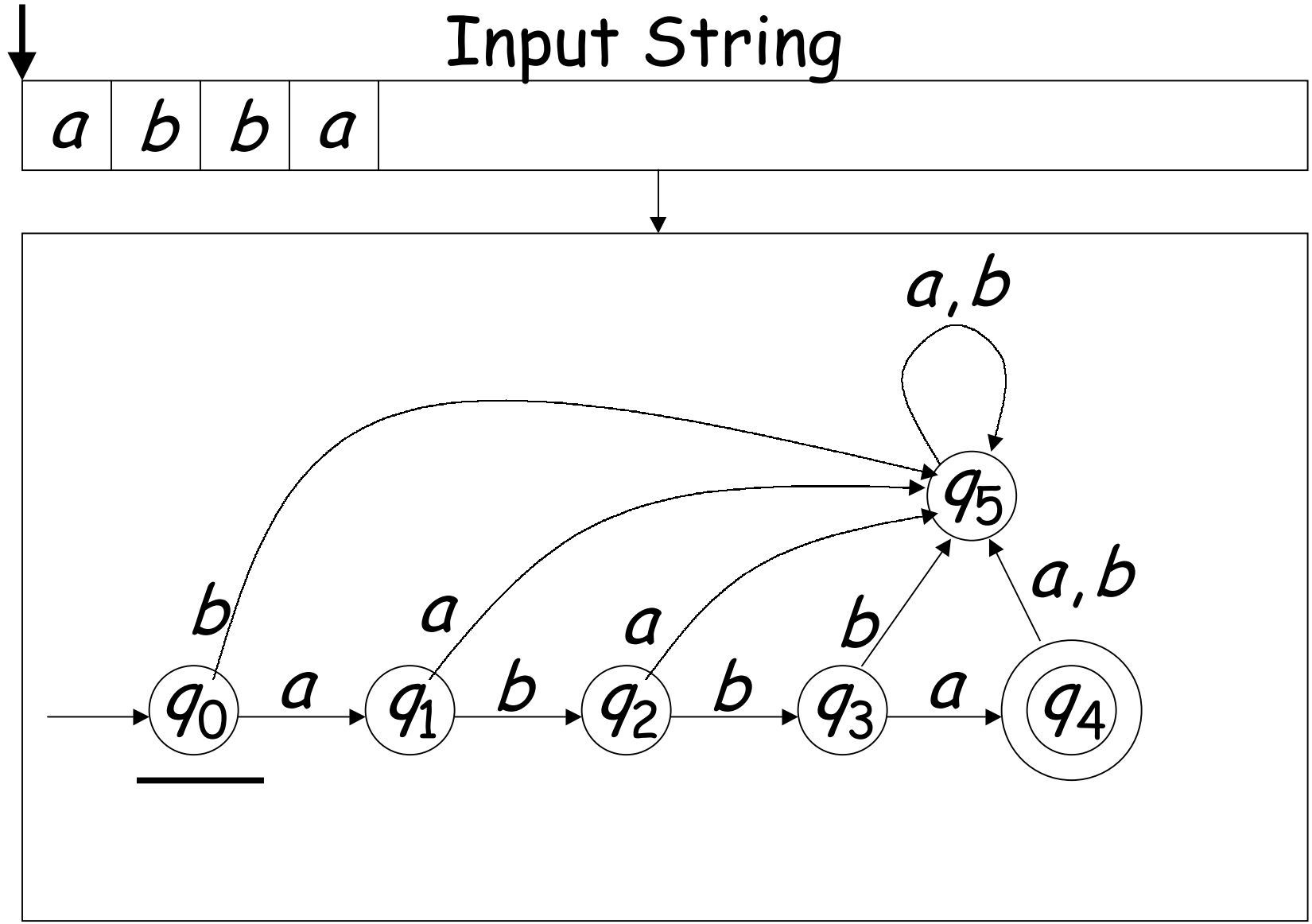
Finite Accepter



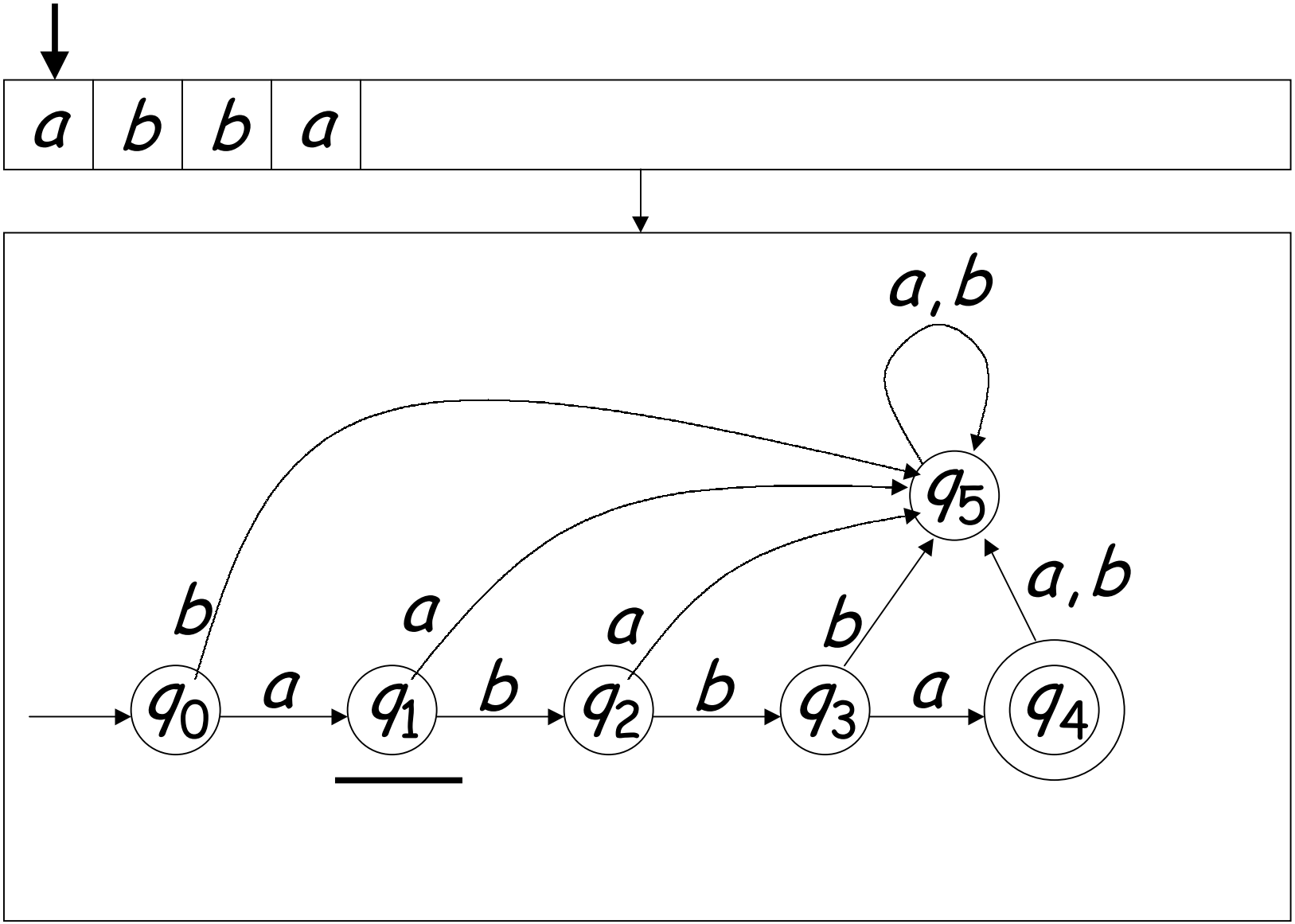
Transition Graph

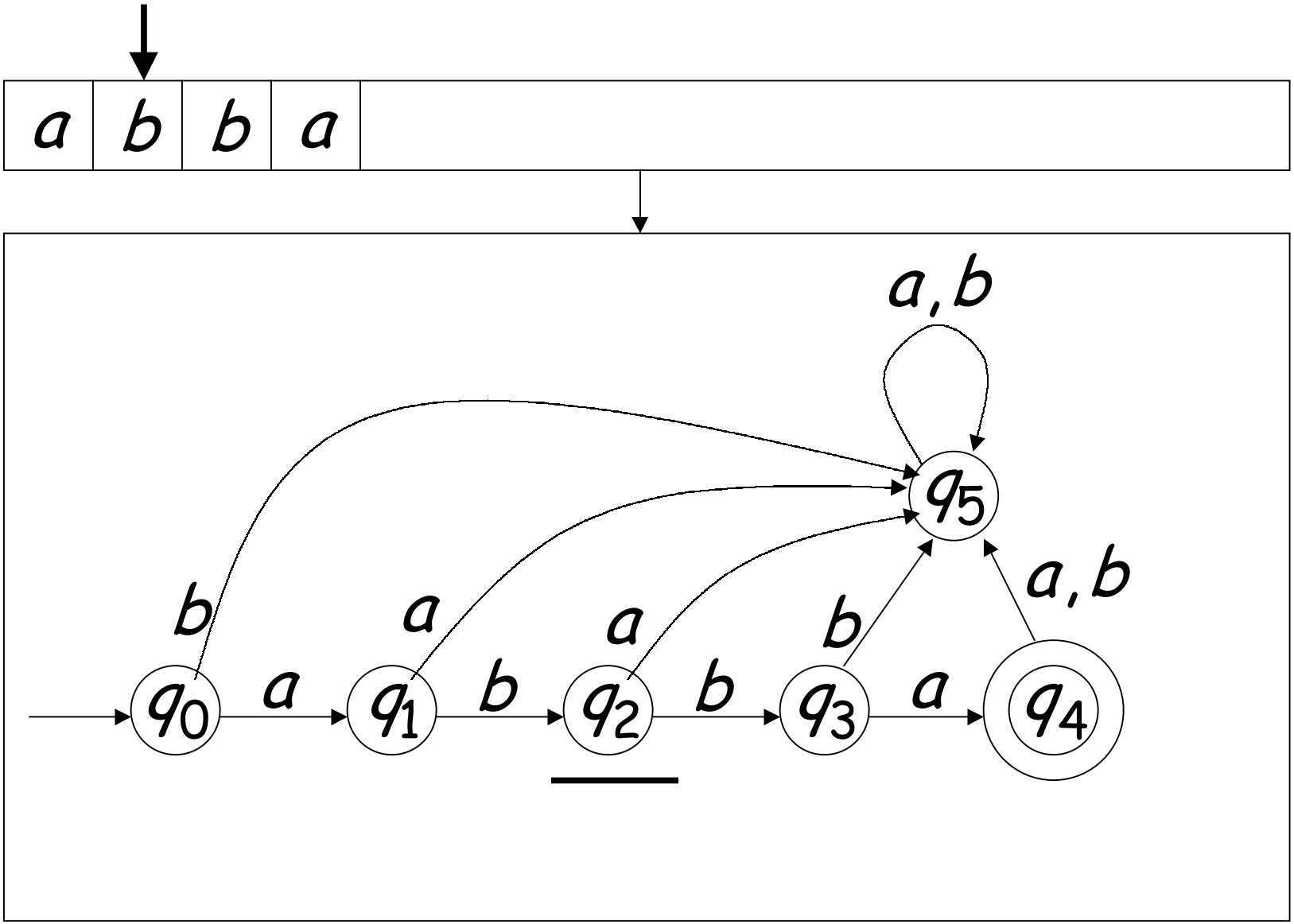


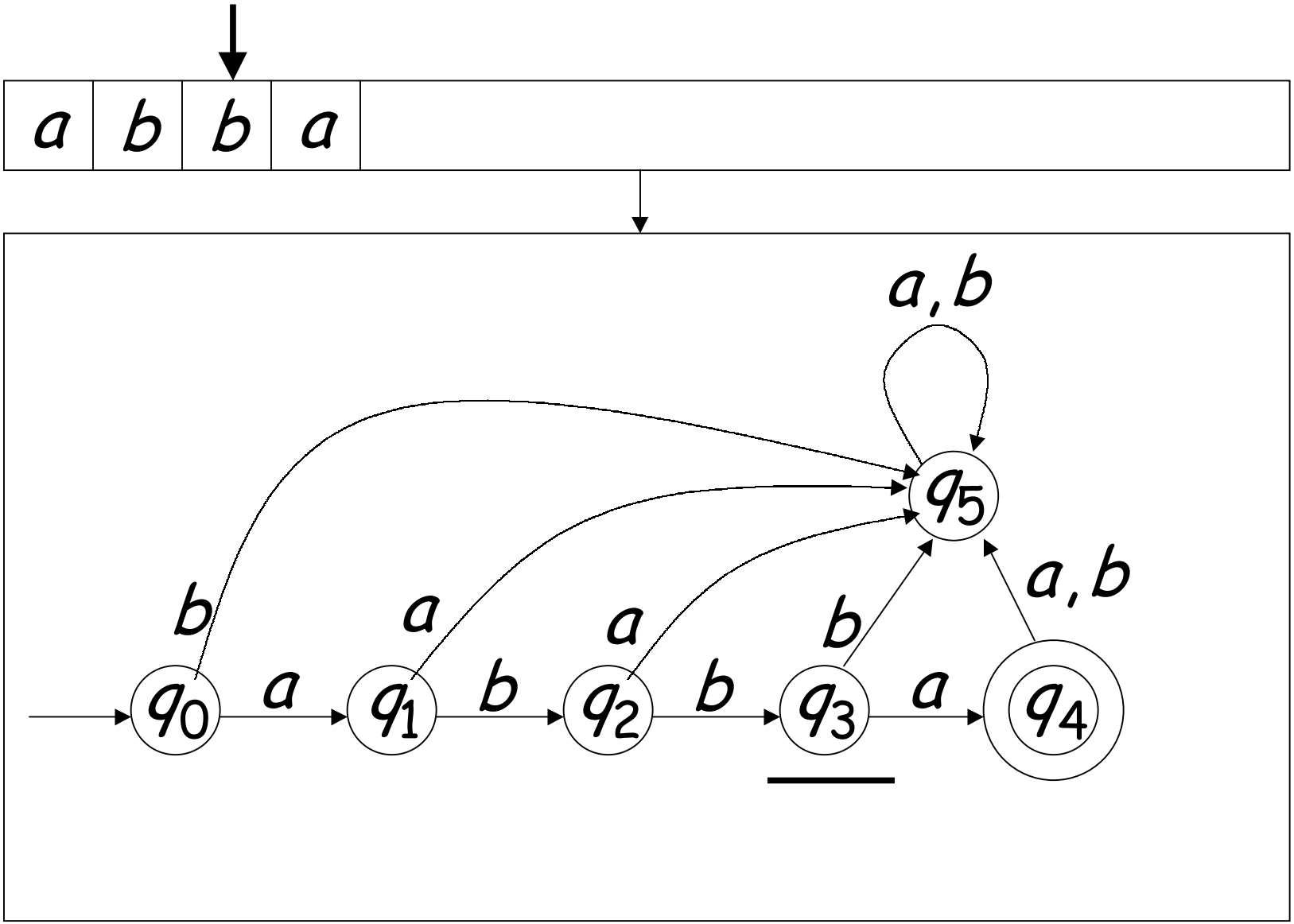
Initial Configuration

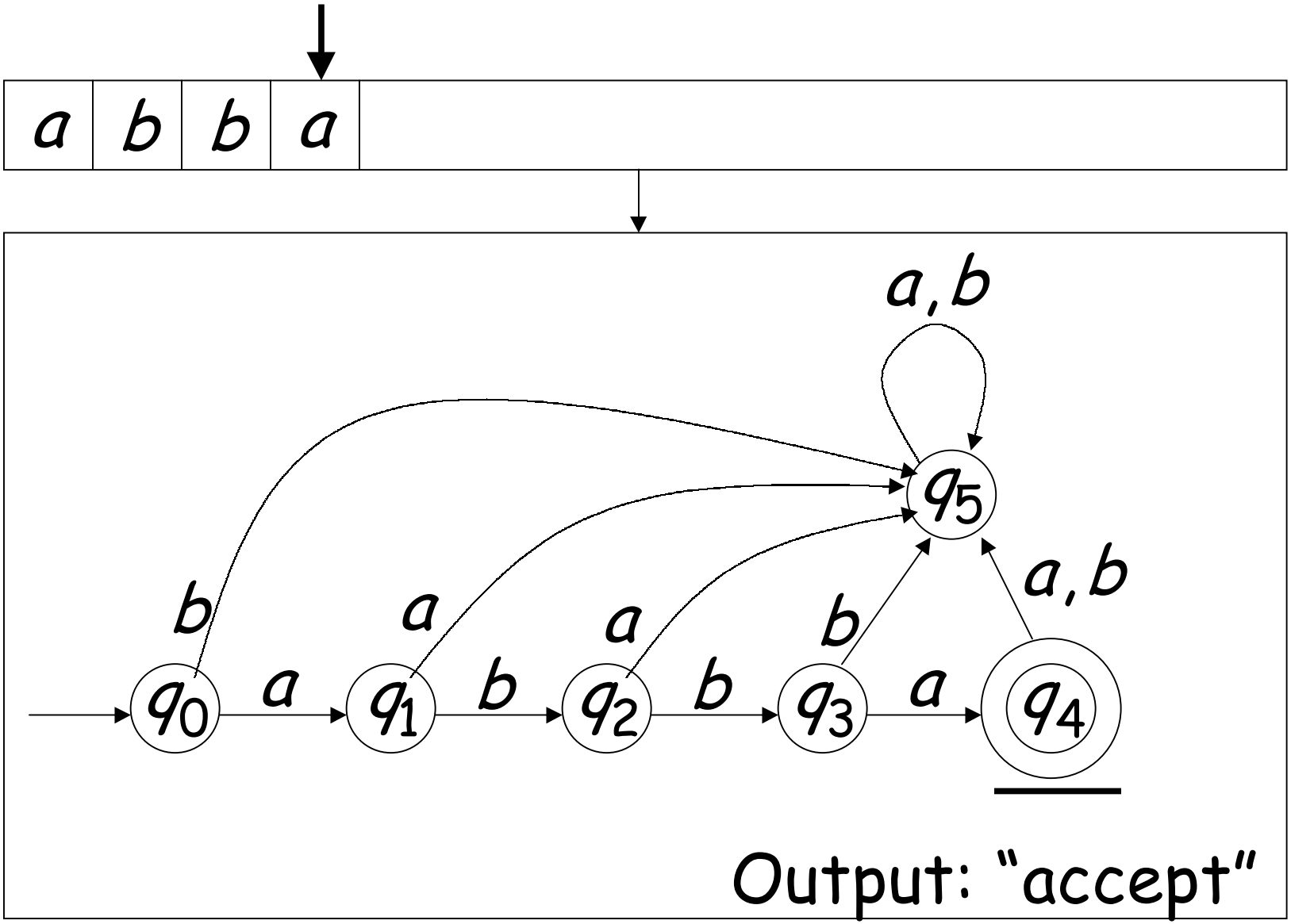


Reading the Input

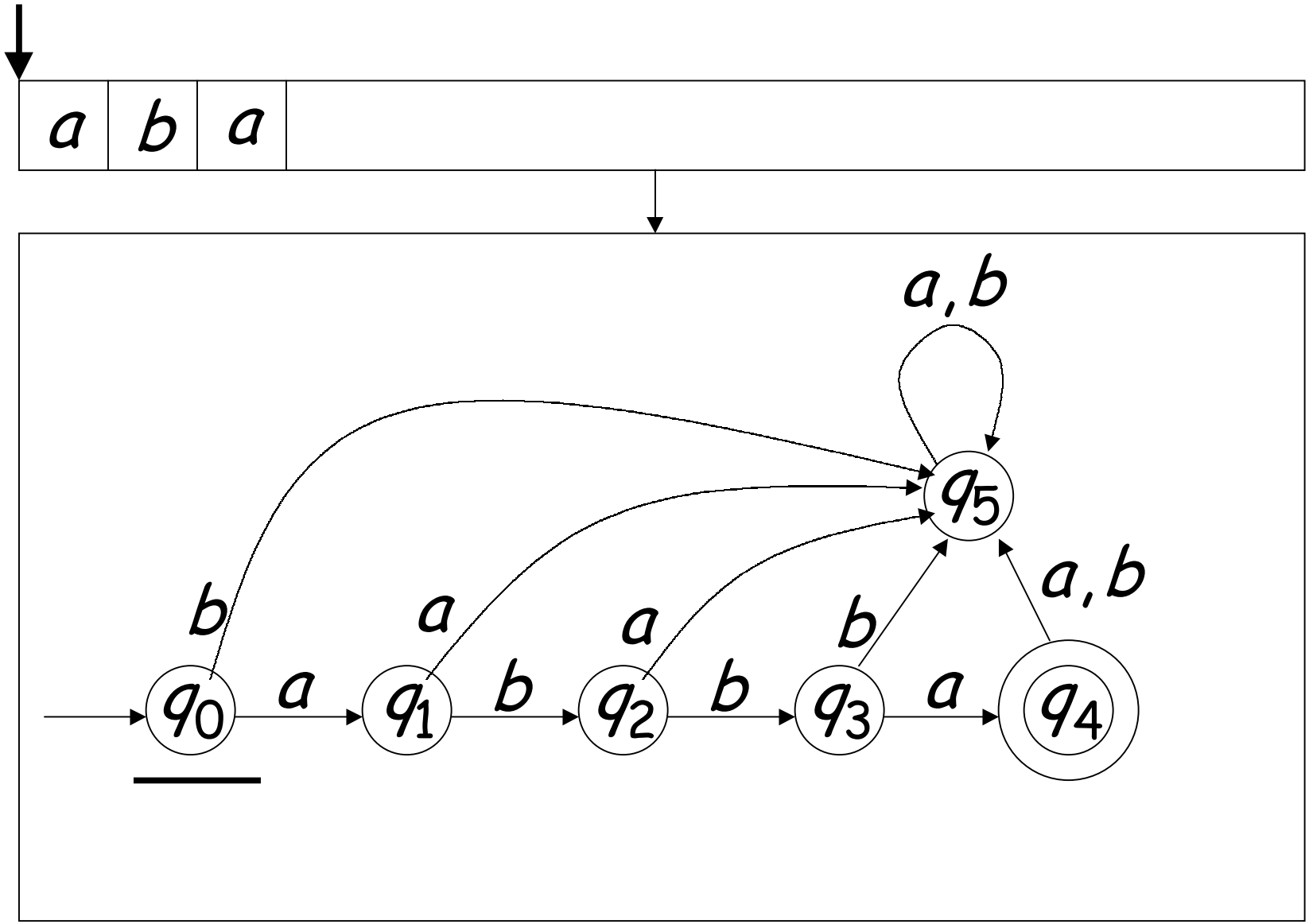


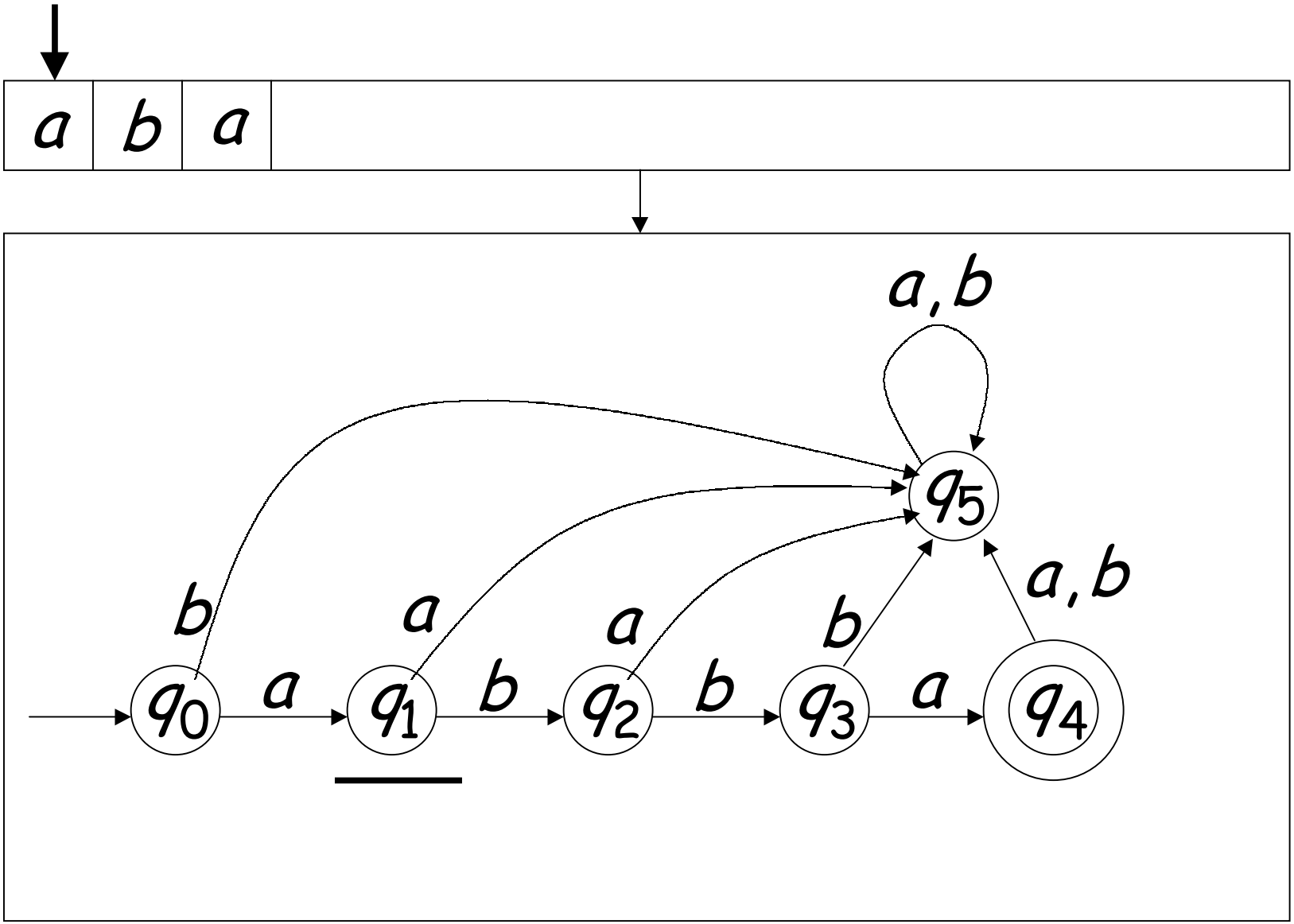

a b b a
a, b
q5
b
a
a
b
a, b
q0
a
q1
b
q2
b
q3
a
q4

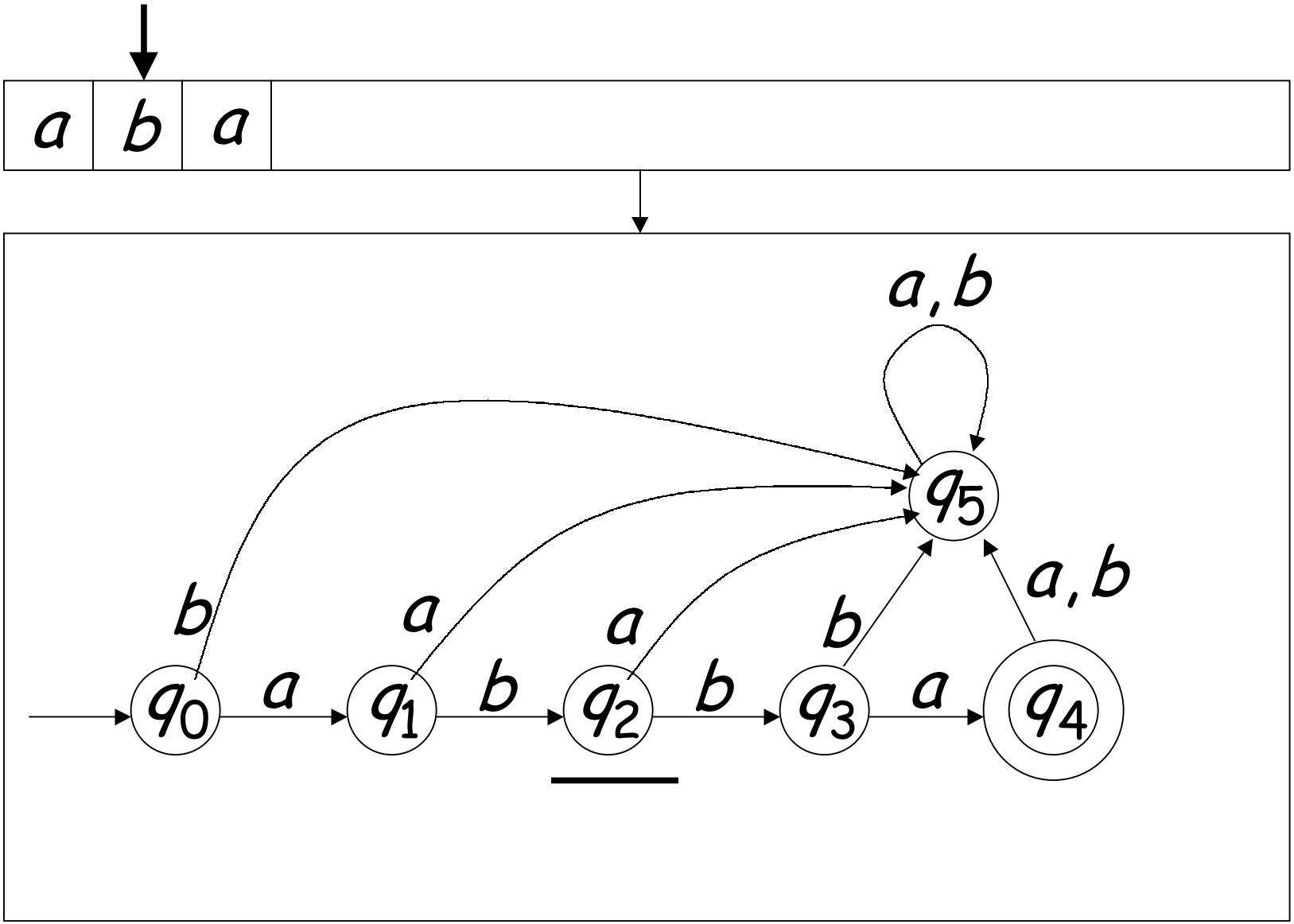

a b b a
a, b
q5
b
a
a
b
a, b
q0
a
q1
b
q2
b
q3
a
q4

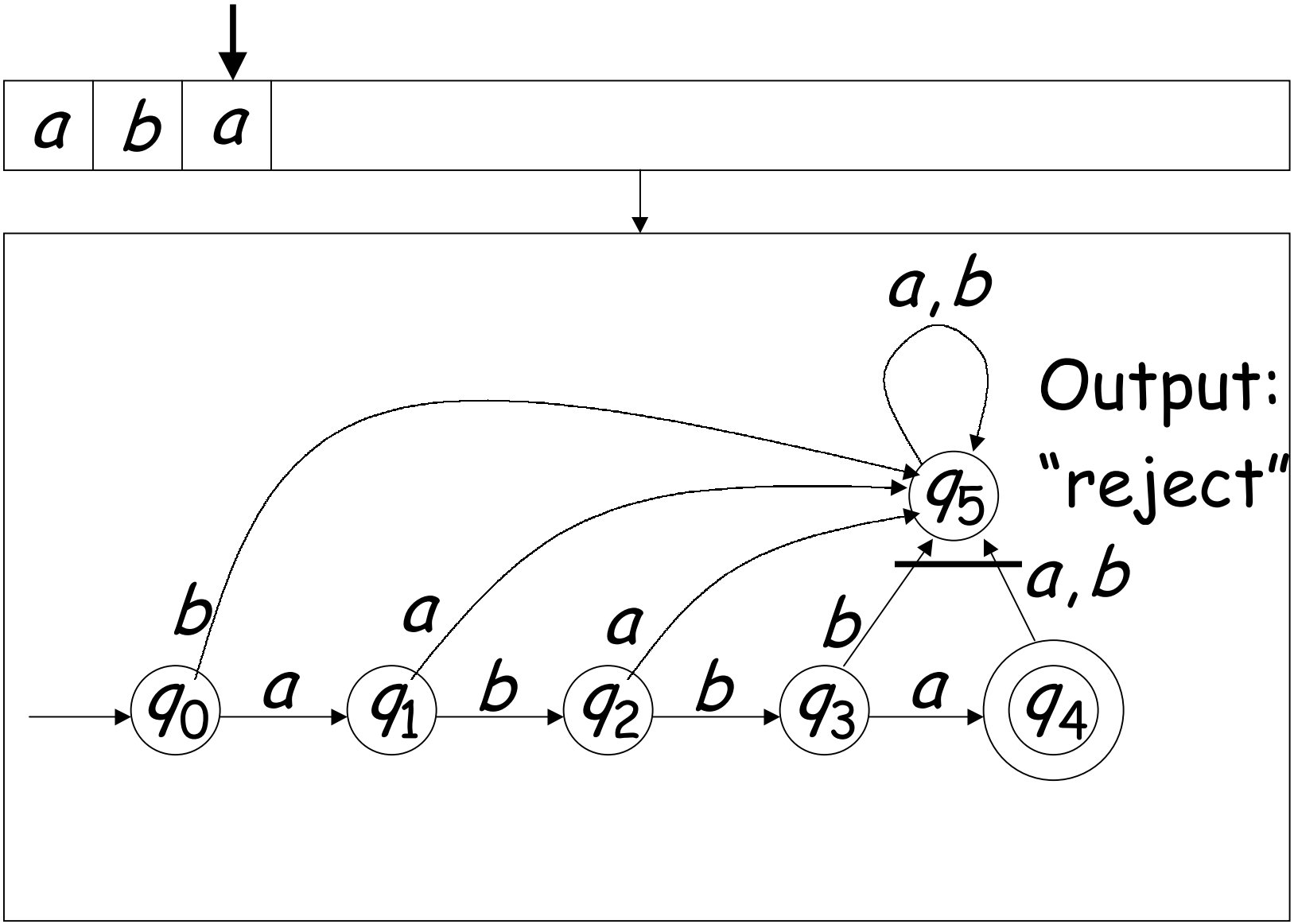

a b b a
a, b
q5
b
a
a
b
a, b
q0
a
q1
b
q2
b
q3
a
q4
Output: "accept"

Rejection




a b a
a, b
q5
b
a
a
b
a, b
q0
a
q1
b
q2
b
q3
a
q4


a
b
a
a, b
q5
b
a
a
b
a, b
q0
a
q1
b
q2
b
q3
a
q4


a b a
a,b
Output:
"reject"
q5
a,b
b
a
a
b
q0
a
q1
b
q2
b
q3
a
q4

Formalities

Deterministic Finite Accepter (DFA)

$$M = (Q, \Sigma, \delta, q_0, F)$$

Q : set of states

Σ : input alphabet

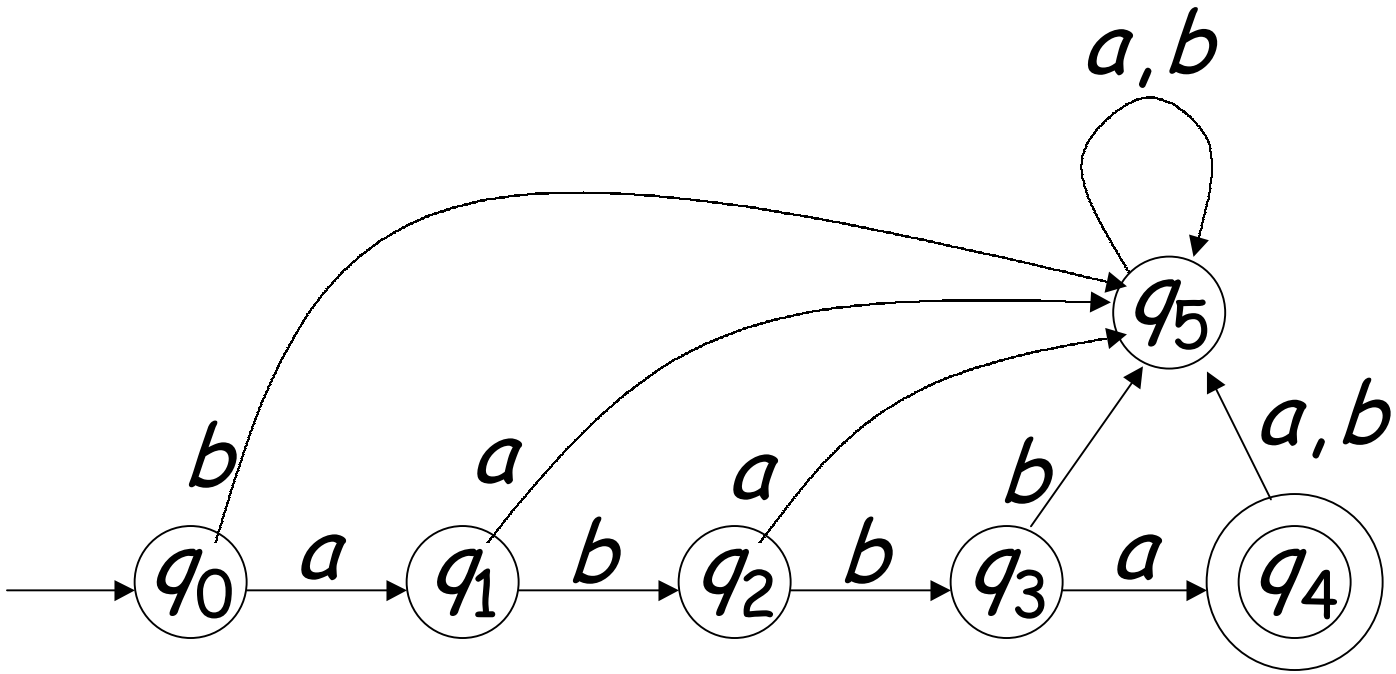
δ : transition function

q_0 : initial state

F : set of final states

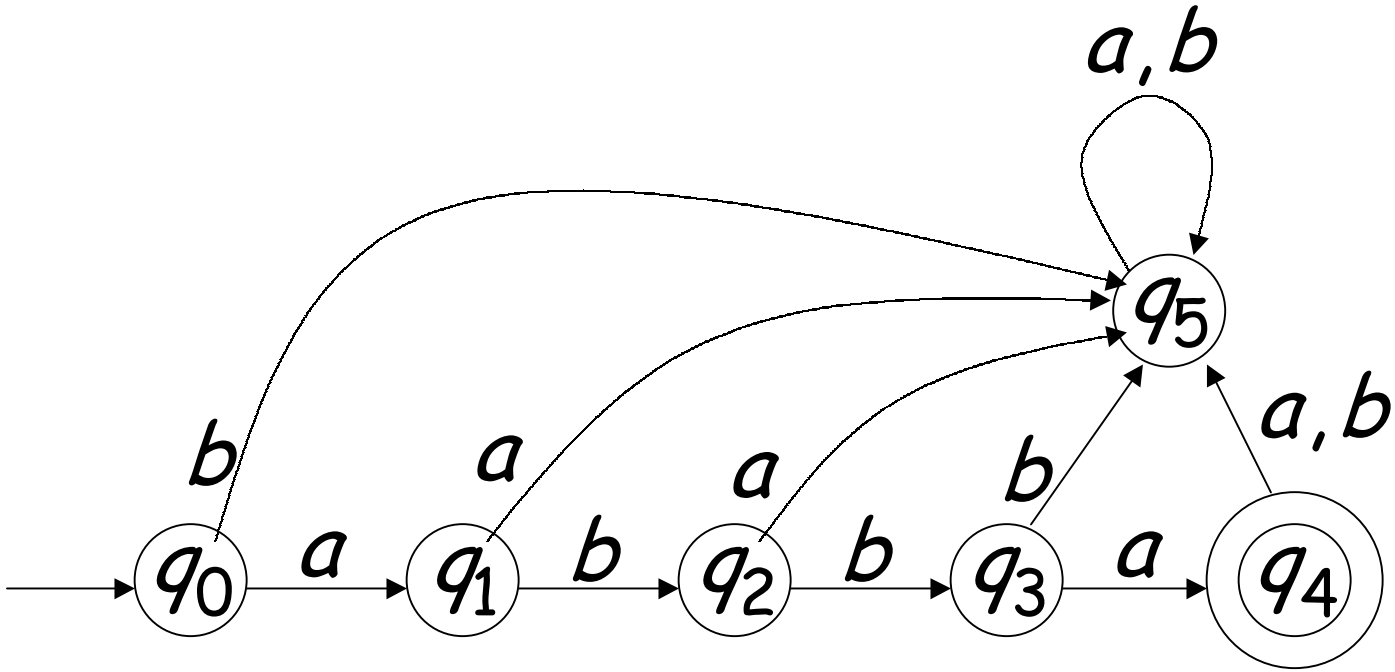
Input Aplhabet Σ

$$\Sigma = \{a, b\}$$

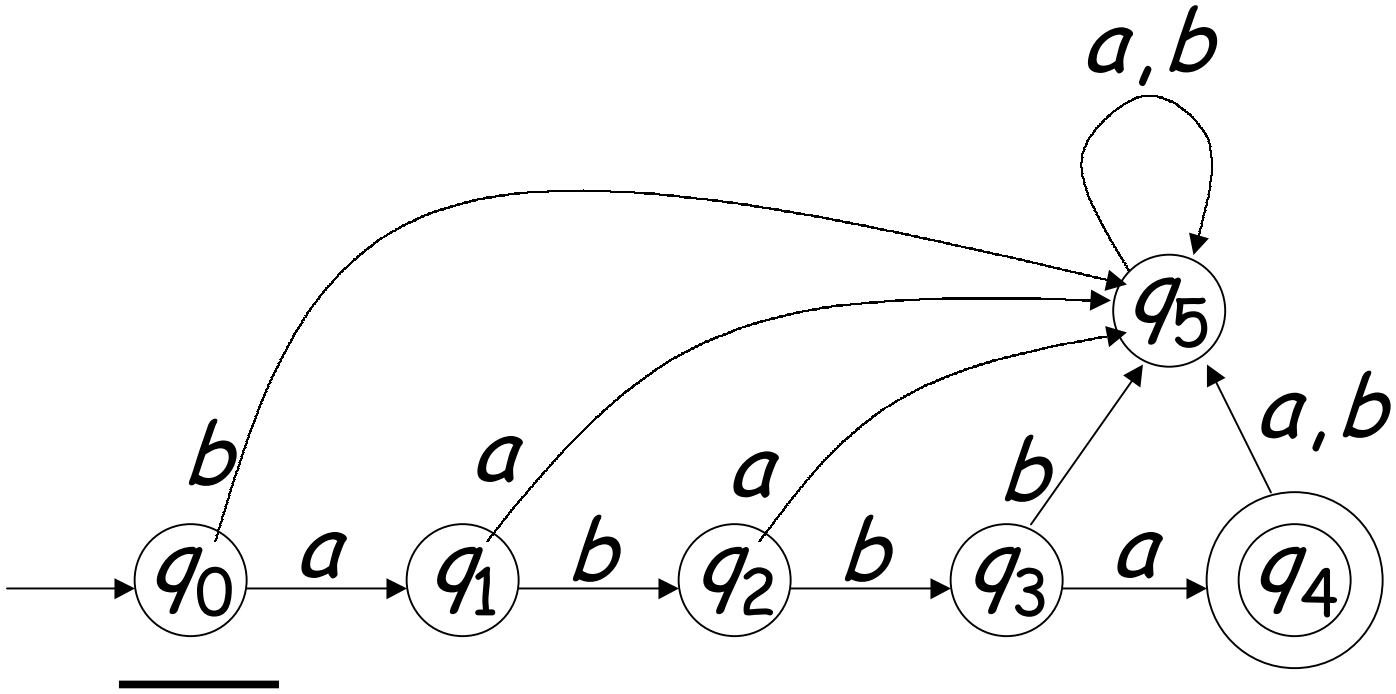


Set of States Q

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$$

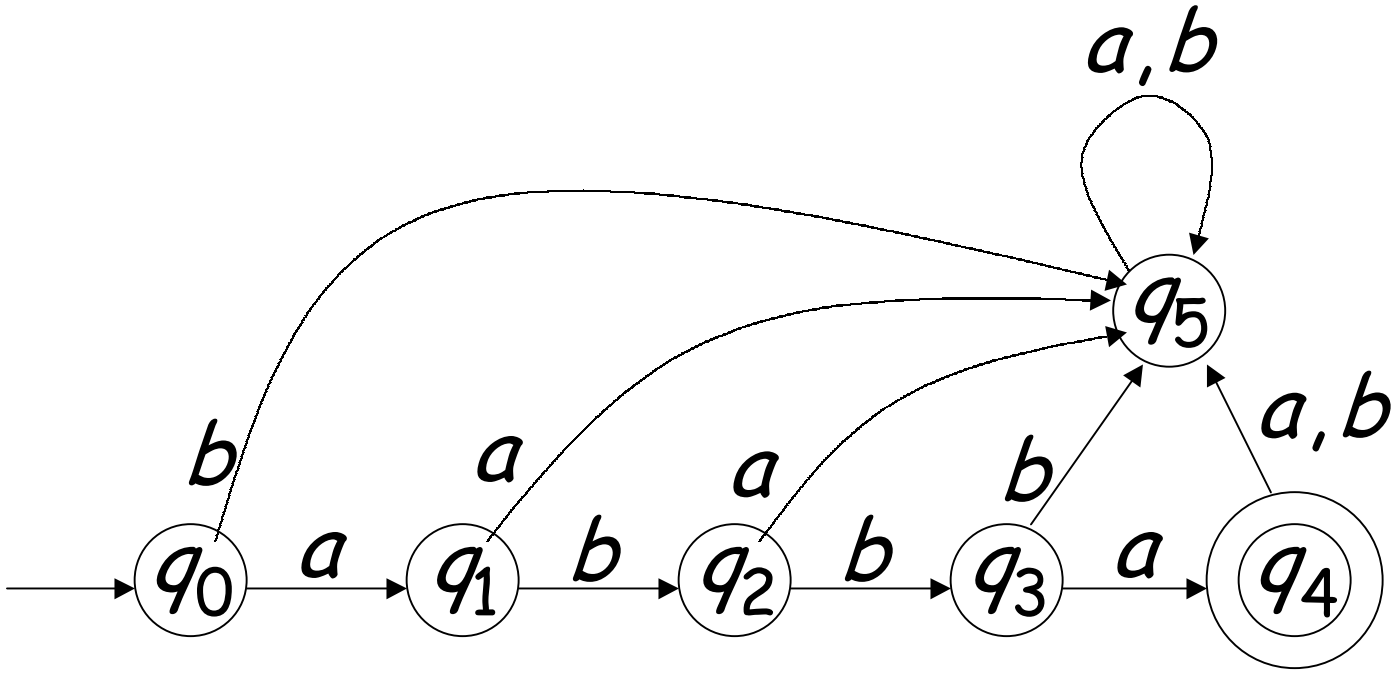


Initial State q_0



Set of Final States F

$F = \{q_4\}$

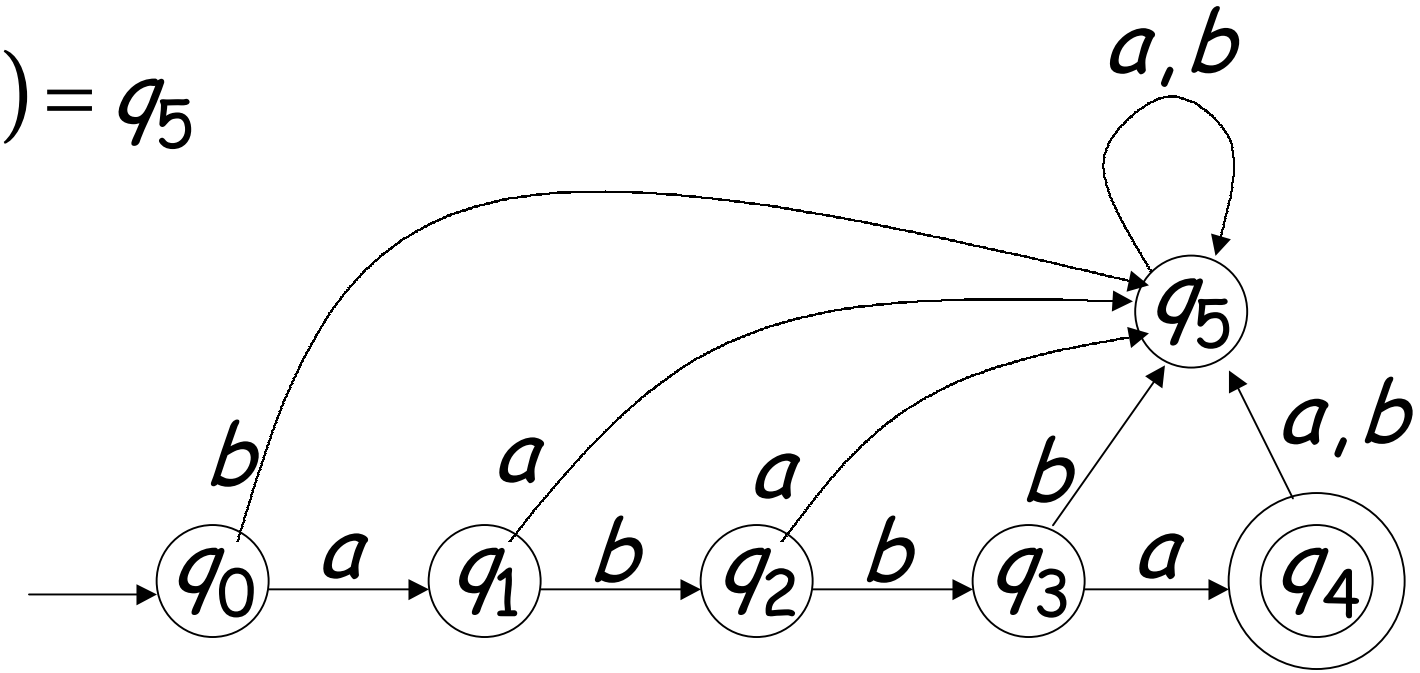


Transition Function δ

$$\delta : Q \times \Sigma \to Q$$

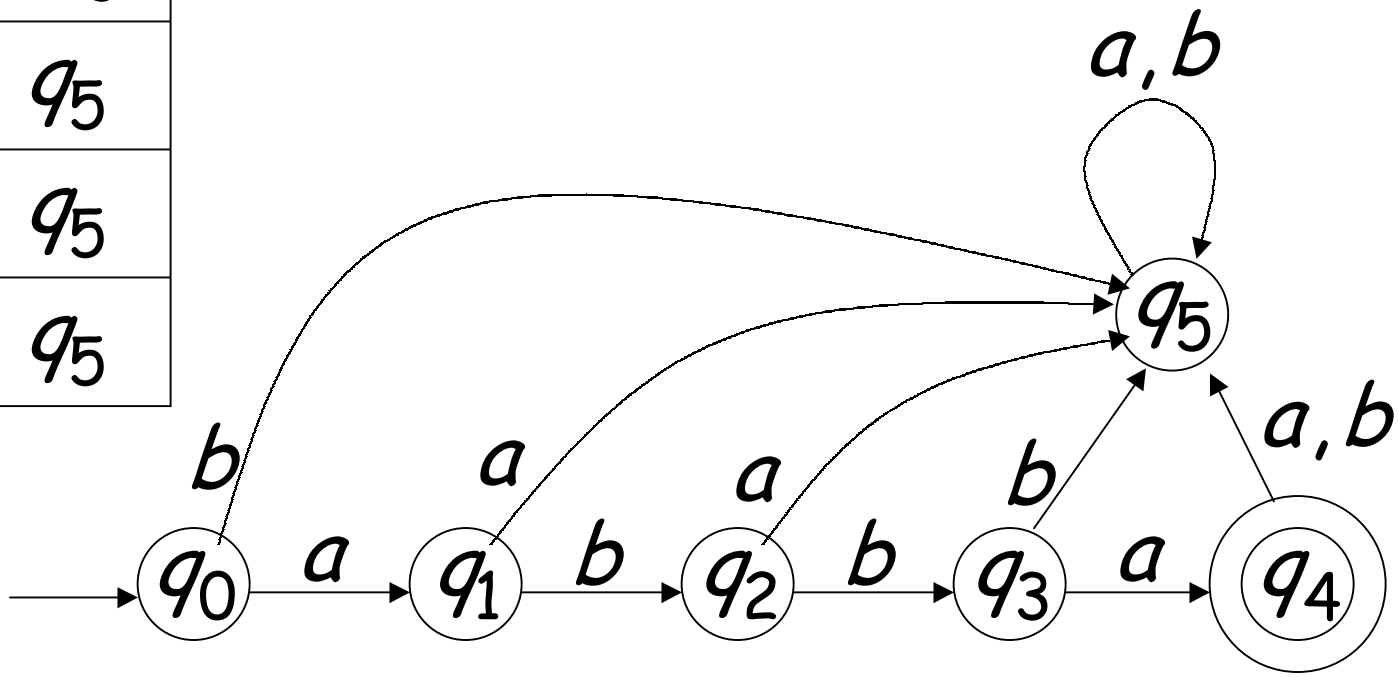
$$\delta(q_0, a) = q_1$$

$$\delta(q_0, b) = q_5$$



Transition Function δ

δ	a	b
q_0	q_1	q_5
q_1	q_5	q_2
q_2	q_2	q_3
q_3	q_4	q_5
q_4	q_5	q_5
q_5	q_5	q_5



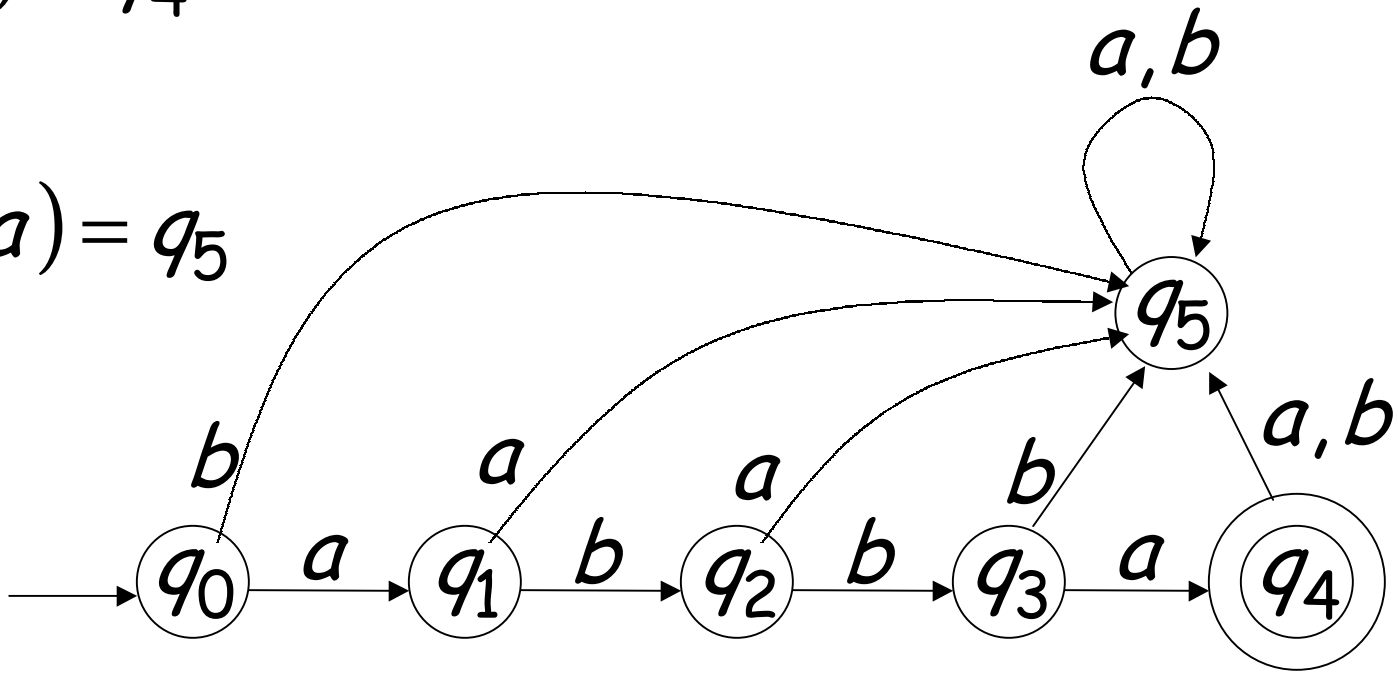
Extended Transition Function δ^*

$$\delta^* : Q \times \Sigma^* \rightarrow Q$$

$$\delta^*(q_0, ab) = q_2$$

$$\delta^*(q_0, abba) = q_4$$

$$\delta^*(q_0, abbaa) = q_5$$



Recursive Definition

$$\delta^*(q,\lambda) = q$$

$$\delta^*(q,wa) = \delta(\delta^*(q,w),a)$$

$$\delta^*(q_0,ab) =$$

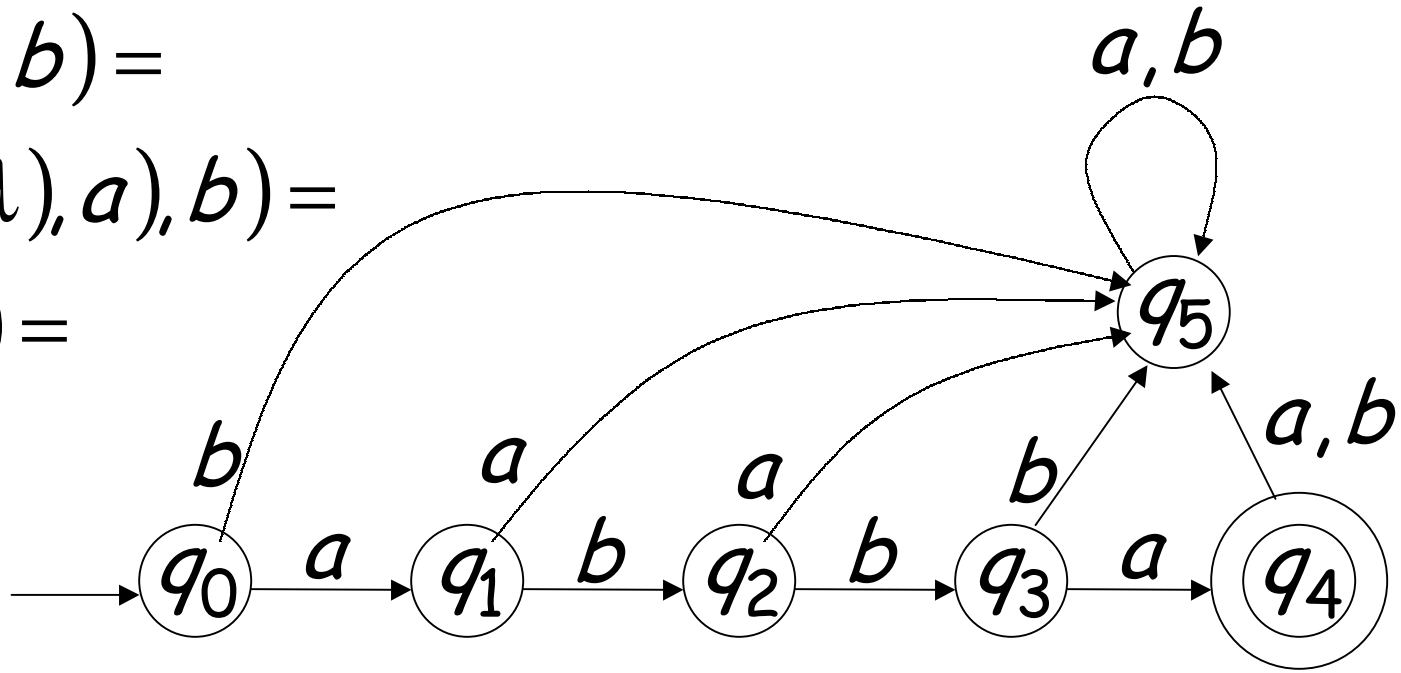
$$\delta(\delta^*(q_0,a),b) =$$

$$\delta(\delta(\delta^*(q_0,\lambda),a),b) =$$

$$\delta(\delta(q_0,a),b) =$$

$$\delta(q_1,b) =$$

$$q_2$$



Languages Accepted by DFAs

Take DFA M

Definition:

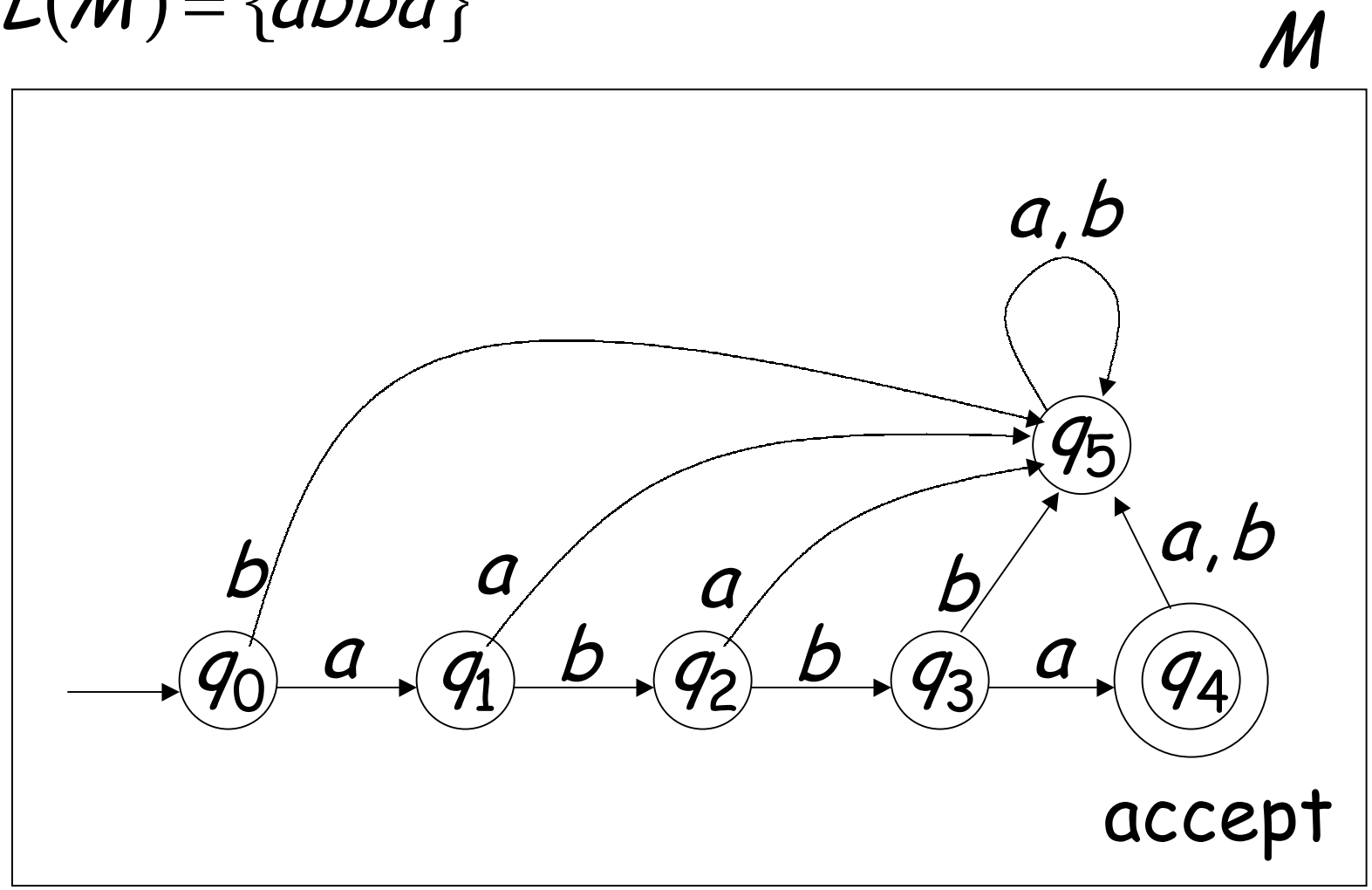
The language $L(M)$ accepted by M
contains all input strings accepted by M

In other words:

$L(M)$ = { strings that drive M to a final state}

Example

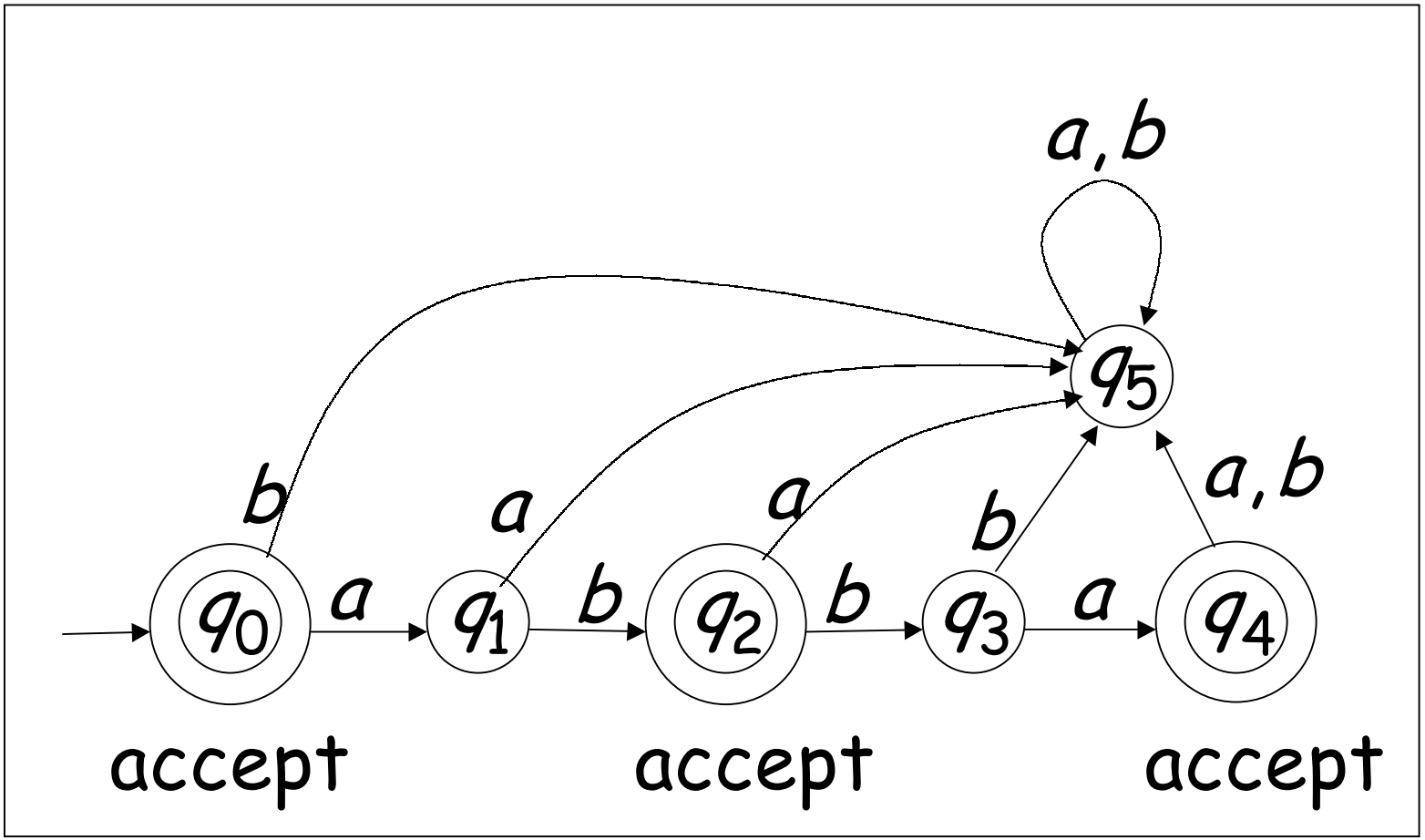
$L(M) = \{abba\}$



Another Example

$L(M) = \{\lambda, ab, abba\}$

M

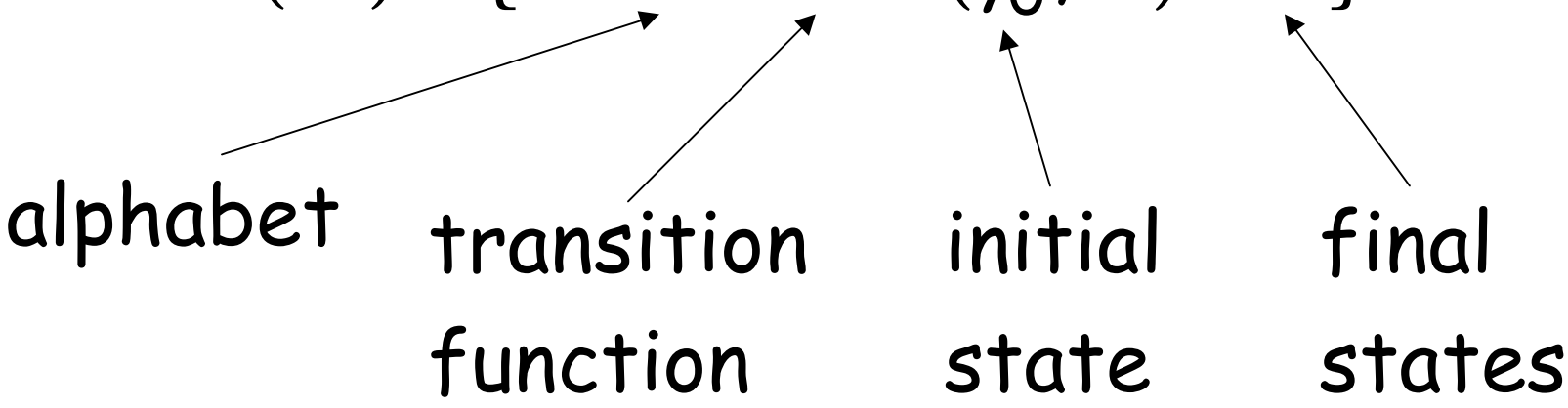


Formally

For a DFA $M = (Q, \Sigma, \delta, q_0, F)$

Language accepted by M :

$$L(M) = \{w \in \Sigma^* : \delta^*(q_0, w) \in F\}$$



Observation

Language accepted by M:

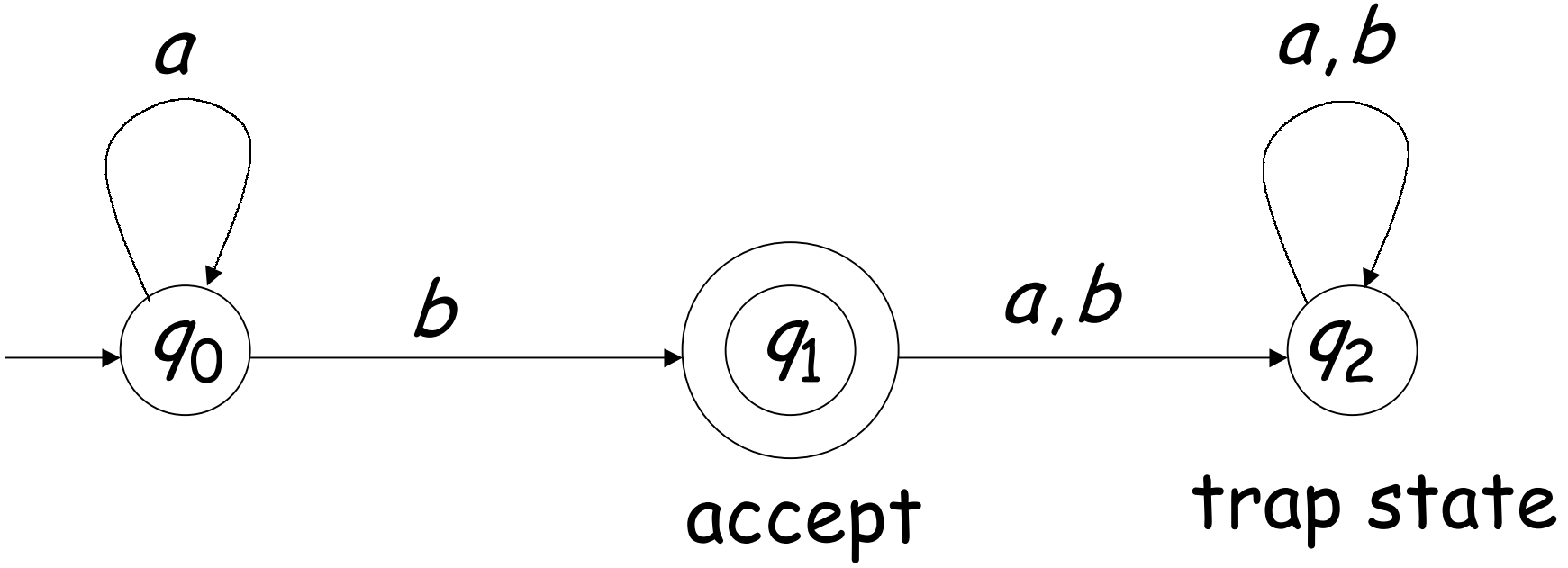
$$L(M) = \{w \in \Sigma^* : \delta^*(q_0, w) \in F\}$$

Language rejected by M:

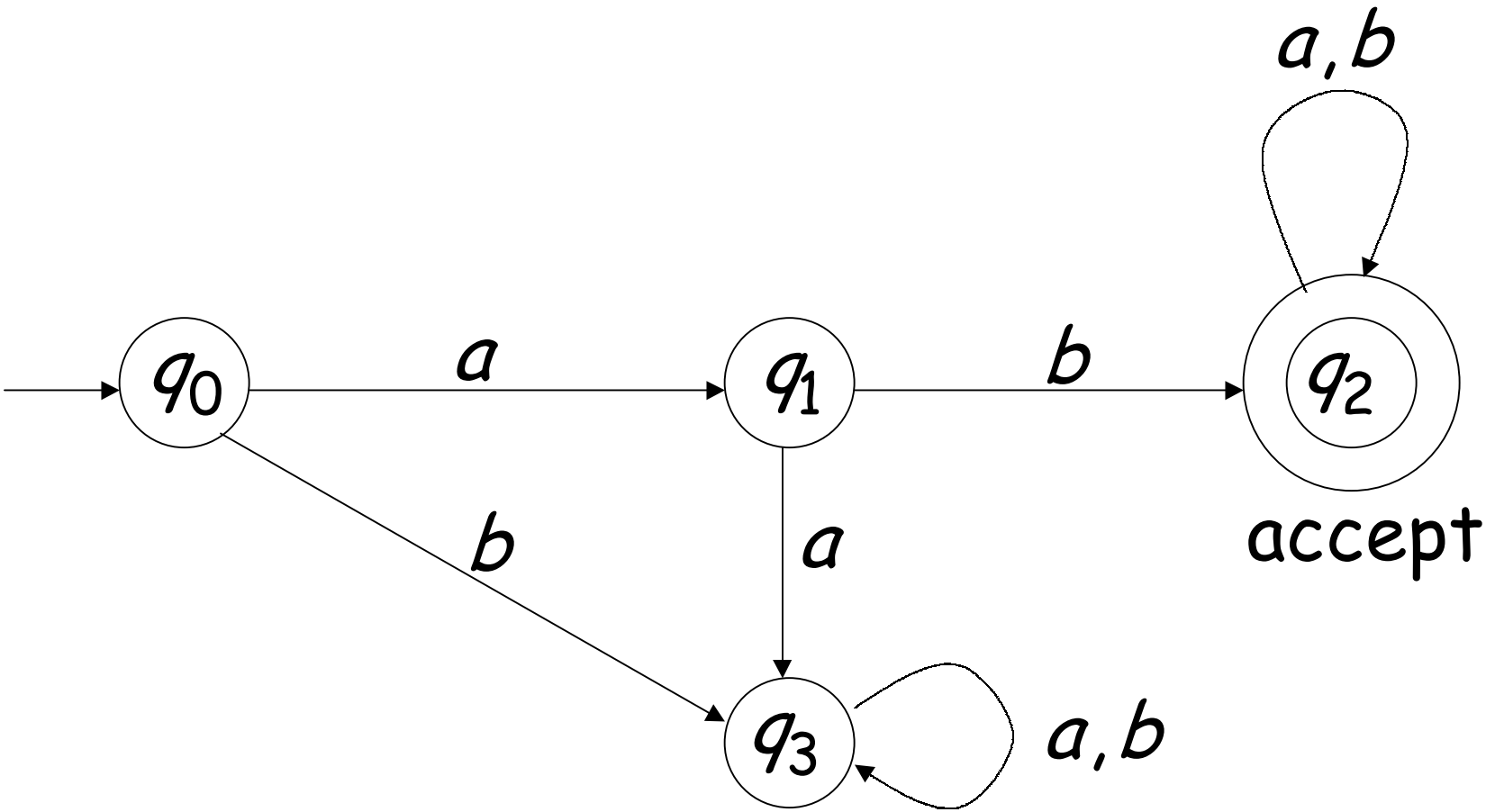
$$\overline{L(M)} = \{w \in \Sigma^* : \delta^*(q_0, w) \notin F\}$$

More Examples

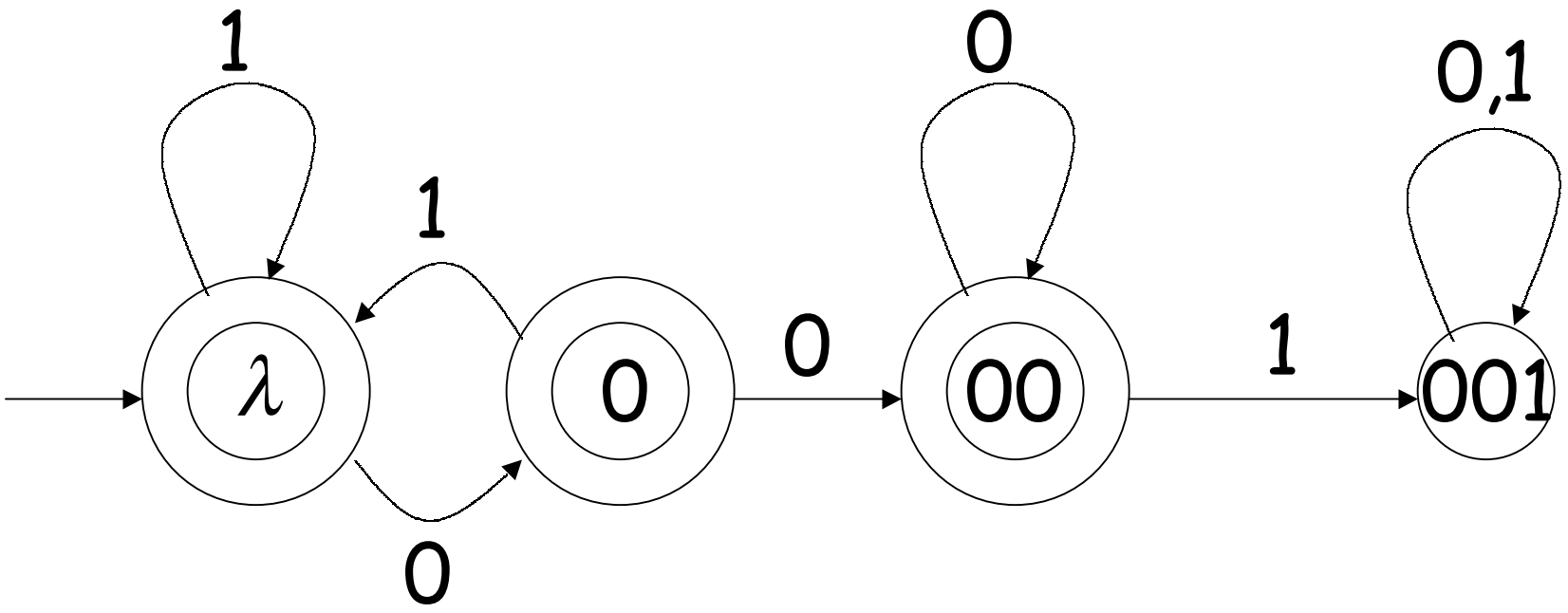
$$L(M) = \{a^n b : n \geq 0\}$$



$L(M)$ = { all substrings with prefix ab }



$L(M)$ = { all strings without substring 001 }



Regular Languages

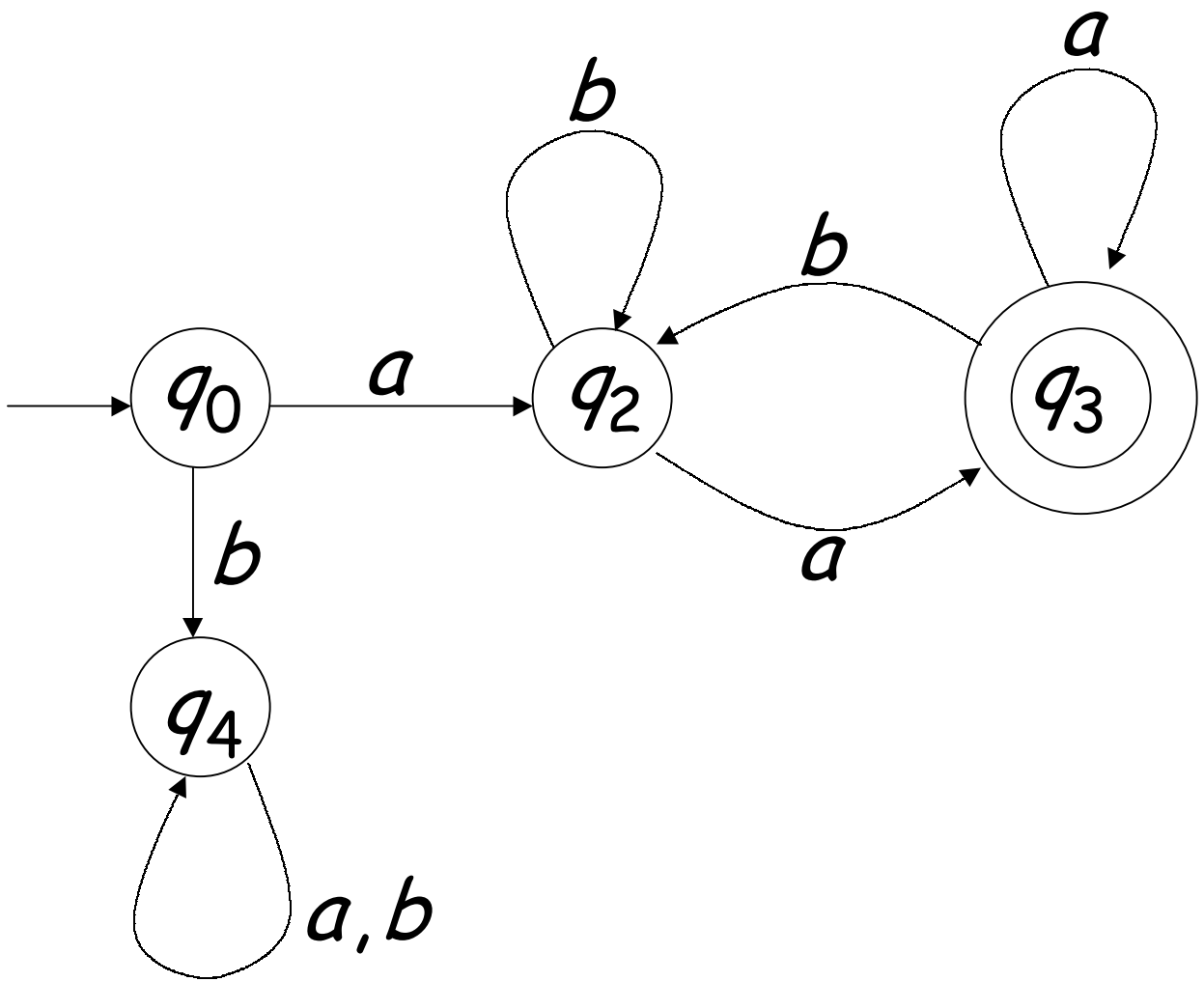
Definition:

A language L is regular

if there is a DFA M such that

$$L = L(M)$$

All the regular languages form a family

Example

The language $L = \{awa : w \in \{a,b\}^*\}$ is regular: